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DISTRIBUTION FUNCTIONS IN LIGHT OF THE UNCERTAINTY PRINCIPLE*

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Abstract – Here, we have considered the Husimi and Wigner distribution functions. We have, in particular, shown how the uncertainty principle works for the two cases of simple and damped harmonic oscillators when either of these two distributions are used. The conclusion shows that the Husimi distribution function remains non-negative through the phase space but it does not always satisfies the uncertainty principle. On the other hand the Wigner distribution function while becoming negative in certain regions of the phase space do not violate the uncertainty principle.

Keywords – Distribution functions, uncertainty principle, damped harmonic oscillator

1. INTRODUCTION

A few years after the advent of quantum mechanics, Wigner [1], by introducing distribution function (WDF), undertook the study of the quantum corrections needed for the calculation of various physical quantities appearing in classical statistical mechanics. Since then, the WDF, and more generally the phase space representation of quantum mechanics have been used extensively in various fields such as quantum chemistry, quantum optics [2, 3] and many-body physics [4].

The most well-known phase space-representations are those of Glauber [2], Husimi [5], Margenau et al [6], and others. It can be shown that these different distribution functions are related to each other by suitable canonical transformations in an extended phase space, while each of these distribution functions is associated with a particular ordering rule of non commutative operators. However, not all of these canonical transformations in the classical level correspond to unitary transformations in the quantum level [7-10], therefore, they are not entirely equivalent. The question which may be raised in this respect is that in what sense a given distribution is preferable. Of course the final verdict should come from observations involving the product of non-commuting observables. However, in the absence of such evidence, one may resort to the ones which are relatively less problematic. In this paper, we will compare the Husimi distribution function (HDF) with the Wigner distribution function (WDF) and will point out the problems encountered in each case.

It is widely known that the WDF can assume negative values in some regions of the phase space and oscillates violently with the wavelength $\hbar$ in those regions. To avoid this difficulty, many authors

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have tried to define a positive distribution function. In particular, Husimi's positive distribution function is an attempt in that direction. However, the use of HDF is not warranted for the entire range of the parameters involved in that distribution. In fact, we will show that although the HDF surmounts the non-negativity problem throughout the phase space, it does, however, result in the violation of the Heisenberg uncertainty relations.

In particular, we have considered the cases of simple and damped harmonic oscillators and found that contrary to the case of WDF, the Heisenberg uncertainty principle for HDF does not always hold.

In section 2, WDF and HDF are reviewed. In section 3, each of these distribution functions is examined in light of the uncertainty principle. Finally, section 4 is devoted to the concluding remarks.

2. REVIEW OF WDF AND HDF

In 1932 Wigner introduced the distribution function, $W(p,q)$, defined by [1]

$$W(p,q) = \frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} \psi^*(q+y)\psi(q-y)e^{2ipy/\hbar}dy, \tag{1}$$

where the normalized wave function, ψ, is the solution of the Schrodinger equation and p and q are the phase space variables. The generalization of WDF to higher dimensions is straight-forward. For details and properties of this distribution one can consult the review article by Hillary et al [11] and the references therein. The motivation for working with $W(p,q)$ and other distributions is an attempt to express the quantum expectation values in the form of averages of classical observables, which are defined by means of some probability distribution functions in phase space. For instance, one may quantize a given classical observable function, $A(p,q)$, by Weyl's ordering rule to obtain a quantum operator $\hat{A}(\hat{p},\hat{q})$ in the Hilbert space of wave functions ψ, and then obtain the quantum expectation value, $<\hat{A}>_Q$, through WDF by

$$<\hat{A}>_Q \equiv <\psi|\hat{A}|\psi> = \iint A(p,q)W(p,q)dpdq, \tag{2}$$

while associated with the symmetric ordering rule of non-commutative operators.

Despite the fact that $W(p,q)$ does not remain non-negative everywhere in phase space, it provides a nice way of formulating the quantum mechanics as a probabilistic theory.

Husimi used the method of Gaussian smoothing of the Wigner function to obtain the following non-negative distribution function.

$$P(p,q) = \iint W(p,q)\exp[-\frac{\alpha}{\hbar^2}(p'-p)^2 - \frac{1}{\alpha}(q'-q)^2]dp'dq', \tag{3}$$

where $W(p,q)$ is given by Eq. (1) and α is a positive parameter [12]. Using Eq. (3) one finds

$$P(p,q) = \exp-\{\frac{\alpha}{4}\frac{\partial^2}{\partial q^2} + \frac{\hbar^2}{4\alpha}\frac{\partial^2}{\partial q^2}\}W(q,p). \tag{4}$$

It can be shown that $P(q,p)$ is positive definite and normalizable everywhere in the phase space. However, it seems that some discrepancies still exist. This is shown in the following section.

3. WDF AND HDF AND THE UNCERTAINTY PRINCIPLE

The uncertainty relation Γ may be written as

$$\Gamma_Q = (\Delta p)^2 (\Delta q)^2 - \frac{< qp + pq >_Q^2}{4} \geq \frac{\hbar^2}{4}, \tag{5}$$

where $(\Delta p)^2 =< p^2 >_Q - < p >_Q^2$, $(\Delta q)^2 =< q^2 >_Q - < q >_Q^2$ and p and q are the momentum and the coordinate operators, respectively. The quantum mechanical averages are denoted by the subscript Q so that they can be distinguished from the phase space averages defined later. In the phase space representation of quantum mechanics, p and q are C-numbers and commute with each other. Therefore, using Eq. (2) as a general characteristic of phase space distribution functions, Eq. (5) assumes the following form:

$$\Gamma_{ph} = (\Delta p)^2 (\Delta q)^2 - < qp >_{ph}^2 \geq \frac{\hbar^2}{4}, \tag{6}$$

where the subscript *ph* denotes the average obtained by means of the phase space distribution functions.

Let us firstly examine Eq. (6) for the simple harmonic oscillator when HDF and WDF are used. The Wigner function for the *n*th excited state in this case is

$$W_n = \frac{1}{\pi \hbar} e^{-2H/\hbar\omega} L_n(\frac{4H}{\hbar\omega}), n = 0,1,2,\ldots, \tag{7}$$

where $H = p^2/2m + m\omega^2 q^2/2$ is the Hamiltonian and $L_n(x)$ is the Laguerre polynomial of order n. W_n in Eq. (7) is quadratic in both q and p for all values of *n*. Therefore, we have $< qp >_{W_n} = 0$, for all *n*. Furthermore, it can be shown that

$$\Gamma_{W_n} = (\Delta p)^2 (\Delta q)^2 - < qp >_{W_n}^2 = \frac{\hbar^2}{4}. \tag{8}$$

Equation (8) shows that the WDF has the minimum uncertainty which is its familiar property. The positive distribution of Eq. (4) is also quadratic in both q and p. Thus, the quantity $< qp >_{pn}$ is zero for this distribution. It can be shown that

$$\Gamma_{ph} = (\Delta p)^2 (\Delta q)^2 - < qp >_{ph}^2 = \frac{\hbar}{4}(1 + \frac{\alpha m\omega}{\hbar})(1 + \frac{\hbar}{\alpha m\omega}) \geq \frac{\hbar^2}{4}, \tag{9}$$

for any $\alpha > 0$. Thus, for the simple harmonic oscillator neither HDF nor WDF have any problem with the Heisenberg uncertainty principle. Now, let us see what happens if we consider the case of a damped harmonic oscillator. In this case, when the ground state of the oscillator is considered, both of the distributions assume the following form [12, 13].

$$g(q, p) = A \exp[-\alpha_1 q^2 - \alpha_2 q^2 + 2\alpha_3 pq]. \tag{10}$$

For $g(q, p) = W_\circ(q, p)$,

$A = \frac{1}{\pi\hbar}, \alpha_1 = \frac{m\omega^2}{\hbar\Omega} e^{\gamma t}, \alpha_2 = \frac{1}{\hbar\Omega m} e^{-\gamma t}$, and $\Omega^2 = \omega^2 - \gamma^2/4$,

with γ as damping constant and ω as the undamped oscillator frequency. For $g(q, p) = P_\circ(q, p)$,

$$A = (\frac{m\Omega\alpha}{\hbar})^{1/2}(\frac{1}{\pi\hbar})\{(1+\alpha\lambda)(e^{-\gamma t} + m\Omega\alpha/\hbar)\}^{-1/2},$$

$$\alpha_1 = \lambda/(1+\alpha\lambda), \alpha_2 = \alpha/[(1+\alpha\lambda)\hbar^2], \alpha_3 = \kappa/(1+\alpha\lambda),$$

$$\lambda = m\Omega/\hbar \frac{e^{\lambda t}\{e^{-\gamma t} + (\alpha m\omega^2/\hbar\Omega)\}}{e^{\lambda t} + \alpha m\Omega/\hbar}, \kappa = \frac{1}{2}\gamma m\alpha/\hbar^2 (e^{-\gamma t} + m\Omega\alpha/\hbar).$$

Using Eq. (10), it is easy to show that

$$< qp >_g = \frac{A\alpha_3\pi}{2(\alpha_1\alpha_2 - \alpha_3^2)^{3/2}}, \tag{11a}$$

$$< q >_g = < p >_f = 0, \tag{11b}$$

$$< q^2 >_g = \frac{1}{2\hbar}\frac{\alpha_2}{(\alpha_1\alpha_2 - \alpha_3^2)^{3/2}}, \tag{11c}$$

$$< p^2 >_g = \frac{1}{2\hbar}\frac{\alpha_1}{(\alpha_1\alpha_2 - \alpha_3^2)^{3/2}}. \tag{11d}$$

and as a result

$$\Gamma_{W_0} \equiv (\Delta p)^2(\Delta q)^2 - < qp >_{W_0}^2 = \frac{\hbar^2}{4}. \tag{12}$$

Therefore, $W_\circ$ satisfies the uncertainty principle just as an equality. However, by the same procedure we obtain the following expression for $\Gamma_{P_\circ}$:

$$\Gamma_{P_0} \equiv (\Delta p)^2(\Delta q)^2 - < qp >_{P_0}^2 = \frac{\hbar^2}{4} f(\beta,\sigma,c), \tag{13a}$$

where

$$f(\beta,\varsigma,c) = 1 + \beta\{\sigma + \frac{1}{\beta} - \frac{c(\sigma-1)(1+2\beta+\beta^2\sigma)^3}{[(1+\beta\sigma)-(\sigma-1)\beta(1+\beta)^3 c]^3}\}, \tag{13b}$$

$\beta = \frac{\alpha m\Omega}{\hbar}e^{\gamma t}, \sigma = \frac{\omega^2}{\Omega^2}, \Omega^2 = \omega^2 - \gamma^2/4$ and $c = e^{-4\gamma t}$.

According to Eq. (13a), $P_\circ$ will satisfy the uncertainty relation, Eq. (6), if $f \geq 1$. However, as Fig. 1 shows this is not always true. In this Fig., $f(\beta,\sigma,c)$ is plotted as a function of β for $c = 0.8$ and $\sigma = 2$. It is seen that for some specific values of β, f not only becomes smaller than 1, but also diverges to $-\infty$. Moreover, the point is pursued for the excited states $P_n(q,p)$ and $W_n(q,p)$ of the damped harmonic oscillator. W_n for this case is [11].

$$W_n = \frac{(-1)^n}{\pi\hbar} e^{-\frac{2}{\hbar\Omega}(H+\frac{1}{2}\gamma pq)} L_n[4(H + \frac{1}{2}\gamma pq)], \tag{14}$$

where $L_n(x)$ is the Laguerre polynomial of order *n* and H is the Kanai Hamiltonian [10]. Using the series form of the Laguerre polynomial, it is more useful to write Eq. (14) as

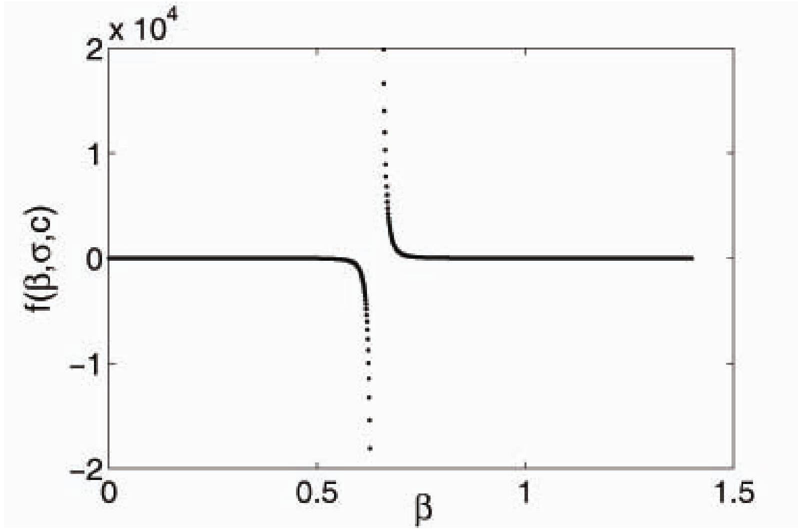

Fig. 1. Plot of the function *f*(β) in terms of β for damped oscillator when σ=2 and *c*= 0.8.
The divergence of *f* for β≈ 0.66 shows the failure of uncertainty principle by HDF

$$W_n = \frac{(-1)^n}{\pi\hbar}\sum_{s=0}^{n}\frac{2^{n-s}\,n!}{[(n-s)!^2\,s!}\left(\frac{d}{d\xi}\right)^{n-s} e^{-\xi x}\Big|\xi = 1, \tag{15}$$

where $x = H + \frac{1}{2}\gamma\, pq$. Equation (15) easily gives

$$< qp >_{W_n} = \left(\sum_{s=0}^{n}(-1)^s\,\frac{2^{n-s}\,n!(n-s+1)}{[(n-s)!s!}\right) < qp >_{W_n}, \tag{16}$$

where $< qp >_{W_n}$ and $< qp >_{W_o}$ are expectation values obtained by using W_n and $W_\circ$, respectively. After some manipulations one arrives at

$$\Gamma_{W_n} = \left(\sum_{s=0}^{n}\frac{(-1)^s\,2^{n-s}\,n!(n-s+1)}{[(n-s)!s!}\right)^2 \Gamma_{W_\circ} \geq \frac{\hbar^2}{4}. \tag{17}$$

Therefore, when WDF is used for the damped harmonic oscillator, the uncertainty principle is not violated. However, in the case of HDF, the uncertainty principle for the excited states of a damped harmonic oscillator, like the ground state, could be violated. Let us write Eq. (4) in the form of

$$P_n = FW_n, \tag{18}$$

where F is the operator $\exp\left(\frac{\alpha}{4}\partial^2/\partial q^2 + \frac{\hbar^2}{4\alpha}\partial^2/\partial p^2\right)$. It is illustrative to work out the problem for P_1 first. The generalization to the excited states is straight-forward. Thus,

$$P_1 = FW_1 = -(1 + 2\frac{d}{d\xi})W_\circ(\xi)\big|\xi = 1, \tag{19}$$

where $W_\circ(\xi) = \frac{1}{\pi\hbar}\exp[\frac{-2\xi}{\hbar\Omega}(H + \frac{1}{2}\gamma\, pq)]$. Therefore, we get

$$< qp >_{P_1} = -(1+2\frac{d}{d\xi})\frac{1}{d\xi}\Big|_{\xi=1} < qp >_{P_0} = 3 < qp >_{P_0}, \tag{20a}$$

$$< p^2 >_{P_1} = 3 < p^2 >_{P_0}, \tag{20b}$$

and

$$< q^2 >_{P_1} = 3 < q^2 >_{P_0}, \tag{20c}$$

$$\Gamma_{p_1} = 3\Gamma_{p_\circ} \ . \tag{21}$$

It is obvious that for the higher excited states, Γ_{p_n} are positive multiples of $\Gamma_{p_\circ}$, and therefore, the functional nature of Γ_{p_n} is the same as Γ_{P_0}, for higher excited states. Hence the divergency will persist for these states as well.

4. CONCLUSIONS

As mentioned before, WDF assumes negative values in some parts of the phase space. Although some authors [7, 14] do not consider this point as a weakness of WDF, pointing out that they give the expectation values of quantum mechanical observables properly. However, others have continued their search for non-negative distribution functions. Notable among these has been Husimi's work for defining HDF. But it is interesting to note that HDF, by resolving the negativity problem throughout the phase space, encounters a new drawback. That is, in contrast to WDF, the uncertainty relation is violated for damped harmonic oscillator using HDF.

It seems that the problem discussed here can be traced to the basic difference between the nature of WDF and HDF. That is, the evolution operator, known as Wigner's operator [1], is hermitian, while that of HDF's introduced in Eq. (4) is not. This is because the corresponding canonical transformation in the classical extended phase space is not a unitary transformation in the quantum level [7].

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