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# A STUDY ON TIME-LIKE COMPLEMENTARY RULED SURFACES IN THE MINKOWSKI *n*-SPACE*

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**Abstract** – For the first time in the Minkowski space, an $R_1^n$ time-like complementary ruled surface is described and relations are given connected with an asymptotic and tangential bundle of the time-like complementary ruled surface. Furthermore, theorems are given related to edge space, central space and central ruled surface of this complementary ruled surface.

**Keywords –** Minkowski space, ruled surfaces, complementary ruled surfaces

## 1. INTRODUCTION

$(k+1)$-dimensional ruled surfaces and their invariants have been studied in *n*-dimensional Euclidean space $E^n$ [1-3]. In reference [4], complementary ruled surfaces and the definitions and theorems for these surface edges and central spaces have been investigated.

Taking Minkowski space $R_1^n$ instead of Euclidean space $E^n$, $(k+1)$-dimensional ruled surfaces and the definition and properties of their edge and central spaces have been explained in references [5, 6]. Furthermore, in reference [7], $(k-m+1)$-dimensional time-like central ruled surfaces have been investigated for the first time in Minkowski space $R_1^n$.

In this study we have studied time-like complementary ruled surfaces and their tangential and asymptotic bundles in $R_1^n$, and their related results have been obtained. Theorems related to edge spaces, which are produced when the dimension of tangential bundle is equal to the dimension of asymptotic bundle, have been given. Moreover, for the case of inequality for the dimensions of these bundles, central space is produced and their theorems are also given.

## 2. PRELIMINARIES

We shall assume throughout the paper that all manifolds, maps, vector fields, etc,... are differentiable of class $C^\infty$. Let $V$ be a real vector space. A bilinear form on $V$ is a bilinear function

$$\langle\ ,\ \rangle : V \times V \to R$$

and we consider only the symmetric case: $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$ for all $\vec{v}, \vec{w} \in \vec{V}$. If a symmetric bilinear form $\langle\ ,\ \rangle$ on $V$ is positive (negative) definite provided $\vec{v} \neq 0$ implies $\langle \vec{v}, \vec{v} \rangle > 0$ $(< 0)$ and non-degenerate provided $\langle \vec{v}, \vec{w} \rangle = 0$ for all $\vec{w} \in \vec{V}$ implies $\vec{v} = 0$. If $\langle\ ,\ \rangle$ is a symmetric bilinear form on $V$, then for any subspace $W$ of $V$ the restriction $\langle\ ,\ \rangle\big|_{(W \times W)}$, denoted merely by $\langle\ ,\ \rangle\big|_W$, is again

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symmetric and bilinear. The index of a symmetric bilinear form $\langle\,,\rangle$ on $V$ is the largest integer that is the dimension of a subspace $W \subset V$ on which $\langle\,,\,\rangle|_W$ is negative definite.

Let $R^n$ be the *n*-dimensional vector space. The following symmetric, bilinear and non-degenerate metric tensor is called the Lorentz metric on $R^n$.

$$\langle X,Y\rangle = \sum_{i=1}^{n-1} x_i y_i - x_n y_n \ , \ X=(x_1,x_2,...,x_n),\ Y=(y_1,y_2,...,y_n)$$

$R^n$ together with the Lorentz metric is called the *n*-dimensional Minkowski space and denoted by $R_1^n$. A curve $\alpha$ in $R_1^n$ is time-like curve if $\langle\dot{\alpha},\dot{\alpha}\rangle<0$, where $\dot{\alpha}$ is the velocity vector of $\alpha$. Further, the basic definitions and theorems related to the Minkowski space $R_1^n$ have been found in [8].

Let $\alpha$ be a time-like curve and $\{e_1(t),e_2(t),....,e_k(t)\}$ be an orthonormal set of vectors spanning the $k$-dimensional subspace $E_k(t)$ of $T_{R_1^n}(\alpha(t))$. We get a $(k+1)$-dimensional surface in $R_1^n$ if the subspace $E_k(t)$ moves along the curve $\alpha$. This surface is called a $(k+1)$-dimensional time-like ruled surface and denoted by $M$. The curve $\alpha$ is called the base curve and the subspace $E_k(t)$ is called the generating space at point $\alpha(t)$. A parameterization of this ruled surface is as follows

$$\phi(t,u_1,u_2,...,u_k)=\alpha(t)+\sum_{i=1}^{k}u_i e_i(t).$$

Taking the derivative of $\phi$ with respect to *t* and $u_i$ $(1\le i\le k)$ we get

$$\phi_t=\dot{\alpha}(t)+\sum_{i=1}^{k}u_i\dot{e}_i(t)$$

$$\phi_{u_i}=e_i(t),\ 1\le i\le k.$$

In this paper, we will assume that

$$\left\{\dot{\alpha}(t)+\sum_{i=1}^{k}u_i\dot{e}_i(t),e_1(t),e_2(t),...,e_k(t)\right\}$$

is a linear independent system, and the subspace $E_k(t)$ is a space-like subspace. We call

$$Sp\{e_1,e_2,...,e_k,\dot{e}_1,\dot{e}_2,...,\dot{e}_k\}$$

asymptotic bundle of $M$ with respect to $E_k(t)$ and denoted by $A(t)$. We have

$$\dim A(t)=k+m,\ 0\le m\le k.$$

There exists an orthonormal basis of $A(t)$ which we denote as follows

$$\{e_1,e_2,.....,e_k,a_{k+1},a_{k+2},....,a_{k+m}\}.$$

The space

$$Sp\{e_1,e_2,...,e_k,\dot{e}_1,\dot{e}_2,...,\dot{e}_k,\dot{\alpha}\}$$

is called the tangential bundle of $M$ with respect to $E_k(t)$ and it is denoted by $T(t)$. It can be easily seen that

$$k+m \le \dim T(t) \le k+m+1,\ 0 \le m \le k .$$

Because of $E_k(t)$ is a space-like subspace and $\alpha$ is a time-like curve, if $\dim T(t) = k+m$, then the asymptotic bundle $A(t)$ of $M$ is a time-like subspace in $R_1^n$. If $\dim T(t) = k+m+1$, then the asymptotic bundle $A(t)$ of *M* is a space-like subspace of $R_1^n$. It is expressed that the tangential bundle $T(t)$ of the time-like ruled surface $M$ is always a time-like subspace in $R_1^n$, [6].

**Theorem 1.** Let $M$ be a $(k+1)$-dimensional time-like ruled surface in $R_1^n$ and $A(t)$ is an asymptotic bundle of *M*. We can choose an orthonormal base $\{e_1(t), e_2(t), ...., e_k(t)\}$ of $E_k(t)$ such that the following relations are held,

$$\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij} e_j + \kappa_i a_{k+i},\ \kappa_i > 0,\ 1 \le i \le m$$

$$\dot{e}_s = \sum_{j=1}^{k} \alpha_{sj} e_j,\ m+1 \le s \le k .$$

If $\dim T(t) = k+m+1$ we find that

$$\dot{\alpha} \notin A(t) = Sp\{e_1, e_2, ..., e_k, a_{k+1}, a_{k+2}, ..., a_{k+m}\}.$$

In this case

$$\{e_1, e_2 ....., e_k, a_{k+1}, a_{k+2} ...., a_{k+m}, a_{k+m+1}\}$$

is an orthonormal base of $T(t)$. Hence, we can write for $\eta_{m+1} \ne 0$

$$\dot{\alpha} = \sum_{i=1}^{k} \zeta_i e_i + \sum_{v=1}^{m} \eta_v a_{k+v} + \eta_{m+1} a_{k+m+1} . \tag{1}$$

For the arbitrary base curve $P(t)$, it is quite possible to write

$$P(t) = \alpha(t) + \sum_{i=1}^{k} u_i(t) e_i(t).$$

Taking the derivative of the last equation and if we substitute equation (1) and Theorem1, we reach the following equation where the derivative vector $\dot{P}(t)$ of points $P$ which satisfy

$$\kappa_s u_s + \eta_s = 0,\ 1 \le s \le m$$

are in the space $Sp\left\{e_1, e_2, ..., e_k, \dot{\alpha}\right\}$. Because $\kappa_s > 0$, $(1 \le s \le m)$, there are $m$ solutions of the given system of equations, which uniquely determine $u_s$, $(1 \le s \le m)$. The remaining $k-m$ numbers of $u_i$ can be chosen arbitrarily. Therefore, the points $P(t)$, which are found by these remaining arbitrarily chosen $u_i$'s, form a $(k-m)$-dimensional subspace of $E_k(t)$. This subspace is called the central space of *M* and is denoted by $Z_{k-m}(t)$. Since $E_k(t)$ is a space-like subspace, the central space $Z_{k-m}(t)$ is a space-like subspace, [6].

Now we define a $(k-m+1)$-dimensional ruled surface in the following way. The above found central space becomes the new generating space which moves along the base curve $\alpha$ of $M$. Because we considered $\alpha$ to be a time-like curve, and $Z_{k-m}(t)$ to be a space-like subspace, the central ruled surface is a time-like ruled surface, [6]. The central ruled surface is denoted by $\Omega$.

**Theorem 2.** Let $M$ be a $(k+1)$-dimensional time-like ruled surface in $R_1^n$ and $\Omega$ be a $(k-m+1)$-dimensional time-like central ruled surface contained in $M$. Now we complete the orthonormal frame $\{e_1(t), e_2(t), ...., e_k(t)\}$ of generating space $E_k(t)$ by an arbitrary orthonormal frame $\{a_{k+1}, a_{k+2}, ...., a_{k+m}, a_{k+m+1}, a_{k+m+2}, ..., a_n\}$ of the orthogonal complement, called a complementary orthonormal frame. From the orthogonality conditions we obtain by differentiation, [7].

$$\dot{a}_{k+i} = -\kappa_i e_i + \sum_{i=1}^{m} \tau_{i\ell} a_{k+\ell} + w a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \gamma_{i\lambda} a_{k+m+\lambda}$$

$$\dot{a}_{k+m+1} = -\sum_{\ell=1}^{m} w_\ell a_{k+\ell} - \sum_{\lambda=2}^{n-k-m} \beta_\lambda a_{k+m+\lambda} \tag{2}$$

$$\dot{a}_{k+m+\xi} = \sum_{\ell=1}^{m} w_{\xi\ell} a_{k+\ell} + \beta_\xi a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \beta_{\xi\lambda} a_{k+m+\lambda}\ ,\ 2 \le \xi \le n-k-m\ .$$

A leading curve $\alpha$ of $M$ is a leading curve of the central ruled surface $\Omega$ if its tangent vector has the form

$$\dot{\alpha} = \sum_{j=1}^{k} \varsigma_j e_j + \eta_{m+1} a_{k+m+1} \tag{3}$$

for $\eta_{m+1} \neq 0$, $a_{k+m+1}$ is a unit vector well defined up to the sign with the property that $\{e_1, e_2, ...., e_k, a_{k+1}, a_{k+2}, ..., a_{k+m}, a_{k+m+1}\}$ is an orthonormal frame of the tangent bundle of $M$.

Any leading curve $P(t)$ with respect to the leading curve $\alpha(t)$ is given by

$$P(t) = \alpha(t) + \sum_{s=1}^{k-m} x_{m+s}(t) e_{m+s}(t)$$

differentiating the last equation and using Theorem 1, equation (3) we have

$$\dot{P}(t) = \sum_{\ell=1}^{m} \left(\varsigma_\ell + x_{m+s} \alpha_{(m+s)\ell}\right) e_\ell + \sum_{s=1}^{k-m} \left(\varsigma_{m+s} + \dot{x}_{m+s}\right) e_{m+s} + \eta_{m+1} a_{k+m+1}\ .$$

The points $P(t)$ satisfying the equations $\varsigma_{m+s} + \dot{x}_{m+s} = 0$, $(1 \le s \le k-m)$ constitute orthogonal trajectory of $\Omega$ time-like central ruled surface.

## 3. TIME-LIKE COMPLEMENTARY RULED SURFACES

For $\forall t \in I$, orthonormal base $\{a_{k+m+2}, a_{k+m+3}, ...., a_n\}$, which completes orthonormal base of $\{e_1, e_2, ...., e_k, a_{k+1}, a_{k+2}, ..., a_{k+m}, a_{k+m+1}\}$ of tangential bundle $T(t)$ of $M$ to the base of $\{e_1, e_2, ...., e_k, a_{k+1}, a_{k+2}, ..., a_{k+m}, a_{k+m+1}, ..., a_n\}$ of $R_1^n$, is called complementary orthonormal base. Since the space generated $\{e_1, e_2, ...., e_k, a_{k+1}, ..., a_{k+m}, a_{k+m+1}\}$ is a time-like subspace, the space generated $\{a_{k+m+2}, a_{k+m+3}, ...., a_n\}$ is a space-like subspace. The orthonormal system

$\{a_{k+m+2}, a_{k+m+3}, ...., a_n\}$ spans a $(n-k-m-1)$-dimensional subspace of tangent space $T_{R_1^n}(\alpha(t))$ at the point of $\alpha(t) \in R_1^n$. This subspace is denoted by $F(t)$ and is given by

$$F(t) = Sp\{a_{k+m+2}(t), a_{k+m+3}(t), ..., a_n(t)\}.$$

As space-like subspace $F(t)$ moves along the time-like curve $\alpha$, using leading curve $\alpha$ of $(k+1)$-dimensional time-like ruled surface $M$ with central ruled surface $\Omega$, it generates a $(n-k-m)$-dimensional surface which is not contained by $M$ in $R_1^n$. This surface is called a $(n-k-m)$-dimensional time-like complementary ruled surface of $(k+1)$-ruled surface $M$ and is denoted by $\Psi_\alpha$. For this ruled surface we can give the following parameterization

$$\Psi_\alpha(t, x_2, x_3, ..., x_{n-k-m}) = \alpha(t) + \sum_{\lambda=2}^{n-k-m} x_\lambda a_{k+m+\lambda}(t)$$

The set of

$$X : JxF^{n-k-m-1} \to E^n$$

$$(t, x_2, x_3, ..., x_{n-k-m}) \to \mathrm{X}(t, x_2, x_3, ..., x_{n-k-m}) = \sum_{\lambda=2}^{n-k-m} x_\lambda a_{k+m+\lambda}(t)$$

produced by $a_{k+m+\lambda}(t)$, $(2 \le \lambda \le n-k-m)$ vectors of $F(t)$ space-like subspace is called $(n-k-m)$-dimensional directional conical of time-like complementary ruled surface $\Psi_\alpha$ where

$$rank\left[\mathrm{X}_t, \mathrm{X}_{x_2}, ..., \mathrm{X}_{x_{n-k-m}}\right] = rank\left[\sum_{\lambda=2}^{n-k-m} x_\lambda \dot{a}_{k+m+\lambda}, a_{k+m+2}, ..., a_n\right] = n-k-m.$$

**Theorem 3.** Let $\Psi_\alpha$ be a $(n-k-m)$-dimensional time-like complementary ruled surface in $R_1^n$ and $F(t)$ the generating space of $\Psi_\alpha$. We can find an interval $J$, such that $t_0 \in J \subset I$ then there exists a unique orthonormal base $\{a_{k+m+2}(t_0), a_{k+m+3}(t_0), ....., a_n(t_0)\}$ of $F(t)$ which satisfies

$$\left\langle \dot{\bar{a}}_{k+m+\xi}, \bar{a}_{k+m+\lambda} \right\rangle = 0,\ 2 \le \xi, \lambda \le n-k-m. \tag{4}$$

**Proof:** Because $F(t)$ is a space-like subspace of the Minkowski space $R_1^n$, we have for the $\{a_{k+m+2}(t), a_{k+m+3}(t), ...., a_n(t)\}$,

$$\left\langle a_{k+m+\xi}, a_{k+m+\lambda} \right\rangle = \delta_{\xi\lambda} \ \ ,\ \delta_{\xi\lambda} = \begin{cases} 1, & \xi = \lambda \\ 0, & \xi \ne \lambda \end{cases} \ ,\ 1 \le \xi, \lambda \le n-k-m.$$

Let $b_{(k+m+\lambda)(k+m+h)}$, $(1 \le \lambda, h \le n-k-m)$ be the functions which are defined as solutions of the system of differential equations

$$\dot{b}_{(k+m+\lambda)(k+m+h)} + \sum_{\xi=1}^{n-k-m} b_{(k+m+\lambda)(k+m+\xi)} \left\langle \dot{a}_{k+m+\xi}, a_{k+m+h} \right\rangle = 0 \tag{5}$$

and

$$\bar{a}_{k+m+\lambda} = \sum_{\xi=1}^{n-k-m} b_{(k+m+\lambda)(k+m+\xi)} a_{k+m+\xi}.$$

In this case

$$\dot{\bar{a}}_{k+m+\lambda} = \sum_{\xi=1}^{n-k-m} \dot{b}_{(k+m+\lambda)(k+m+\xi)} a_{k+m+\xi} + \sum_{\xi=1}^{n-k-m} b_{(k+m+\lambda)(k+m+\xi)} \dot{a}_{k+m+\xi}$$

and therefore we get

$$\left\langle \dot{\bar{a}}_{k+m+\lambda}, a_{k+m+h} \right\rangle = \sum_{\xi=1}^{n-k-m} \dot{b}_{(k+m+\lambda)(k+m+\xi)} \left\langle a_{k+m+\xi}, a_{k+m+h} \right\rangle + \sum_{\xi=1}^{n-k-m} b_{(k+m+\lambda)(k+m+\xi)} \left\langle \dot{a}_{k+m+\xi}, a_{k+m+h} \right\rangle.$$

For $1 \le s \le n-k-m$,

$$\left\langle \dot{\bar{a}}_{k+m+\lambda}, \bar{a}_{k+m+s} \right\rangle = \sum_{h=1}^{n-k-m} b_{(k+m+s)(k+m+h)} \left\langle \dot{\bar{a}}_{k+m+\lambda}, a_{k+m+h} \right\rangle = 0 .$$

In conclusion we find

$$\frac{d\left\langle \bar{a}_{k+m+\lambda}, \bar{a}_{k+m+\xi} \right\rangle}{dt} = \left\langle \dot{\bar{a}}_{k+m+\lambda}, \bar{a}_{k+m+\xi} \right\rangle + \left\langle \bar{a}_{k+m+\lambda}, \dot{\bar{a}}_{k+m+\xi} \right\rangle = 0 .$$

If we compute the values of solution (5), we get an orthogonal matrix $\left[ b_{(k+m+\lambda)(k+m+h)}(t_0) \right]$ and the base $\left\{ \bar{a}_{k+m+\lambda}(t_0) \right\}$, $(1 \le \lambda \le n-k-m)$ is orthogonal too. Therefore, for each point $t$ it will be orthogonal, that is the condition

$$\begin{aligned} \left\langle \bar{a}_{k+m+\lambda}, \bar{a}_{k+m+s} \right\rangle &= \sum_{\xi=1}^{n-k-m} b_{(k+m+\lambda)(k+m+\xi)} b_{(k+m+s)(k+m+\xi)} \\ &= \delta_{(k+m+\lambda)(k+m+s)} \end{aligned}$$

is satisfied. This yields an orthonormal base with

$$\left\langle \dot{\bar{a}}_{k+m+\xi}, \bar{a}_{k+m+\lambda} \right\rangle = 0 , \ 2 \le \xi, \lambda \le n-k-m .$$

If we take equations (2) and (5) into consideration we find the following

$$\beta_{\xi\lambda} = 0 , \ 2 \le \xi, \lambda \le n-k-m . \tag{6}$$

The vectors subspace of

$$Sp\left\{ a_{k+m+2}(t), a_{k+m+3}(t), \ldots, a_n(t), \dot{a}_{k+m+2}(t), \dot{a}_{k+m+3}(t) \ldots, \dot{a}_n(t) \right\}$$

which is spanned by

$$\left\{ a_{k+m+2}(t), a_{k+m+3}(t), \ldots, a_n(t), \dot{a}_{k+m+2}(t), \dot{a}_{k+m+3}(t) \ldots, \dot{a}_n(t) \right\}$$

is known as asymptotic bundle of $\Psi_\alpha$ with respect to $F(t)$ and denoted as $A_\alpha(t)$. Denoting velocity vector $\dot{\alpha}$ as leading curve $\alpha$ of $(n-k-m)$-dimensional time-like complementary ruled surface $\Psi_\alpha$, the vectors subspace of

$$Sp\left\{ a_{k+m+2}(t), a_{k+m+3}(t), \ldots, a_n(t), \dot{a}_{k+m+2}(t), \dot{a}_{k+m+3}(t), \ldots, \dot{a}_n(t), \dot{\alpha}(t) \right\}$$

which is spanned by the set of

$$\{a_{k+m+2}(t), a_{k+m+3}(t), ..., a_n(t), \dot{a}_{k+m+2}(t), \dot{a}_{k+m+3}(t) ..., \dot{a}_n(t), \dot{\alpha}(t)\}$$

is called tangential bundle of $\Psi_\alpha$ with respect to $F(t)$ and denoted as $T_\alpha(t)$.

As generating space $F(t)$ is a space-like subspace and $\alpha$ is time-like curve

$$\langle a_{k+m+\xi}, a_{k+m+\lambda} \rangle = \delta_{\xi\lambda}, \; \delta_{\xi\lambda} = \begin{cases} 1, & \xi = \lambda \\ 0, & \xi \neq \lambda \end{cases}, \; 1 \leq \xi, \lambda \leq n-k-m$$

$$\langle \dot{\alpha}, \dot{\alpha} \rangle < 0 .$$

If the condition

$$\dim A_\alpha(t) = n-k-m-1+s, \; 0 \leq s \leq n-k-m-1$$

is satisfied for the asymptotic bundle of $A_\alpha(t)$, then

$$n-k-m-1+s \leq \dim T_\alpha(t) \leq n-k-m+s .$$

Let us assume that $\dim T_\alpha(t) = n-k-m-1+s$. Thus $\dim T_\alpha(t) = \dim A_\alpha(t)$. In other words, the orthonormal bases of $A_\alpha(t)$ and $T_\alpha(t)$ are the same. Therefore we can write

$$\dot{\alpha} \in \{a_{k+m+2}(t), a_{k+m+3}(t), ..., a_n(t), \dot{a}_{k+m+2}(t), \dot{a}_{k+m+3}(t), ..., \dot{a}_n(t)\}.$$

In this case, taking equation (6) into consideration with (2) differentiation equations yields

$$\dot{a}_{k+m+\xi} = \sum_{\ell=1}^{m} \omega_{\xi\ell} a_{k+\ell} + \beta_\xi a_{k+m+1}, \; 2 \leq \xi \leq n-k-m. \tag{7}$$

This last equation can be written in the matrix form as

$$\begin{bmatrix} \dot{a}_{k+m+2} \\ \dot{a}_{k+m+3} \\ . \\ . \\ \dot{a}_{n-1} \\ \dot{a}_n \end{bmatrix} = \begin{bmatrix} \omega_{21} & \omega_{22} & . . . . . . . & \omega_{2m} & \beta_2 \\ \omega_{31} & \omega_{32} & . . . . . . . & \omega_{3m} & \beta_3 \\ . & . & & . & . \\ . & . & & . & . \\ \omega_{(n-k-m-1)1} & \omega_{(n-k-m-1)2} & . . . . . . . & \omega_{(n-k-m-1)m} & \beta_{n-k-m-1} \\ \omega_{(n-k-m)1} & \omega_{(n-k-m)2} & . . . . . . . & \omega_{(n-k-m)m} & \beta_{n-k-m} \end{bmatrix} \begin{bmatrix} a_{k+1} \\ a_{k+2} \\ . \\ . \\ a_{k+m} \\ a_{k+m+1} \end{bmatrix}.$$

Hence, we obtain the following equation

$$rank \begin{bmatrix} \omega_{\ell\xi} \\ \beta_\xi \end{bmatrix} = rank[\omega_{\ell\xi}] + 1, \; \begin{matrix} 2 \leq \xi \leq n-k-m \\ 1 \leq \ell \leq m \end{matrix}. \tag{8}$$

From this it is clear that

$$\zeta_\nu = 0. \tag{9}$$

This means that time-like curve $\alpha$ is orthogonal trajectory of $(k+1)$-dimensional time-like ruled surface. Furthermore, as $F(t)$ is space-like subspace and $\alpha$ is time-like curve, there occurs a time-like vector in between vectors $\dot{a}_{k+m+\xi}$, $(2 \le \xi \le n-k-m)$. This means that asymptotic bundle $A_\alpha(t) = T_\alpha(t)$, and this equation shows $A_\alpha(t)$ is time-like subspace in $R_1^n$.

Therefore we can give the following theorem.

**Theorem 4.** Let $\Psi_\alpha$ be $(n-k-m)$-dimensional time-like complementary ruled surface in $R_1^n$ and $T_\alpha(t)$ be tangential bundle of $\Psi_\alpha$. If $\dim T_\alpha(t) = \dim A_\alpha(t)$, then $A_\alpha(t)$ asymptotic bundle is time-like subspace.

Let arbitrary leading curve of time-like complementary ruled surface be

$$Z(t) = \alpha(t) + \sum_{\lambda=2}^{n-k-m} z_\lambda(t) a_{k+m+\lambda}(t). \tag{10}$$

If $Z(t)$ is differentiated with respect to $t$, we get

$$\dot{Z}(t) = \dot{\alpha}(t) + \sum_{\lambda=2}^{n-k-m} \left( \dot{z}_\lambda(t) a_{k+m+\lambda}(t) + z_\lambda(t) \dot{a}_{k+m+\lambda}(t) \right).$$

Here, $Z(t)$ points which satisfy the following equations

$$\sum_{\lambda=2}^{n-k-m} z_\lambda(t) \omega_{\lambda\ell} = 0, \; 1 \le \ell \le m \tag{11}$$

and

$$\sum_{\lambda=2}^{n-k-m} z_\lambda(t) \beta_\lambda + \eta_{m+1} = 0 \tag{12}$$

generate edge space of $\Psi_\alpha$. For $\forall t_1 \in I$, as $\eta_{m+1} \ne 0$, edge space of $\Psi_\alpha$ will never contain leading curve point of $\alpha(t_1)$. Since generating space $F(t)$ is a space-like subspace, edge space is also a space-like subspace.

Thus, we can give the following theorem.

**Theorem 5.** Let $\Psi_\alpha$ be $(n-k-m)$-dimensional time-like complementary ruled surface in $R_1^n$ and $A_\alpha(t)$ and $T_\alpha(t)$ be the asymptotic and tangential bundles of $\Psi_\alpha$, respectively. If $\dim T_\alpha(t) = \dim A_\alpha(t)$, then edge space of $\Psi_\alpha$ is also a space-like subspace.

Now, let us assume that $\alpha(t)$ leading curve is not an orthogonal trajectory of $(k+1)$-dimensional time-like ruled surface $M$ in the $I_0 \subset I$ interval, i.e., one of the equations of (8) and (9) is not valid. Therefore, the time-like complementary ruled surface $\Psi_\alpha$ has a central ruled surface in $I_0 \subset I$. This ruled surface is denoted by $\Omega_\alpha$.

For the arbitrary leading curve of time-like complementary ruled surface given by equation (10), if the following equation is valid

$$\left\langle \dot{Z}(t), \dot{a}_{k+m+\lambda}(t) \right\rangle = 0, \; 2 \le \lambda \le n-k-m \tag{13}$$

then $Z(t)$ curve becomes a leading curve of central ruled surface $\Omega_\alpha$.

If the equations

$$\dot{Z}(t)=\sum_{\nu=1}^{k}\zeta_{\nu}e_{\nu}+\sum_{\lambda=2}^{n-k-m}z_{\lambda}\left(\sum_{\ell=1}^{m}\omega_{\lambda\ell}a_{k+\ell}\right)+\left(\sum_{\xi=2}^{n-k-m}z_{\xi}\beta_{\xi}+\eta_{m+1}\right)a_{k+m+1}+\sum_{\lambda=2}^{n-k-m}\dot{z}_{\lambda}a_{k+m+\lambda}$$

and

$$\dot{a}_{k+m+\xi}=\sum_{\ell=1}^{m}\omega_{\xi\ell}a_{k+\ell}+\beta_{\xi}a_{k+m+1}$$

are substituted into equation (13), for the parameter $z_{\lambda}$, $(2\leq\lambda\leq n-k-m)$ we find the following linear equation system

$$\sum_{\sigma=1}^{m}(z_{\lambda}\omega_{\lambda\sigma})\omega_{\xi\sigma}+\beta_{\xi}\left(\sum_{\lambda=2}^{n-k-m}z_{\lambda}\beta_{\lambda}+\eta_{m+1}\right)=0\,. \tag{14}$$

Thus, from equation (11) and (12), one can easily see that time-like complementary ruled surface $\Psi_{\alpha}$ is independent of leading curve $\alpha(t)$ of $\Omega_{\alpha}$. In equation (14), if $\beta_{\xi}=0$ $(2\leq\lambda\leq n-k-m)$ for $t_1\in I_0$, then leading curve point $\alpha(t_1)$ is in central space of $\Psi_{\alpha}$.

Now we can give the following theorems.

**Theorem 6.** Let $M$ be $(k+1)$-dimensional time-like ruled surface with central ruled surface $\Omega$, and $\Psi_{\alpha}$ be the $(n-k-m)$-dimensional time-like complementary ruled surface of $M$. If equation (9) is not valid, but (8), then central ruled surface $\Omega_{\alpha}$ of complementary ruled surface $\Psi_{\alpha}$ is defined by equations (11) and (12).

**Theorem 7.** Let $M$ be $(k+1)$-dimensional time-like ruled surface with central ruled surface $\Omega$ in $R_1^n$ and $\Psi_{\alpha}$ be the $(n-k-m)$-dimensional time-like complementary ruled surface of $M$. Central ruled surface $\Omega_{\alpha}$ of complementary ruled surface $\Psi_{\alpha}$ is time-like.

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