Iranian Journal of Science & Technology, Transaction A, Vol. 29, No. A2
Printed in The Islamic Republic of Iran, 2005

EXISTENCE RESULTS FOR A CLASS OF SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS*

G. A. AFROUZI** AND M. JAFARI

Department of Mathematics, Faculty of Basic Sciences
Mazandaran University, Babolsar, I. R. of Iran

Abstract – We consider the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u(x) = \lambda f(u(x)); & x \in \Omega \\ u(x) = 0; & x \in \partial\Omega \end{cases}$$

where $\lambda > 0$ is a parameter, Ω is a bounded region in R^N with a smooth boundary, and f is a smooth function. We prove, under some additional conditions, the existence of a positive solution for λ large. We prove that our solution u for λ large is such that $\| u \| := \sup_{x \in \Omega} | u(x) | \to \infty$ as $\lambda \to \infty$. Also, in the case of $N = 1$, we use a bifurcation theory to show that the solution is unstable.

Keywords – Semilinear elliptic problem, positive solution, unstable solution

1. INTRODUCTION

Here we consider the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u(x) = \lambda f(u(x)); & x \in \Omega \\ u(x) = 0; & x \in \partial\Omega \end{cases} \qquad (1)\ \&\ (2)$$

where $\lambda > 0$ is a constant, Ω is a bounded region in R^N with a smooth boundary and f is a smooth function.

First we state the following Theorems, then we establish these Theorems in Section 2.

Theorem 1. 1. If $f(0) < 0$, $\lim_{t\to\infty} f(t)/t = 0$, and f is a smooth function such that $f'(t)$ is bounded below, then, there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_o$, problem (1) and (2) have a solution u where $u \leq 0$ in Ω.

Remark 1. 1. We assume that there exists $c > 0$, $M > 0$ such that

*Received by the editor April 12, 2003 and in final revised form February 7, 2005
**Corresponding author

$$f(x) \geq M, \qquad \forall x \geq 0 \tag{3}$$

and prove that for problem (1) and (2) there is a non-negative solution for large λ. This result has been established in [1], but our method further established that our solution u is such that $\| u \| \to +\infty$ as $\lambda \to +\infty$. Here we use a recent result by Clement and Sweers [2] to create an appropriate sub-solution.

Theorem 1. 2. Assume the hypotheses of Theorem 1.1, and furthermore assume that f satisfies (3). Then there exists a λ^* such that for $\lambda > \lambda^*$ the problem (1), (2) has a non-negative solution u such that $\| u \| \to +\infty$ as $\lambda \to +\infty$.

Theorem 1. 3. Let $f(0) = 0$, $f'(0) < 0$, $\lim_{t\to\infty} f(t)/t = 0$ and f is eventually increasing.

Also, assume that there exists $\beta > 0$ such that $f(t) < 0$ for $t \in (0, \beta)$ and $f(t) > 0$ for $t > \beta$. Then, there exists $\overline{\lambda}$ such that for $\lambda \geq \overline{\lambda}$, the problem (1), (2) has at least two positive solutions.

Theorem 1. 4. Let f satisfy the same hypotheses as in Theorem 1.3. Then there exist λ^* such that for $\lambda < \lambda^*$, the problem (1), (2) has no positive solutions.

Remark 1. 2. Theorem 1.3 is established by using sub-super solutions arguments and results from the so-called semipositone problems. Theorem 1.4 follows easily from the fact that f is negative near zero and is sublinear at infinity.

Remark 1. 3. We note here that if u is a non-negative ($u \geq 0$ for $x \in \Omega$), non-trivial solution then u is necessarily positive ($u > 0$ for $x \in \Omega$). This follows from the fact that there exists $c(\lambda) > 0$ such that $\lambda f(u) + c(\lambda)u \geq 0, \forall u \geq 0$ and so $u \geq 0$ satisfies $-\Delta u + c(\lambda)u \geq 0, \forall x \in \Omega$ and hence $u > 0$ by the maximum principle [3].

Remark 1. 4. Here we deal with sublinear nonlinearities satisfying $f(0) = 0$ and $f'(0) < 0$. For an existence result via the variational method for the superlinear nonlinearities satisfying $f(0) = 0, f'(0) < 0$ and f being superlinear at infinity [4], and for an instability result for convex nonlinearities see [5].

Remark 1. 5. We note that for classes of nonlinearities of f, that $f(0) = 0, f'(0) < 0$, and there exist $\beta_2 > \beta_1 > 0$ such that $f(t) < 0$ for $t \in (0, \beta_1) \cup (\beta_2, +\infty)$ and $f(t) > 0$ for $t \in (\beta_1, \beta_2)$. Theorems 1.3 and 1.4 can be easily established by using the ideas given in this paper.

Now we consider the boundary value problem

$$u''(x) + \lambda f(u(x)) = 0; \quad x \in (-1,1) \tag{4}$$

$$u(-1) = 0 = u(1). \tag{5}$$

Where λ is a positive parameter (the case of $N = 1$).

For any solution $u(x)$ of (4), (5) let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. $w(x) > 0$ satisfies

$$w'' + \lambda f'(u)w + \mu w = 0; \quad x \in (-1,1), \quad w(-1) = 0 = w(1) \tag{6}$$

Recall that solution $u(x)$ of (4) and (5) is called unstable if $\mu < 0$, otherwise it is stable. Recall also that a solution of (4) and (5) is called degenerate (or singular) if for $\mu = 0$ there is a non-trivial solution of (6). It is easy to see that for a positive degenerate solution any solution w of (6) is of one sign, i.e. $\mu = 0$ can only be the principal eigenvalue. In fact, if u is a positive degenerate solution, then u is an even function, $u' < 0$ in (0,1) and u' satisfies $(u')'' + \lambda f'(u)u' = 0$. Then by Sturm comparison Lemma, w must be of one sign. It follows that unstable solutions are non-degenerate.

$$\text{Let } F(u) = \int_0^u f(t)dt, \qquad h(u) = 2F(u) - uf(u).$$

We establish the following results in section 3.

Theorem 1. 5. Assume that $f \in C^1[0,\infty), f(0) > 0$, and for some $\alpha > \beta > 0$ we have

$$h'(u) \geq 0; \;\; 0 < u < \beta, \;\; h'(u) \leq 0; \;\; \beta < u < \alpha, \tag{7}$$

$$h(\alpha) \leq 0. \tag{8}$$

Then the solution of (4) and (5) with $u(0) = \alpha$ is unstable if it exists.

Remark 1. 6. Theorem 1.5 is stated in a way that we assume the existence of a solution with $u(0) = \alpha$. In fact, if $f(u) > 0$ for all $u \in [0,\alpha]$, then for any $d \in (0,\alpha]$, there exists a unique $\lambda(d)$ such that (4) and (5) have a positive solution with $\lambda = \lambda(d)$ and $u(0) = d$, see [6].

Remark 1. 7. It is easy to see that condition (7) holds if

$$f''(u) > 0; \;\; 0 < u < \alpha \tag{9}$$

and (8) is also satisfied. So Theorem 1.5 still is true if we replace (7) with (9).

Remark 1. 8. It is well- known that if for some $\beta > 0$, $f(u) > 0$ and $h'(u) \geq 0$ or $0 \leq u \leq \beta$, then the solutions of (4) and (5) with $u(0) = d$ and $0 < d \leq \beta$ are all stable ([7], Theorem 6.2). Thus Theorem 1.5 implies that if f is convex and positive, and satisfies (8), then the unique degenerate solution u satisfies $\beta < u(0) < \alpha$.

Remark 1. 9. Note that any solution of (4) and (5) is symmetric with respect to any point $x_0 \in (-1,1)$ such that $u'(x_0) = 0$, so any positive solution of (4) and (5) is a reflection extension of a monotone decreasing solution of

$$u'' + \lambda f(u) = 0; \qquad x \in (0,1) \tag{10}$$

$$u'(0) = u(1) = 0; \tag{11}$$

$$u'(x) < 0; \qquad x \in (0,1). \tag{12}$$

So the study of all positive solutions is reduced to the study of (10)-(12). On the other hand, all solutions of (10)-(12) can be parameterized by their initial values $u(0) = \rho$.

In fact, by integrating the equation we obtain

$$u'(x) = -\sqrt{2\lambda[F(\rho) - F(u(x))]}; \qquad x \in (0,1), \tag{13}$$

where $u(0) = \rho$, and

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\rho \frac{du}{\sqrt{f(\rho) - F(u)}} := G(\rho). \tag{14}$$

So for each $\rho > 0$, there is at most one (if the integral in (14) is well defined and convergent) λ such that (10)-(12) have a solution. Thus the solution set of (10)-(12) can be represented by $\lambda = \lambda(\rho) = [G(\rho)]^2$, which we call bifurcation diagram.

Remark 1.10. In view of Remark 1.9 and equation (6), at a degenerate solution, we have w as a nontrivial solution of the linearized equation [8, 9]

$$w'' + \lambda f'(u)w = 0; \quad x \in (0,1), \quad w'(0) = 0 = w(1). \tag{15}$$

Lemma 1.1. Suppose that $(\lambda(\rho), u(., \rho))$ is a degenerate solution of (10)-(12), and w is the corresponding solution of linearized equation (15). Then $w(x) \neq 0$ for $x \in [0,1)$, so we can choose w as positive in $[0,1)$.

Lemma 1.2. Assume that $f \in C^2[0,+\infty), f(0) < 0, f(u) < 0$ for $u \in (0,b)$ for some $b > 0, f(b) = 0$ and $f'(b) > 0$, and there exists $\theta > b$ such that $f(\theta) > 0, F(u) < 0$ for $u \in (0,\theta)$, and

$$F(\theta) = 0. \tag{16}$$

Then $u(.,\theta)$ is an unstable solution.

Lemma 1. 3. ([10], [11]) we have

$$\int_0^1 f(u(x,\rho))w(x)dx = \frac{u'(1)w'(1)}{2\lambda(\rho)}.$$

Although it is possible that $u'(1) = 0$ for a solution $u(.)$ of (10)-(12), (in fact, $u_x(1,\rho) = 0$ if and only if $\rho = \theta$), we can show that $u'(1) < 0$ if $u(.)$ is a degenerate solution.

2. PROOFS OF THEOREMS 1. 1-1. 4

Proof of Theorem 1. 1. Suppose u is a solution of (1) and (2) such that it is positive somewhere in Ω. Then there exists $\Omega^* \subset \Omega$ such that $u > 0$ in Ω^*, $u = 0$ on $\partial\Omega^*$. Let $\lambda_1(\Omega^*)$ be the smallest eigenvalue of

$$-\Delta\phi = \lambda_1\phi; \qquad x \in \Omega^*$$

$$\phi = 0; \qquad x \in \partial\Omega^*$$

and $\phi > 0$ in Ω^*, a corresponding eigenfunction. Now by assumptions in Theorem1. 1, there exists a $\beta > 0$ such that

$$f(t) \le \beta t; \qquad \forall t \ge 0.$$

Now u satisfies

$$-\Delta u - \lambda\beta u = \lambda\{f(u) - \beta u\}; \qquad x \in \Omega^*.$$

Hence multiplying by ϕ and integrating over Ω^* we have,

$$\int_{\Omega^*} (-\Delta u - \lambda\beta u)\phi dx = \int_{\Omega^*} (\lambda_1(\Omega^*) - \lambda\beta)u\phi dx \le 0.$$

This is impossible if $\lambda < \lambda_1(\Omega^*)/\beta$. But we know that $\lambda_1 = \lambda_1(\Omega) \le \lambda_1(\Omega^*)$ if $\Omega^* \subset \Omega$. Hence the result holds for $\lambda_0 = \lambda_1(\Omega)/\beta$.

Proof of Theorem 1. 2. Consider the boundary value problem

$$-\Delta w(x) = \lambda g(w(x)); \qquad x \in \Omega \tag{17}$$

$$w(x) = 0; \qquad x \in \partial\Omega \tag{18}$$

where

$$g(s) = g(s,\alpha,\delta) := \begin{cases} [\frac{M}{(\alpha-1)c}]s - \frac{M}{\alpha-1}; & 0 \le s \le c \\ -\frac{M(s-c)(s-\alpha c)}{[(\alpha-1)c]^2}; & c \le s \le (\alpha+1)c/2 \\ \frac{-M(-s[(\alpha+1)c-\delta])(s-\delta)}{[s\delta-(\alpha+1)c]^2}; & s \ge (\alpha+1)c/2. \end{cases}$$

Here $\alpha > 1$ and $\delta > \alpha c$.

Note that $g \in C^1([0,\infty))$, $\max g = g(\frac{(\alpha+1)c}{2}) = \frac{M}{4}$ and

$$g'(c) = \frac{M}{(\alpha-1)c} \to +\infty \qquad as \qquad \alpha \to 1^+.$$

Thus by (3) there exists $\alpha = \alpha_0$ such that

$$f(s) \ge g(s); \qquad \forall s \ge \alpha_0. \tag{19}$$

Now, for $\alpha = \alpha_0$ there exists a δ_0 such that

$$\int_0^s g(s)ds > 0$$

for every $\delta \geq \delta_0$. Let $g(s, \alpha_0, \delta)$ with $\delta \geq \delta_0$. Then by the result of Clement and Sweers ([2, Theorem 2]), problem (17) and (18) have a non-negative solution $0 \leq w < \delta$ for λ large, say for $\lambda \geq \hat{\lambda}(\delta)$, such that $\| w \| \to \delta$ as $\lambda \to +\infty$. But by (19) this solution is a sub-solution of (1), (2).

Now let $v(x)$ to be the unique positive solution of

$$-\Delta v(x) = 1; \qquad x \in \Omega$$

$$v(x) = 0; \qquad x \in \partial\Omega$$

and consider $\phi(x) = Jv(x)$; $J > 0$ (to be chosen). Then ϕ satisfies

$$-\Delta \phi(x) = J; \qquad x \in \Omega$$

$$\phi(x) = 0; \qquad x \in \partial\Omega.$$

But by $\lim_{t \to +\infty} f(t)/t = 0$, there exists a $J_0 > 0$ such that for $J > J_0$,

$$-\Delta \phi(x) = J \geq \lambda f(Jv(x)); \qquad x \in \Omega$$

and thus $\phi(x)$ will be a super-solution for (1) and (2). Consequently, given $\lambda \geq \hat{\lambda}(\delta)$, there exists $J_0(\lambda)$ such that $J > J_0(\lambda)$

$$\phi(x) = Jv(x)$$

will be a super-solution of (1), (2) satisfying

$$\phi(x) \geq w.$$

Hence there exists a solution u for $\lambda \geq \hat{\lambda}(\delta)$ such that $w \leq u \leq \phi(x)$. But $\| w \| \to \delta$ as $\lambda \to +\infty$ and δ can be chosen arbitrarily large. Hence Theorem 1.2 is proven.

Remark 2. 1. We first recall the following sub-super solutions result which will be used to establish Theorem 1.3.

Theorem 2. 1. Suppose there exists a sub-solution φ_1, a strict super-solution ϕ_1, a strict sub-solution φ_2 and a super-solution ϕ_2 for the problem (1), (2) such that $\varphi_1 < \phi_1 < \phi_2$, $\varphi_1 < \varphi_2 < \phi_2$ and φ_2 is not less than or equal to ϕ_1 [12].

Then the problem (1), (2) has at least three distinct solutions u_s (s=1, 2, 3) such that

$$\varphi_1 \leq u_1 < u_2 < u_3 \leq \varphi_2.$$

Note that a weaker form of Theorem 2.1 (under the assumption $\phi_1 \leq \varphi_2$) was established in [13]. However, we require this stronger version to establish our multiplicity result.

Proof of Theorem 1. 3. Clearly $\varphi_1 \equiv 0$ is a solution to (1), (2). Consider $\phi_1(x) = \varepsilon v(x)$ where $v(x) > 0$; $x \in \Omega$ is an eigenfunction satisfying

$$-\Delta v(x) = \lambda_1 v(x); \qquad x \in \Omega \tag{20}$$

$$v(x) = 0; \qquad x \in \partial\Omega \tag{21}$$

corresponding to the principal eigenvalue $\lambda_1 > 0$. Now $H(z) = \lambda_1 z - \lambda f(z) > 0$ for small positive z since $f'(0) < 0$. Thus $-\Delta\phi_1 = \lambda_1 \varepsilon v(x) > \lambda f(\varepsilon v(x))$ for $x \in \Omega$ if $\varepsilon > 0$ is small, and hence ϕ_1 is a strict super-solution of (1) and (2).

Next consider a C^1 function g as in Theorem 1.2 such that $g(u) < f(u)$ for all $u \geq 0$. This is clearly possible by the hypotheses on f. Let $\varphi_2 = \varphi_2(x,\lambda)$ be a positive solution for large λ described in Theorem 1.2 of $-\Delta w = \lambda g(w)$ in Ω, $w = 0$ on $\partial\Omega$. Then $-\Delta\varphi_2 = \lambda g(\varphi_2) < \lambda f(\varphi_2)$ for $x \in \Omega$, and hence φ_2 is a strict sub-solution of (1), (2) for large λ.

Finally, consider $\phi_2(x) = MZ(x)$ where $Z(x)$ is the unique positive solution of $-\Delta w = 1$ in Ω, $w \equiv 0$ on $\partial\Omega$, and $M > 0$ is a constant.

Then $-\Delta\phi_2(x) = M \geq \lambda f(MZ(x))$ for $x \in \Omega$, provided that $M \geq M_1(\lambda)$ for some $M_1(\lambda)$ large enough so that $M \geq \lambda f(M \| Z \|_\infty)$, which is possible since $\lim_{u\to\infty} \frac{f(u)}{u} = 0$ and f is eventually increasing. Now also choose $M \geq M_2(\lambda)$ where $M_2(\lambda)$ is large enough so that $MZ(x) > \varphi_2(x,\lambda)$ and $MZ(x) > \phi_1(x)$ for $x \in \Omega$, which is possible since $Z(x) > 0$ for $x \in \Omega$ and $\frac{\partial Z}{\partial n} < 0$ for $x \in \partial\Omega$ where n denotes the outward normal. Choose $M \geq \max\{M_1(\lambda), M_2(\lambda)\}$. Further, choose $\varepsilon > 0$ small enough so that the set $S = \{x \in \Omega : \varphi_2(x) - \phi_1(x) > 0\}$ is non-empty.

Now applying Theorem 2.1, the existence of at least two distinct positive solutions for λ large easily follows. In particular, a positive solution u_1 such that $\varphi_2(x) \leq u_1(x) \leq \phi_2(x)$ for $x \in \Omega$, and a second positive solution u_2 such that $0 < u_2(x) \leq \phi_2(x)$ for $x \in \Omega$, $S_1 = \{x \in \Omega : u_2(x) - \phi_1(x) > 0\} \neq \phi$ and $S_2 = \{x \in \Omega : u_2(x) - \varphi_2(x) < 0\} \neq \phi$ exist.

Proof of Theorem 1. 4. Let u be a positive solution of (1), (2). Multiplying (19) by u and (1) by v, where $v(x)$ is as defined in problem (19) and (20), and subtracting we obtain

$$\int_\Omega (\lambda f(u) - \lambda_1 u) v dx = 0. \tag{22}$$

Here we have used the fact that $\int_\Omega [(-\Delta u)v - (-\Delta v)u] dx = 0$, which easily follows by applying Green's identity and boundary conditions. But since $f(0) = 0$, $\lim_{u\to 0} \frac{f(u)}{u}$ exists and $\lim_{u\to\infty} \frac{f(u)}{u} = 0$, there exists $K > 0$ such that $f(u) \leq Ku$ for all $u \geq 0$. Thus, if λ is small enough so that $\frac{\lambda_1}{\lambda} > K$, equation (22) cannot hold. Hence, for λ small, the problem (1) and (2) has no positive solution and Theorem 1.4 is proven.

3. PROOFS OF THEOREM 1. 5 AND LEMMAS 1.1 AND 1. 2

Proof of Theorem 1. 5.

We have $h(0) = 0$, $h'(0) = f(u) - uf'(u)$, $h'(0) = f(0) > 0$. It follows that $h(u)$ is unimodular on $[0,\alpha]$, and it takes its positive maximum at $u = \beta$. Define $x_0 \in (0,1)$ by $u(x_0) = \beta$. We then conclude

$$\begin{cases} f(u(x)) - u(x) f'(u(x)) \leq 0 & on \quad (0, x_0), \\ f(u(x)) - u(x) f'(u(x)) \geq 0 & on \quad (x_0, 1). \end{cases} \tag{23}$$

We also remark that by condition (8),

$$\int_0^1 [f(u)-uf'(u)]u'(x)dx = \int_0^1 \frac{d}{dx} h(u(x))dx = -h(\alpha) \ge 0. \tag{24}$$

Assume now that $u(x)$ is stable, i.e., $\mu \ge 0$ in (6). Without loss of generality, we assume that $w > 0$ in $(-1,1)$. By the maximum principle, $u'(1) < 0$, so near $x = 1$ we have $-u'(x) > w(x)$. Since $-u'(0) = 0$, while $w(0) > 0$, the functions $w(x)$ and $-u'(x)$ change their order at least once on $(0,1)$. We claim that the functions $w(x)$ and $-u'(x)$ change their order exactly once on $(0,1)$. Observe that $-u'(x)$ satisfies

$$(-u')'' + \lambda f'(u)(-u') = 0 \tag{25}$$

on $(0,1)$. Let $x_3 \in (0,1)$ be the largest point where $w(x)$ and $-u'(x)$ change the order. Assuming the claim to be false, let x_2, with $0 < x_2 < x_3$, be the next point where the order changes. We have $w > -u'$ on (x_2, x_3), and the opposite inequality to the left of x_2. Since $w(0) > -u'(0)$, there is another point $x_1 < x_2$, where the order is changed. We multiply (6) by $-u'$, multiply (25) by w, subtract and integrate from x_1 to x_2, then we obtain

$$w(x_2)[w'(x_2) + u''(x_2)] - w(x_1)[w'(x_1) + u''(x_1)] + \mu \int_{x_1}^{x_2} (-u'(x))w(x)dx = 0, \tag{26}$$

since $w(x_i) = -u'(x_i)$ for $i = 1,2$. Let $t(x) = w(x) - (-u'(x))$. Then $t(x) \le 0$ for $x \in (x_1, x_2)$ and $t(x) \ge 0$ for $x \in (x_2, x_3)$. Thus $t(x_1) = w'(x_1) + u''(x_1) \le 0$ and $t(x_2) = w'(x_2) + u''(x_2) \ge 0$. Because $w(x) > 0$ and $-u'(x) > 0$ on $(0,1)$, we get a contradiction in (26).

Since the point of changing of order is unique, by scalling $w(x)$ we can achieve

$$\begin{cases} -u'(x) \le w(x) & on \quad (0, x_0), \\ -u'(x) \ge w(x) & on \quad (x_0, 1). \end{cases} \tag{27}$$

Using (23), (27), and also (24), we have

$$\int_0^1 [f(u)-uf'(u)]w(x)dx < \int_0^1 [f(u)-uf'(u)](-u'(x))dx \le 0. \tag{28}$$

On the other hand, multiplying equation (6) by u, Eqs. (4) and (5) by w, subtracting and integrating over $(0,1)$, we have

$$\int_0^1 [f(u)-uf'(u)]w(x)dx = \frac{\mu}{\lambda}\int_0^1 uwdx \ge 0,$$

which contradicts (28). So $\mu < 0$.

Proof of Lemma 1. 1. The function $u_x(x,\rho)$ satisfies

$$v'' + \lambda f'(u)v = 0; \quad x \in (0,1), \quad v(0) = 0, \quad v'(x) < 0; \quad x \in (0,1). \tag{29}$$

Suppose that w has a zero $x_0 \in (0,1)$. Since w and u_x satisfy the same differential equation (not the same boundary conditions), then by the Sturm comparison Lemma, there is a zero of u_x in the interval $(x_0,1)$, that is a contradiction. So w is of one sign in $[0,1)$.

Proof of Lemma 1. 2. We recall from (6) that a solution $(\lambda(\rho), u(.,\rho))$ of (10)-(12) is stable if the principal eigenvalue μ_1 of

$$\phi'' + \lambda(\rho) f'(u(.,\rho))\phi = -\mu_1 \phi; \quad x \in (0,1), \quad \phi'(0) = \phi(1) = 0, \tag{30}$$

is non-negative, otherwise it is unstable. Let ϕ be the eigenfunction corresponding to μ_1, the principal eigenvalue for $u = u(.,\theta)$. From the equation of u_x and (30), we obtain

$$[\phi' u_x - (u_x)' \phi]_0^1 + \mu_1 \int_0^1 \phi u_x dx = 0. \tag{31}$$

Using the boundary conditions and $u_x(1,\theta) = 0$, we have

$$u_{xx}(0,\theta)\phi(0) + \mu_1 \int_0^1 \phi u_x dx = 0. \tag{32}$$

We can assume that $\phi(x) > 0$ for $x \in [0,1)$, and we also have $u_{xx}(0,\theta) = -\lambda f(\theta) < 0$ and $u_x \leq 0$, thus $\mu_1 < 0$.

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