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SOME SPECTRAL PROPERTIES OF STURM-LIOUVILLE PROBLEM WITH TRANSMISSION CONDITIONS[*]

M. KADAKAL[1**], O. SH. MUKHTAROV[2] AND F. S. MUHTAROV[3]

[1]Ondokuz Mayis University, Science and Arts Faculty, Department of Mathematics
55139 Kurupelit, Samsun, Turkey, E-mail: mkadakal@omu.edu.tr,
mkadakal@yahoo.com
[2]Gaziosmanpasa University, Science and Arts Faculty, Department of Mathematics
60000-Tokat-Turkey, E-mail: omukhtarov@yahoo.com
[3]Azerbaijan Science Academy, Mathematics and Mechanic
Institute, Baku-Azerbaijan

Abstract – We investigate the boundary-value problem generated by the Sturm-Liouville equation with discontinuous coefficients, eigenparameter dependent boundary conditions and transmission conditions at the point of discontinuity. With a different approach we introduce an adequate Hilbert space formulation, investigate some properties of eigenvalues, Green's function and resolvent operator, and find simple conditions on the coefficients which guarantee the self-adjointness of the considered problem.

Keywords – Sturm-Liouville problem, eigenvalue, eigenfunction, Green's function, resolvent operator

1. INTRODUCTION

The investigation of boundary-value problems for which the eigenvalue parameter appears in the boundary conditions originates from the work of [1]. There is quite substantial literature on such types of problems. Here we mention the results of [2-15] and the corresponding references cited therein.

Basically, boundary-value problems with continuous coefficients and without transmission conditions have been studied. However, in this study we investigate one discontinuous problem with eigen-dependent boundary conditions and with special type transmission conditions. These kinds of problems arise in the theory of heat and mass transfer, in diffraction problems and various physical transfer problems [2, 9, 14, 15] (and corresponding references cited therein).

By using the techniques of [2, 4 and 11] and some new approaches, we construct special type initial-value problems (3.15)-(3.17) and (3.18)-(3.20) and special type solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ of the considered problem (1.1)-(1.5) below. We introduce adequate operator formulation in the suitable Hilbert space, construct Green's function, investigate the resolvent operator and prove the self-adjointness of the considered problem.

In this paper, we shall study the discontinuous Sturm-Liouville problem consisting of the differential equation

$$\tau u := -u'' + q(x)u = \lambda u \tag{1.1}$$

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[**]Corresponding author

on $(-1,0)\cup(0,1)$, with boundary condition at $x=-1$

$$L_1u := \alpha_1 u(-1)+\alpha_2 u'(-1)=0, \tag{1.2}$$

transmission conditions at the point of discontinuity $x=0$

$$L_2u := \gamma_1 u(-0)-\delta_1 u(+0)=0 \tag{1.3}$$

$$L_3u := \gamma_2 u'(-0)-\delta_2 u'(+0)=0, \tag{1.4}$$

and eigenparameter dependent boundary condition at $x=1$

$$L_4(\lambda)u := \lambda\big(\beta_1' u(1)-\beta_2' u'(1)\big)+\big(\beta_1 u(1)-\beta_2 u'(1)\big)=0 \tag{1.5}$$

in the space $H^2(-1,0)\times H^2(0,1)=\left\{ f\in L^2(-1,0)\times L^2(0,1) \,\middle|\, f, f'\in L^2(-1,0)\times L^2(0,1)\right\}$, where $q(\bullet)$ is a given real-valued function, which is continuous in $[-1,0]$ and $[0,1]$ (that is, continuous in $[-1,0)$ and $(0,1]$ and has finite limits $q(\pm 0)=\lim\limits_{x\to\pm 0} q(x)$); λ is a complex spectral parameter; $\alpha_i, \beta_i, \beta_i', \gamma_i, \delta_i, (i=1,2)$ are real numbers. Furthermore, below we shall assume that $|\alpha_1|+|\alpha_2|\neq 0$, $|\gamma_i|+|\delta_i|\neq 0$ $(i=1,2)$, $\gamma_1\gamma_2-\delta_1\delta_2=0$ and $\rho=\beta_1'\beta_2-\beta_1\beta_2'>0$.

2. THE OPERATOR FORMULATION

In this section, we shall introduce a suitable Hilbert space and a symmetric linear operator defined in this space in such a way that the considered problem (1.1)-(1.5) can be represented as the eigenvalue problem of this operator.

Let us introduce the Hilbert space $H := L^2(-1,0)\times L^2(0,1)\mathrm{x}\mathbb{C}$ for which the inner product of the elements

$$F=\begin{pmatrix} f(\bullet) \\ f_1 \end{pmatrix}\in H,\ G=\begin{pmatrix} g(\bullet) \\ g_1 \end{pmatrix}\in H,$$

is defined by

$$< F, G >_H = \int_{-1}^{1} f(x)\overline{g(x)}dx + \frac{1}{\rho} f_1\overline{g_1}$$

where $f(\bullet), g(\bullet)\in L^2(-1,0)\times L^2(0,1)$; $f_1, g_1\in\mathbb{C}$. Note that here under $\int_{-1}^{1}$, we mean $\lim\limits_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{-1}^{-\varepsilon}+\int_{\varepsilon}^{1}\right)$.

For convenience, we shall use the following notations for $f\in H^2(-1,0)\times H^2(0,1)$:

$$R_1(f) := \beta_1 f(1)-\beta_2 f'(1)$$

$$R_1'(f) := \beta_1' f(1)-\beta_2' f'(1)$$

By using the standard construction [2, 3, 8, 10] we define the operator A mapping the Hilbert space H into itself with domain of definition

$$D(A)=\left\{F=\begin{pmatrix} f \\ R_1'(f)\end{pmatrix}\middle|\ f\in H^2(-1,0)\times H^2(0,1),\ L_1f=L_2f=L_3f=0\right\} \tag{2.1}$$

and action law

$$AF:=\begin{pmatrix} \tau f \\ -R_1(f)\end{pmatrix},\ F\in D(A)\ , \tag{2.2}$$

Now we can rewrite the considered problem (1.1)-(1.5) in the operator form as

$$\begin{pmatrix} \tau f \\ -R_1(f)\end{pmatrix}=\begin{pmatrix} \lambda f \\ \lambda R_1'(f)\end{pmatrix},\quad i.e.\quad AF=\lambda F,\ F\in D(A) \tag{2.3}$$

in Hilbert space H. Consequently, the eigenvalues of the operator A and the boundary-value-problem (1.1)-(1.5) are the same.

By direct calculation we obtain the following Lemma.

Lemma 2. 1. If the functions $f(\bullet)$ and $g(\bullet)$ are differentiable on the interval $[0,1]$ then

$$R_1(f)R_1'(g)-R_1'(f)R_1(g)=(\beta_1'\beta_2-\beta_2'\beta_1)W(f,g;1)$$

where, as usual, $W(f,g;x)$ is denoted as the Wronskians $f(x)g'(x)-f'(x)g(x)$.

Theorem 2. 2. The operator A given by (2.1), (2.2) is densely defined and symmetric in the Hilbert space H.

Proof: First, we prove that the domain $D(A)$ is dense in the Hilbert space H. For this suppose that there is an element

$$U_0=\begin{pmatrix} u_0(\bullet) \\ u_1\end{pmatrix}\in H$$

which is orthogonal to all

$$F=\begin{pmatrix} f(\bullet) \\ R_1'(f)\end{pmatrix}\in D(A)$$

in the Hilbert space H, i.e.

$$<F,U_0>_H=\int_{-1}^{1} f(x)\overline{u_0(x)}dx+\frac{1}{\rho}R_1'(f)u_1=0 \tag{2.4}$$

for all $F\in D(A)$.

Denote by $\left(C^\infty(-1,0)\times C^\infty(0,1)\right)_{-1,\pm 0,1}$ the set of all functions f defined on $[-1,0)\cup(0,1]$ for which $f(-1)=f'(-1)=f(-0)=f'(-0)=f(+0)=f'(+0)=f(1)=f'(1)=0$ and let

$\left(C^{\infty}(-1,0)\times C^{\infty}(0,1)\right)_{-1,\pm 0,1}\times\{0\}$ be the set of all two-component elements of the form $\begin{pmatrix} f(\cdot) \\ 0 \end{pmatrix}$, where $f\in\left(C^{\infty}(-1,0)\times C^{\infty}(0,1)\right)_{-1,\pm 0,1}$. It is obvious that

$$\left(C^{\infty}(-1,0)\times C^{\infty}(0,1)\right)_{-1,\pm 0,1}\times\{0\}\subset D(A) \tag{2.5}$$

Then from (2.4) and (2.5) it follows immediately that

$$<F,U_0>=0 \text{ for all } \quad F\in\left(C^{\infty}(-1,0)\times C^{\infty}(0,1)\right)_{-1,\pm 0,1}\times\{0\} \tag{2.6}$$

Consequently,

$$\int_{-1}^{1} f(x)\overline{u_0(x)}dx=0 \text{ for all } f\in\left(C^{\infty}(-1,0)\times C^{\infty}(0,1)\right)_{-1,\pm 0,1} \tag{2.7}$$

Let, as usual, $C^{\infty}(a,b)$ denote the set of infinitely differentiable functions with a compact support on (a,b). It is obvious that

$$C_0^{\infty}(-1,0)\times C_0^{\infty}(0,1)\subset\left(C^{\infty}(-1,0)\times C^{\infty}(0,1)\right)_{-1,\pm 0,1}.$$

From this and from the well-known fact that $C_0^{\infty}(a,b)$ is dense in the Hilbert space $L^2(a,b)$ [15, p.96] it follows that the set $C_0^{\infty}(-1,0)\times C_0^{\infty}(0,1)$ is dense in $L^2(-1,0)\times L^2(0,1)$ and the set $\left(C^{\infty}(-1,0)\times C^{\infty}(0,1)\right)_{-1,\pm 0,1}$ is dense in the Hilbert space $L^2(-1,0)\times L^2(0,1)$.

Therefore, (2.7) means that $u_0(\cdot)$ is orthogonal to the subspace $\left(C^{\infty}(-1,0)\times C^{\infty}(0,1)\right)_{-1,\pm 0,1}$ which is dense everywhere in the Hilbert space $L^2(-1,0)\times L^2(0,1)$, so $u_0(\cdot)$ is null element of $L^2(-1,0)\times L^2(0,1)$. Putting $u_0(\cdot)=0$ in (2.4) we have

$$\frac{1}{\rho}R_1'(f)u_1=0 \tag{2.8}$$

for all $f\in L^2(-1,0)\times L^2(0,1)$, such that

$$\begin{pmatrix} f(\bullet) \\ R_1'(f) \end{pmatrix}\in D(A).$$

Now choose $F_0=\begin{pmatrix} f_0(\bullet) \\ R_1'(f_0) \end{pmatrix}$ so that $R_1'(f_0)=1$ (for example, $f_0=\frac{1}{\beta_1}$), from (2.8) we have $u_1=0$. Hence $U_0=\begin{pmatrix} u_0(\bullet) \\ u_1 \end{pmatrix}$ is the null element of the Hilbert space H. Thus, the orthogonal complement of $D(A)$ consists of only the null element, and therefore is dense in the Hilbert space H.

Now let

$$F=\begin{pmatrix} f(\bullet) \\ R_1'(f) \end{pmatrix} \quad \text{and} \quad G=\begin{pmatrix} g(\bullet) \\ R_1'(g) \end{pmatrix}$$

be arbitrary elements of $D(A)$.

Two successive integrations by parts lead to:

$$< AF,G >_H - < F,AG >_H = W(f,\overline{g};-0) - W(f,\overline{g};-1) + W(f,\overline{g};1) - \\ - W(f,\overline{g};+0) + \frac{1}{\rho}\left(R_1'(f)R_1(\overline{g}) - R_1(f)R_1'(\overline{g})\right) \quad (2.9)$$

Since each of the functions f and $\overline{g}$ satisfies the boundary conditions (1.2), it follows that

$$W(f,\overline{g};-1) = 0. \quad (2.10)$$

By applying the transmission conditions (1.3) and (1.4) we get

$$\gamma_1\gamma_2 W(f,g;-0) = \delta_1\delta_2 W(f,g;+0). \quad (2.11)$$

Putting (2.10) and (2.11) in the (2.9), recalling that $\rho = \beta_1'\beta_2 - \beta_1\beta_2'$ and applying Lemma2.1 gives the required equality

$$< AF,G >_H = < F,AG >_H$$

for all $F,G \in D(A)$.

Corollary 2. 3. All the eigenvalues of the considered problem (1.1)-(1.5) are real.

3. CONSTRUCTION OF SOME AUXILIARY SOLUTIONS OF THE PROBLEM

First, by using the special procedure we shall define two auxiliary solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ of the equation (1.1) as

$$\phi(x,\lambda) = \begin{cases} \phi_{1\lambda}(x) & \text{for} \quad x \in [-1,0) \\ \phi_{2\lambda}(x) & \text{for} \quad x \in (0,1] \end{cases}, \quad \chi(x,\lambda) = \begin{cases} \chi_{1\lambda}(x) & \text{for} \quad x \in [-1,0) \\ \chi_{2\lambda}(x) & \text{for} \quad x \in (0,1] \end{cases}.$$

Let $\phi_{1\lambda}(x)$ be the solution of the following initial-value problem

$$-u'' + q(x)u = \lambda u \ , \quad x \in (-1,0) \quad (3.1)$$

$$u(-1) = \alpha_2 \quad (3.2)$$

$$u'(-1) = -\alpha_1 \quad (3.3)$$

By virtue of the Theorem1.5 in [11] this problem has a unique solution, $u = \phi_{1\lambda}(x)$, which is an entire function of $\lambda \in \mathbb{C}$ for every fixed $x \in [-1,0]$.

Now, we shall consider the differential equation

$$-u'' + q(x)u = \lambda u \ , \quad x \in (0,1) \quad (3.4)$$

together with eigenparameter dependent initial conditions

$$u(0) = \frac{\gamma_1}{\delta_1}\phi_{1\lambda}(0) \quad (3.5)$$

$$u'(0) = \frac{\gamma_2}{\delta_2}\phi'_{1\lambda}(0). \tag{3.6}$$

Let us prove that this initial-value problem has a unique solution $u = \phi_{2\lambda}(x)$, which also is an entire function of parameter $\lambda \in \mathbb{C}$ for every fixed $x \in [0,1]$. To prove this we construct the sequence $\phi_{2,n}(x,\lambda)$, $n = 0,1,...,$ by the recurrence formulas

$$\phi_{2,n+1}(x,\lambda) = \frac{\gamma_1}{\delta_1}\phi_{1\lambda}(0) + \frac{\gamma_2}{\delta_2}\phi'_{1\lambda}(0)x + \int_0^x (q(t)-\lambda)\phi_{2,n}(t,\lambda)(x-t)dt \quad , \quad n = 0,1,2,... \tag{3.7}$$

where for $\phi_{2,0}(x,\lambda)$ we set $\phi_{2,0}(x,\lambda) = 0$. Since $\phi_{1\lambda}(0)$ and $\phi_{2\lambda}(0)$ are entire functions of parameter $\lambda \in \mathbb{C}$, each term of the sequence $\{\phi_{2,n}(x,\lambda)\}$ is so for every fixed $x \in [0,1]$.

Let us construct the series

$$\sum_{n=1}^{\infty}\left(\phi_{2,n}(x,\lambda) - \phi_{2,n-1}(x,\lambda)\right).$$

Let

$L := \max_{x\in(0,1]}|q(x)|$, $M(\lambda) := \left|\frac{\gamma_1}{\delta_1}\phi_{1\lambda}(0)\right| + \left|\frac{\gamma_2}{\delta_2}\phi_{2\lambda}(0)\right|$ and $M_R := \max_{|\lambda|\le R}|M(\lambda)|$,

where $R > 0$ is arbitrary real number. It is easy to show that,

$$\left|\phi_{2,2}(x,\lambda) - \phi_{2,1}(x,\lambda)\right| \le \int_0^x (L+R)M_R(x-t)dt$$

$$= \frac{1}{2}(L+R)M_R x^2$$

and

$$\left|\phi_{2,n+1}(x,\lambda) - \phi_{2,n}(x,\lambda)\right| \le (L+R)\int_0^x \left|\phi_{2,n}(t,\lambda) - \phi_{2,n-1}(t,\lambda)\right|(x-t)dt, \ n = 2,3,... \tag{3.8}$$

in the closed sphere $\{\lambda \in \mathbb{C} \,\|\lambda| \le R\}$. Applying these recurrence formulas successively we can obtain that

$$\left|\phi_{2,n+1}(x,\lambda) - \phi_{2,n}(x,\lambda)\right| \le \frac{M_R(L+R)^{n+1}x^{2n+2}}{(2n+2)!}. \tag{3.9}$$

Consequently, the series converges uniformly with respect to λ if $|\lambda| \le R$, and with respect to x over $(0,1]$.

Denote

$$\phi_{2\lambda}(x) := \sum_{n=1}^{\infty}\left(\phi_{2,n}(x,\lambda) - \phi_{2,n-1}(x,\lambda)\right), \tag{3.10}$$

i.e. $\phi_{2\lambda}(x) := \lim_{n\to\infty}\phi_{2,n}(x,\lambda)$.

It is obvious that this function is analytical in the open domain $\{\lambda \in \mathbb{C} \,||\lambda| < R\}$. Consequently, $\phi_{2\lambda}(x)$ is an entire function of λ for fixed x, since $R > 0$ is arbitrary. Since each term of the last series is the entire function of $\lambda \in \mathbb{C}$ and the series converges uniformly with respect to λ in the open domain $\{\lambda \in \mathbb{C} \,||\lambda| < R\}$ and with respect to x over $(0,1]$, then so is also the sum of this series, i.e. $\phi_{2\lambda}(x)$.

Further, using (3.7) we have for $n \geq 2$ that

$$\phi'_{2,n}(x,\lambda) - \phi'_{2,n-1}(x,\lambda) = \int_0^x (q(t)-\lambda)(\phi_{2,n-1}(t,\lambda) - \phi_{2,n-2}(t,\lambda))dt\,, \tag{3.11}$$

from which it follows that

$$\phi''_{2,n}(x,\lambda) - \phi''_{2,n-1}(x,\lambda) = (q(x)-\lambda)(\phi_{2,n-1}(x,\lambda) - \phi_{2,n-2}(x,\lambda)) \tag{3.12}$$

for $n \geq 2$.

By virtue of (3.9) the series

$$\sum_{n=1}^{\infty} \int_0^x (q(t)-\lambda)(\phi_{2,n-1}(t,\lambda) - \phi_{2,n-2}(t,\lambda))dt \tag{3.13}$$

and

$$\sum_{n=1}^{\infty} (q(x)-\lambda)(\phi_{2,n-1}(x,\lambda) - \phi_{2,n-2}(x,\lambda)) \tag{3.14}$$

are converges, uniformly with respect to λ over $\{\lambda \in \mathbb{C} \,|\, |\lambda| \leq R\}$, and with respect to x over $[0,1]$.

Consequently, since (3.11) and (3.12), the first and second differentiated series

$$\sum_{n=1}^{\infty} (\phi'_{2,n}(x,\lambda) - \phi'_{2,n-1}(x,\lambda)) \text{ and } \sum_{n=1}^{\infty} (\phi''_{2,n}(x,\lambda) - \phi''_{2,n-1}(x,\lambda))$$

also converge with respect to x over [0,1] for every fixed $\lambda \in \mathbb{C}$ and with respect to λ over arbitrary closed sphere $\{\lambda \in \mathbb{C} \,|\, |\lambda| \leq R\}$ for every fixed $x \in [0,1]$.

Finally, using (3.10) and (3.12) we see that

$$\phi''_{2\lambda}(x) = \sum_{n=1}^{\infty} (\phi''_{2,n}(x,\lambda) - \phi''_{2,n-1}(x,\lambda))$$

$$= (q(x)-\lambda) \sum_{n=2}^{\infty} (\phi_{2,n-1}(x,\lambda) - \phi_{2,n-2}(x,\lambda))$$

$$= (q(x)-\lambda)\phi_{2\lambda}(x)\,,$$

so that $\phi_{2\lambda}(x)$ satisfies (3.4). It also satisfies the initial conditions (3.5) and (3.6), since each term of the sequence (3.7) clearly satisfies both the initial conditions (3.5) and (3.6).

Consequently, the function $\phi(x,\lambda)$ satisfies the differential equation (1.1), one of the boundary conditions (namely, the condition (1.2)) and both transmission conditions (1.3) and (1.4).

By using a similar technique we can also prove that the initial-value problem

$$-u'' + q(x)u = \lambda u \ , \quad x \in (0,1) \tag{3.15}$$

$$u(1) = \beta_2'\lambda + \beta_2 \tag{3.16}$$

$$u'(1) = \beta_1'\lambda + \beta_1 \tag{3.17}$$

has a unique solution $u = \chi_{2\lambda}(x)$ which is an entire function of λ for fixed x, and the initial-value problem

$$-u''(x) + q(x)u = \lambda u \ , \ x \in (-1,0) \tag{3.18}$$

$$u(0) = \frac{\delta_1}{\gamma_1}\chi_{2\lambda}(0) \tag{3.19}$$

$$u'(0) = \frac{\delta_2}{\gamma_2}\chi_{2\lambda}'(0) \, . \tag{3.20}$$

has a unique solution $u = \chi_{1\lambda}(x)$ which is an entire function of λ for fixed x.

By virtue of the well-known Abel's formula [16, p. 488] each of the Wronskian's $W(\phi_{1\lambda}, \chi_{1\lambda}; x)$ and $W(\phi_{2\lambda}, \chi_{2\lambda}; x)$ are independent on variable x.

We let

$$\omega_1(\lambda) := W(\phi_{1\lambda}, \chi_{1\lambda}; x)$$

$$\omega_2(\lambda) := W(\phi_{2\lambda}, \chi_{2\lambda}; x)$$

These functions are entire functions of parameter λ, since $\phi_{i\lambda}(\bullet)$ and $\chi_{i\lambda}(\bullet)$ are entire functions of parameter λ.

Lemma 3. 1. The equality

$$\omega_1(\lambda) = \omega_2(\lambda)$$

holds for each $\lambda \in \mathbb{C}$.

Proof: By using the transmission conditions (3.16), (3.17), (3.19) and (3.20), the short calculation gives

$$\gamma_1\gamma_2 W(\phi_{1\lambda}, \chi_{1\lambda}; 0) = \delta_1\delta_2 W(\phi_{2\lambda}, \chi_{2\lambda}; 0) \, ,$$

so $\omega_1(\lambda) = \omega_2(\lambda)$ for each $\lambda \in \mathbb{C}$.

Corollary 3. 2. The zeros of the functions $\omega_1(\lambda)$ and $\omega_2(\lambda)$ are the same.

Corollary 3. 3. The Wronskians of the functions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ are independent of variable $x \in [-1,0) \cup (0,1]$ and is entire function of parameter λ.

Note: Considering this corollary we can define the following entire function $\omega(\lambda)$ as

$$\omega(\lambda) := W(\phi(x,\lambda), \chi(x,\lambda)) \, . \tag{3.21}$$

Theorem 3. 4. The eigenvalues of the boundary-value-problem (1.1)-(1.5) coincide with the zeros of the function $\omega(\lambda)$.

Proof: Let $\omega(\lambda_0)=0$. Then $W(\phi_{1\lambda_0},\chi_{1\lambda_0};x)=0$, and therefore the functions $\phi_{1\lambda_0}(x)$ and $\chi_{1\lambda_0}(x)$ are linearly dependent. i.e.

$$\chi_{1\lambda_0}(x)=k_1\phi_{1\lambda_0}(x),\, x\in[-1,0]$$

for some $k_1\neq 0$. From this it follows that $\chi(x,\lambda_0)$ satisfies the first boundary condition (1.2), so $\chi(x,\lambda_0)$ is an eigenfunction for the eigenvalue λ_0.

Now let $u_0(x)$ be any eigenfunction corresponding to eigenvalue λ_0, but $\omega(\lambda_0)\neq 0$. Then each of the pair $\phi_{1\lambda},\chi_{1\lambda}$ and $\phi_{2\lambda},\chi_{2\lambda}$ would be linearly independent on $[-1,0]$ and $[0,1]$, respectively. Consequently $u_0(x)$ may be represented by

$$u_0(x)=\begin{cases} c_1\phi_{1\lambda_0}(x)+c_2\chi_{1\lambda_0}(x), & x\in[-1,0) \\ c_3\phi_{2\lambda_0}(x)+c_4\chi_{2\lambda_0}(x), & x\in(0,1], \end{cases}$$

where at least one of the constants c_1,c_2,c_3,c_4 is not zero. Considering the true equations

$$L_\nu\big(u_0(x)\big)=0,\, \nu=\overline{1,4} \tag{3.22}$$

as the linear system of equations of the variables c_1,c_2,c_3,c_4, and taking into account (3.5), (3.6), (3.16) and (3.17), it follows that the determinant of this system

$$\begin{vmatrix} 0 & \omega(\lambda_0) & 0 & 0 \\ \gamma_1\phi_{1\lambda_0}(0) & \gamma_1\chi_{1\lambda_0}(0) & -\delta_1\phi_{2\lambda_0}(0) & -\delta_1\chi_{2\lambda_0}(0) \\ \gamma_2\phi'_{1\lambda_0}(0) & \gamma_2\chi'_{1\lambda_0}(0) & -\delta_2\phi'_{2\lambda_0}(0) & -\delta_2\chi'_{2\lambda_0}(0) \\ 0 & 0 & \omega(\lambda_0) & 0 \end{vmatrix} = -\delta_1\delta_2\omega^3(\lambda_0)\neq 0 .$$

Therefore, the system (3.22) has the only trivial solution $c_1=c_2=c_3=c_4=0$. Thus, we get contradiction, which completes the proof.

Thus, we constructed two special solutions for the equation (1.1) so that the eigenvalues of the considered problem (1.1)-(1.5) coincide with the zeros of the Wronskians of those solutions, which is an entire function of $\lambda\in\mathbb{C}$

4. THE GREEN'S FUNCTION AND RESOLVENT OPERATOR

In this section, we will obtain the resolvent of the boundary-value-transmission-problem (1.1)-(1.5) for λ, not an eigenvalue. For this, we will find the solution of the non-homogeneous differential equation

$$-u''+q(x)u=\lambda u-f(x),\ x\in(-1,0)\cup(0,1) \tag{4.1}$$

which satisfies the non-homogeneous boundary-value-transmission conditions

$$L_1(u):=\alpha_1 u(-1)+\alpha_2 u'(-1)=0 \tag{4.2}$$

$$L_2(u):=\gamma_1 u(-0)-\delta_1 u(+0)=0 \tag{4.3}$$

$$L_3(u) := \gamma_2 u'(-0) - \delta_2 u'(+0) = 0 \tag{4.4}$$

$$L_4(\lambda) := (\lambda\beta_1' + \beta_1)u(1) - (\lambda\beta_2' + \beta_2)u'(1) = f_1 . \tag{4.5}$$

We can write the general solution of homogeneous differential equation

$$-u'' + q(x)u = \lambda u \text{ , } x \in (-1,0) \cup (0,1)$$

in the form

$$U(x,\lambda) = \begin{cases} C_1\phi_{1\lambda}(x) + D_1\chi_{1\lambda}(x), & x \in [-1,0) \\ C_2\phi_{2\lambda}(x) + D_2\chi_{2\lambda}(x), & x \in (0,1] \end{cases}$$

where C_1, D_1, C_2 and D_2 are arbitrary constants. By applying the standard method of variation of the constants, we shall search the general solution of the non-homogenous linear differential equation (4.1) in the form

$$U(x,\lambda) = \begin{cases} C_1(x,\lambda)\phi_{1\lambda}(x) + D_1(x,\lambda)\chi_{1\lambda}(x), & x \in [-1,0) \\ C_2(x,\lambda)\phi_{2\lambda}(x) + D_2(x,\lambda)\chi_{2\lambda}(x), & x \in (0,1] \end{cases} \tag{4.6}$$

where the functions $C_1(x,\lambda)$ and $D_1(x,\lambda)$ satisfy the linear system of equations

$$\begin{cases} C_1'(x,\lambda)\phi_{1\lambda}(x) + D_1'(x,\lambda)\chi_{1\lambda}(x) = 0 \\ C_1'(x,\lambda)\phi_{1\lambda}'(x) + D_1'(x,\lambda)\chi_{1\lambda}'(x) = f(x) \end{cases} \tag{4.7}$$

for $x \in [-1,0)$ and the functions $C_2(x,\lambda)$ and $D_2(x,\lambda)$ satisfy the linear system of equations

$$\begin{cases} C_2'(x,\lambda)\phi_{2\lambda}(x) + D_2'(x,\lambda)\chi_{2\lambda}(x) = 0 \\ C_2'(x,\lambda)\phi_{2\lambda}'(x) + D_2'(x,\lambda)\chi_{2\lambda}'(x) = f(x) \end{cases} \tag{4.8}$$

for $x \in (0,1]$. Each of the linear systems of equation (4.7) and (4.8) has a unique solution, since λ is not an eigenvalue and therefore

$$W(\phi_{1\lambda},\chi_{1\lambda};x) = \begin{vmatrix} \phi_{1\lambda}(x) & \chi_{1\lambda}(x) \\ \phi_{1\lambda}'(x) & \chi_{1\lambda}'(x) \end{vmatrix} \neq 0 \text{ and } W(\phi_{2\lambda},\chi_{2\lambda};x) = \begin{vmatrix} \phi_{2\lambda}(x) & \chi_{2\lambda}(x) \\ \phi_{2\lambda}'(x) & \chi_{2\lambda}'(x) \end{vmatrix} \neq 0$$

It is clear that these solutions can be expressed by

$$C_1(x,\lambda) = \frac{1}{\omega(\lambda)}\int_x^0 f(y)\chi_{1\lambda}(y)dy + C_1 \text{ , } x \in [-1,0)$$

$$D_1(x,\lambda) = \frac{1}{\omega(\lambda)}\int_{-1}^x f(y)\phi_{1\lambda}(y)dy + D_1 \text{ , } x \in [-1,0)$$

$$C_2(x,\lambda) = \frac{1}{\omega(\lambda)}\int_x^1 f(y)\chi_{1\lambda}(y)dy + C_2 \text{ , } x \in (0,1]$$

$$D_2(x,\lambda)=\frac{1}{\omega(\lambda)}\int_0^x f(y)\phi_{2\lambda}(y)dy+D_2 \quad, x\in(0,1]$$

respectively, where C_1, D_1, C_2, D_2 are arbitrary constants. Substituting in (4.6), we get the general solution of non-homogeneous linear differential equation (4.1) in the form

$$U(x,\lambda)=\begin{cases}\dfrac{\phi_{1\lambda}(x)}{\omega(\lambda)}\displaystyle\int_x^0 \chi_{1\lambda}(y)f(y)dy+\dfrac{\chi_{1\lambda}(x)}{\omega(\lambda)}\displaystyle\int_{-1}^0 \phi_{1\lambda}(y)f(y)dy\ +C_1\phi_{1\lambda}(x)+D_1\chi_{1\lambda}(x)\,, & x\in[-1,0)\\ \dfrac{\phi_{2\lambda}(x)}{\omega(\lambda)}\displaystyle\int_x^1 \chi_{2\lambda}(y)f(y)dy+\dfrac{\chi_{2\lambda}(x)}{\omega(\lambda)}\displaystyle\int_{-1}^0 \phi_{2\lambda}(y)f(y)dy\ +C_2\phi_{2\lambda}(x)+D_2\chi_{2\lambda}(x)\,, & x\in(0,1]\end{cases} \tag{4.9}$$

Now, we shall find the constants C_1, D_1, C_2 and D_2 by substituting (4.9) in the boundary-value-transmission conditions (4.2)-(4.5):

By using (4.9) we have

$$L_1(U)=\alpha_1 U(-1)+\alpha_2 U'(-1)$$

$$=\frac{1}{\omega(\lambda)}\int_{-1}^0 \chi_{1\lambda}(y)f(y)dy\big(\alpha_1\phi_{1\lambda}(-1)+\alpha_2\phi'_{1\lambda}(-1)\big)+$$

$$+C_1\big(\alpha_1\phi_{1\lambda}(-1)+\alpha_2\phi'_{1\lambda}(-1)\big)+D_1\big(\alpha_1\chi_{1\lambda}(-1)+\alpha_2\chi'_{1\lambda}(-1)\big) \tag{4.10}$$

From (3.2), (3.3) and (3.21), if we take into account the equalities

$$\alpha_1\phi_{1\lambda}(-1)+\alpha_2\phi'_{1\lambda}(-1)=0$$

$$\alpha_1\chi_{1\lambda}(-1)+\alpha_2\chi'_{1\lambda}(-1)=\omega(\lambda)$$

for the solutions $\phi_{1\lambda}(x)$ and $\chi_{1\lambda}(x)$, we obtain

$$L_1(U)=D_1\omega(\lambda)\,. \tag{4.11}$$

Similarly, we have

$$L_2(U)=\gamma_1 U(-0)-\delta_1 U(+0)$$

$$=\frac{\gamma_1}{\omega(\lambda)}\chi_{1\lambda}(-0)\int_{-1}^0 \phi_{1\lambda}(y)f(y)dy+C_1\gamma_1\phi_{1\lambda}(-0)+D_1\gamma_1\chi_{1\lambda}(-0)-$$

$$-\frac{\delta_1}{\omega(\lambda)}\phi_{2\lambda}(+0)\int_0^1 \chi_{2\lambda}(y)f(y)dy-C_2\delta_1\phi_{2\lambda}(+0)-D_2\delta_1\chi_{2\lambda}(+0) \tag{4.12}$$

$$L_3(U)=\gamma_2 U'(-0)-\delta_2 U'(+0)$$

$$=\frac{\gamma_2}{\omega(\lambda)}\chi'_{1\lambda}(-0)\int_{-1}^0 \phi_{1\lambda}(y)f(y)dy+C_1\gamma_2\phi'_{1\lambda}(-0)+D_1\gamma_2\chi'_{1\lambda}(-0)-$$

$$-\frac{\delta_2}{\omega(\lambda)}\phi'_{2\lambda}(+0)\int_0^1\chi_{2\lambda}(y)f(y)dy - C_2\delta_2\phi'_{2\lambda}(+0) - D_2\delta_2\chi'_{2\lambda}(+0) \tag{4.13}$$

$$L_4(U) = (\lambda\beta'_1+\beta_1)U(1) - (\lambda\beta'_2+\beta_2)U'(1)$$

$$= (\lambda\beta'_1+\beta_1)\left(\frac{\chi_{2\lambda}(1)}{\omega(\lambda)}\int_0^1\phi_{2\lambda}(y)f(y)dy + C_2\phi_{2\lambda}(1) + D_2\chi_{2\lambda}(1)\right) -$$

$$-(\lambda\beta'_2+\beta_2)\left(\frac{\chi'_{2\lambda}(1)}{\omega(\lambda)}\int_0^1\phi_{2\lambda}(y)f(y)dy + C_2\phi'_{2\lambda}(1) + D_2\chi'_{2\lambda}(1)\right)$$

$$= \left[(\lambda\beta'_1+\beta_1)\chi_{2\lambda}(1) - (\lambda\beta'_2+\beta_2)\chi'_{2\lambda}(1)\right]\frac{1}{\omega(\lambda)}\int_0^1\phi_{2\lambda}(y)f(y)dy +$$

$$+C_2\left[(\lambda\beta'_1+\beta_1)\phi_{2\lambda}(1) - (\lambda\beta'_2+\beta_2)\phi'_{2\lambda}(1)\right] + D_2\left[(\lambda\beta'_1+\beta_1)\chi_{2\lambda}(1) - (\lambda\beta'_2+\beta_2)\chi'_{2\lambda}(1)\right]$$

From (3.5), (3.6) and (3.21) it follows that

$$L_4(U) = C_2\omega(\lambda) \tag{4.14}$$

Since $U(x,\lambda)$ is a nontrivial solution and $\omega(\lambda)\neq 0$ for λ not an eigenvalue, from (4.2) and (4.11) it follows that

$$D_1 = 0, \tag{4.15}$$

and similarly from (4.14) and (4.5) it follows that

$$C_2 = \frac{f_1}{\omega(\lambda)}. \tag{4.16}$$

On the other hand, taking into account the equalities (4.15), (4.16) and transmission conditions (4.2)-(4.5) we obtain the following linear system of equation with respect to the variables C_1 and D_2:

$$\begin{cases} \gamma_1\phi_{1\lambda}(-0)C_1 - \delta_1\chi_{2\lambda}(+0)D_2 = -\dfrac{\gamma_1}{\omega(\lambda)}\chi_{1\lambda}(-0)\displaystyle\int_{-1}^0\phi_{1\lambda}(y)f(y)dy + \\ \qquad\qquad + \dfrac{\delta_1}{\omega(\lambda)}\phi_{2\lambda}(+0)\displaystyle\int_0^1\chi_{2\lambda}(y)f(y)dy + \dfrac{f_1}{\omega(\lambda)}\delta_1\phi_{2\lambda}(+0) \\ \gamma_2\phi'_{1\lambda}(-0)C_1 - \delta_2\chi'_{2\lambda}(+0)D_2 = -\dfrac{\gamma_2}{\omega(\lambda)}\chi'_{1\lambda}(-0)\displaystyle\int_{-1}^0\phi_{1\lambda}(y)f(y)dy + \\ \qquad\qquad + \dfrac{\delta_2}{\omega(\lambda)}\phi'_{2\lambda}(+0)\displaystyle\int_0^1\chi_{2\lambda}(y)f(y)dy + \dfrac{f_1}{\omega(\lambda)}\delta_2\phi'_{2\lambda}(+0) \end{cases} \tag{4.17}$$

Recalling the definition of the solutions $\phi_{i\lambda}(x)$ and $\chi_{i\lambda}(x)\,(i=1,2)$, then for the determinant of the linear system of equations (4.17), we have

$$\begin{vmatrix} \gamma_1\phi_{1\lambda}(-0) & -\delta_1\chi_{2\lambda}(+0) \\ \gamma_2\phi'_{1\lambda}(-0) & -\delta_2\chi'_{2\lambda}(+0) \end{vmatrix} = \begin{vmatrix} \delta_1\phi_{2\lambda}(+0) & -\delta_1\chi_{2\lambda}(+0) \\ \delta_2\phi'_{2\lambda}(+0) & -\delta_2\chi'_{2\lambda}(+0) \end{vmatrix}$$

$$= -\delta_1\delta_2\left(\phi_{2\lambda}(+0)\chi'_{2\lambda}(+0) - \phi'_{2\lambda}(+0)\chi_{2\lambda}(+0)\right)$$

$$= -\delta_1\delta_2 W\left(\phi_{2\lambda}, \chi_{2\lambda}; +0\right) = -\delta_1\delta_2\omega(\lambda) \tag{4.18}$$

Since the above determinant differs from zero, the linear system of equations (4.17) has a unique solution.

By using the definitions of the functions $\chi_{1\lambda}(x)$ and $\chi_{2\lambda}(x)$ we get

$$-\frac{\gamma_1}{\delta_1}\chi'_{2\lambda}(+0)\chi_{1\lambda}(-0) + \frac{\gamma_2}{\delta_2}\chi_{2\lambda}(+0)\chi'_{2\lambda}(-0) =$$

$$-\frac{\gamma_1}{\delta_1}\chi'_{2\lambda}(+0)\left(\frac{\delta_1}{\gamma_1}\chi_{2\lambda}(+0)\right) + \frac{\gamma_2}{\delta_2}\chi_{2\lambda}(+0)\left(\frac{\delta_2}{\gamma_2}\chi'_{2\lambda}(+0)\right) \tag{4.19}$$

Now, taking this into account and the equality

$$\phi_{2\lambda}(+0)\chi'_{2\lambda}(+0) - \phi'_{2\lambda}(+0)\chi_{2\lambda}(+0) = \omega(\lambda) \tag{4.20}$$

we have from (4.17) that

$$C_1 = \frac{1}{\omega(\lambda)}\int_0^1 \chi_{2\lambda}(y)f(y)dy + \frac{f_1}{\omega(\lambda)}. \tag{4.21}$$

and

$$D_1 = \frac{1}{\omega(\lambda)}\int_{-1}^0 \phi_{1\lambda}(y)f(y)dy \tag{4.22}$$

Substituting (4.15), (4.16), (4.21) and (4.22) in (4.9) we have

$$U(x,\lambda) = \begin{cases} \dfrac{\phi_{1\lambda}(x)}{\omega(\lambda)}\displaystyle\int_x^0 \chi_{1\lambda}(y)f(y)dy + \dfrac{\chi_{1\lambda}(x)}{\omega(\lambda)}\displaystyle\int_{-1}^x \phi_{1\lambda}(y)f(y)dy + \\ \qquad + \dfrac{\phi_{1\lambda}(x)}{\omega(\lambda)}\displaystyle\int_0^1 \chi_{2\lambda}(y)f(y)dy + \dfrac{f_1}{\omega(\lambda)}\phi_{1\lambda}(x), x \in [-1,0) \\ \dfrac{\phi_{2\lambda}(x)}{\omega(\lambda)}\displaystyle\int_x^1 \chi_{2\lambda}(y)f(y)dy + \dfrac{\chi_{2\lambda}(x)}{\omega(\lambda)}\displaystyle\int_0^x \phi_{2\lambda}(y)f(y)dy + \\ \qquad + \dfrac{\chi_{2\lambda}(x)}{\omega(\lambda)}\displaystyle\int_{-1}^0 \phi_{1\lambda}(y)f(y)dy + \dfrac{f_1}{\omega(\lambda)}\phi_{2\lambda}(x),\ x \in (0,1] \end{cases} \tag{4.23}$$

for the $U(x,\lambda)$. We can rewrite the formulae (4.23) as

$$U(x,\lambda) = \frac{\chi(x,\lambda)}{\omega(\lambda)}\int_{-1}^x \phi(y,\lambda)f(y)dy + \frac{\phi(x,\lambda)}{\omega(\lambda)}\int_x^1 \chi(y,\lambda)f(y)dy + \frac{f_1}{\omega(\lambda)}\phi(x,\lambda) \tag{4.24}$$

for $x \in [-1,0) \cup (0,1]$. By introducing Green's function $G(x,y;\lambda)$ by

$$G(x.y;\lambda) = \begin{cases} \dfrac{1}{\omega(\lambda)}\chi(x,\lambda)\phi(y,\lambda), -1 \le y \le x \le 1, x \ne 0, y \ne 0 \\ \dfrac{1}{\omega(\lambda)}\phi(x,\lambda)\chi(y,\lambda), -1 \le x \le y \le 1, x \ne 0, y \ne 0 \end{cases}, \tag{4.25}$$

we can represent the formulae (4.23) in the following form

$$U(x,\lambda) = \int_{-1}^{1} G(x,y;\lambda) f(y) dy + \frac{f_1}{\omega(\lambda)}\phi(x,\lambda). \tag{4.26}$$

Now, we will obtain the resolvent operator

$$R(\lambda, A) := (\lambda I - A)^{-1} : H \to H$$

For this, we must solve the operator equation

$$(\lambda I - A)U = F \quad , \quad F \in H \tag{4.27}$$

where λ is not an eigenvalue. It is easy to see that the operator equation (4.27) is equivalent to the boundary-value-transmission-problem (4.1)-(4.5)

To obtain the resolvent operator, we will use the following Lemma.

Lemma 4. 1. If λ is not an eigenvalue, then the equality

$$R_1'(G(x,\bullet;\lambda)) = \rho\frac{\phi(x,\lambda)}{\omega(\lambda)}$$

holds.

Proof: From the formula (4.25) and the definition of the function $\chi(x,\lambda)$ we have

$$R_1'(G(x,\bullet;\lambda)) = \beta_1'\frac{1}{\omega(\lambda)}\phi(x,\lambda)\chi(1,\lambda) - \beta_2'\frac{1}{\omega(\lambda)}\phi(x,\lambda)\chi'(1,\lambda)$$

$$= \frac{1}{\omega(\lambda)}\phi(x,\lambda)(\beta_1'\chi_{2\lambda}(1) - \beta_2'\chi_{2\lambda}'(1)) = \frac{1}{\omega(\lambda)}\phi(x,\lambda)[\beta_1'(\beta_2'\lambda + \beta_2) - \beta_2'(\beta_1'\lambda + \beta_1)]$$

$$= \frac{1}{\omega(\lambda)}\phi(x,\lambda)(\beta_1'\beta_2 - \beta_2'\beta_1) = \frac{\phi(x,\lambda)}{\omega(\lambda)}\rho$$

By using this lemma, we can rewrite the formula (4.26) as

$$U(x,\lambda) = \int_{-1}^{1} G(x,y;\lambda) f(y) dy + \frac{1}{\rho} R_1'(G(x,\bullet;\lambda)) f_1. \tag{4.28}$$

Thus, we obtain the formula

$$R(\lambda, A)F = \begin{pmatrix} U(x,\lambda) \\ R_1'(U(x,\lambda)) \end{pmatrix} \tag{4.29}$$

for the resolvent operator. If we use the notations

$$\widetilde{G}_{x,\lambda} := \begin{pmatrix} G(x,\bullet;\lambda) \\ R_1'(G(x,\bullet;\lambda)) \end{pmatrix},\ F := \begin{pmatrix} f(x) \\ \dfrac{f_1}{\omega(\lambda)} \end{pmatrix},$$

then, we can express the formula (4.28) in the form

$$U(x,\lambda) = \int_{-1}^{1} G(x,y;\lambda) f(y)dy + \frac{1}{\rho} R_1'(G(x,\bullet;\lambda))f_1$$

$$=< G_{x,\lambda}, \overline{F} >_H \tag{4.30}$$

where

$$\overline{F} := \begin{pmatrix} \overline{f(x)} \\ \dfrac{\overline{f_1}}{\omega(\lambda)} \end{pmatrix}.$$

By using (4.30), the formula (4.29) can be written as

$$R(\lambda, A)F = \begin{pmatrix} < \widetilde{G}_{x,\lambda}, \overline{F} >_H \\ R_1'(< \widetilde{G}_{x,\lambda}, \overline{F} >_H) \end{pmatrix}.$$

Theorem 4. 2. For the resolvent operator $R(\lambda, A): H \to H$ the inequality

$$\|R(\lambda, A)\|_{H \to H} \le \frac{1}{|\operatorname{Im}\lambda|}$$

holds for all λ such that $\operatorname{Im}\lambda \ne 0$.

Proof: Let $\operatorname{Im}\lambda \ne 0$ and denote

$$U = R(\lambda, A)F \tag{4.31}$$

for arbitrary $F \in H$. Then, the equation

$$AU = \lambda U - F$$

holds. From this equation we get

$$< AU, U >_H =< \lambda U - F, U >_H = \lambda < U, U >_H - < F, U >_H$$

and

$$< U, AU >_H =< U, \lambda U - F >_H = - < U, F >_H + \overline{\lambda} < U, U >_H .$$

Since operator A is symmetric, from latter equations it follows that

$$(\lambda - \overline{\lambda}) < U, U >_H = < F, U >_H - < U, F >_H$$

Thus, we find

$$\left|\operatorname{Im}\lambda\right| \left\|U\right\|_H^2 = \left|\operatorname{Im} < F, U >_H\right|. \quad (4.32)$$

On the other hand, by using well-known Cauchy-Schwartz inequality we have

$$\left|\operatorname{Im} < F, U >_H\right| \le \left|< F, U >\right|_H \le \left\|F\right\|_H \left\|U\right\|_H \quad (4.33)$$

From (4.32) and (4.33), we get the inequality

$$\left\|R(\lambda, A)F\right\|_H = \left\|U\right\|_H \le \frac{1}{\left|\operatorname{Im}\lambda\right|}\left\|F\right\|_H.$$

Corollary 4. 3. Operator A is self-adjoint.

Proof: By virtue of Theorem4.2 each non-real $\lambda \in \mathbb{C}$ is a regular point of A. Furthermore $D(A)$ is dense in H (Theorem 2.2) and $(\lambda I - A)D(A) = (\overline{\lambda}I - A)D(A) = H$ for $\operatorname{Im}\lambda \ne 0$. Consequently, A is self-adjoint in the Hilbert space H [17, Theorem 2.2. p.198]).

REFERENCES

1. Birkhoff, G. D. (1908). On the asymptotic character of the solution of the certain linear differential equations containing parameter. *Trans. Amer. Soc.*, *9*, 219-231.
2. Fulton, C. T. (1977). Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. *Proc. Roy. Soc. Edin. 77A*, 293-308.
3. Mukhtarov, O. Sh., Kadakal, M. & Muhtarov, F. S. (2004). Eigenvalues and normalized eigenfunctions of discontinuous Sturm-Liouville problem with transmission conditions. *Reports on Math. Physics*, *54*, 41-56.
4. Naimark, M. N. (1967). *Linear Differential Operators*. New York, Ungar.
5. Walter, J. (1973). Regular eigenvalue problems with eigenvalue parameter in the boundary conditions, *Verlag Math. Z. 133*, 301-312.
6. Binding, P. A., Browne, P. J. & Watson, B. A. (2002). Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter I. *P. Edinburgh Math. Soc. 45*, 631-645 Part 3.
7. Binding, P. A, Browne, P. J. & Watson, B. A. (2002). Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter II. *J. Comput. Appl. Math. 148* (1), 147-168.
8. Mukhtarov, O. Sh., Kadakal, M. & Altınısık, N. (2003). Eigenvalues and eigenfunctions of discontinuous Sturm-Liouville problems with eigenparameter in the boundary conditions. *Indian J. Pure Math*, *34*(3), 501-516.
9. Yakubov, S. (1994). *Completeness of Root Functions of Regular Differential Operators*. New York, Longman, Scientific Technical.
10. Hinton, D. B. (1979). An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition. *Quart. J. Math. Oxford. 30*, 33-42.
11. Titchmarsh, E. C. (1962). *Eigenfunctions Expansion Associated With Second Order Differential Equations* I 2^{nd}. London, Oxford Univ. Press.

12. Yakubov, S. & Yakubov, Y. (1999). Abel Basis of Root Functions of Regular Boundary Value Problems. *Math. Nachr. 197*, 157-187.
13. Schneider, A. (1974). A note on eigenvalue problems with eigenvalue parameter in the boundary conditions, *Math. Z., 136*, 163-167.
14. Shkalikov, A. A. (1983). Boundary value problems for ordinary differential equations with a parameter in boundary condition. *Trudy Sem. Imeny I. G. Petrowsgo*, *9*, 190-229.
15. Yakubov, S. & Yakubov, Y. (2000). *Differential-Operator Equations (Ordinary and Partial Differential Equations)*. Chapman and Hall/CRC, Boca Raton.
16. Shepley, L. R. (1974). *Differential Equations*. John Wiley and Sons, Inc.
17. Lang, S. (1983). *Real Analysis* (Second Edition). Addison-Wesley, Reading, Mass.