“Research Note”

TOPOLOGICAL RING-GROUPOIDS AND LIFTINGS*

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Abstract – We prove that the set of homotopy classes of the paths in a topological ring is a topological ring object (called topological ring-groupoid). Let \( p : \tilde{X} \rightarrow X \) be a covering map and let \( X \) be a topological ring. We define a category \( UTRCov(X) \) of coverings of \( X \) in which both \( X \) and \( \tilde{X} \) have universal coverings, and a category \( UTRGdCov(\pi_1 X) \) of coverings of topological ring-groupoid \( \pi_1 X \), in which \( X \) and \( \tilde{R}_0 = \tilde{X} \) have universal coverings, and then prove the equivalence of these categories. We also prove that the topological ring structure of a topological ring-groupoid lifts to a universal topological covering groupoid.

Keywords – Fundamental groupoids, topological coverings, topological ring-groupoids

1. INTRODUCTION

Let \( X \) be a connected topological group with zero element 0, and let \( p : \tilde{X} \rightarrow X \) be the universal covering map of the underlying space of \( X \). It follows easily from classical properties of lifting maps to covering spaces that for any point \( \tilde{0} \) in \( \tilde{X} \) with \( p(\tilde{0}) = 0 \), there is a structure of topological group on \( \tilde{X} \) such that \( \tilde{0} \) is the zero element and \( p : \tilde{X} \rightarrow X \) is a morphism of topological groups. We say that the structure of the topological group on \( \tilde{X} \) lifts to \( X \) [1]. It is less generally appreciated that this result fails for the non-connected case. R. L. Taylor [2] showed that the topological group \( X \) determines an obstruction class \( k_X \) in \( H^3(\pi_0 X, \pi_1(X,0)) \), and that the vanishing of \( k_X \) is a necessary and sufficient condition for the lifting of the topological group structure on \( X \) to the universal covering so that the projection is a morphism. This result was generalized in terms of group-groupoids and crossed modules [3], and then written in a revised version in [4]. A topological version of that was also given in [5].

The ring version of the above results was proved in [6]. Let \( \tilde{X} \) and \( X \) be connected topological spaces and let \( p : \tilde{X} \rightarrow X \) be a universal covering. If \( X \) is a topological ring with a zero element 0, and \( \tilde{0} \in \tilde{X} \) such that \( p(\tilde{0}) = 0 \), then the ring structure of \( \tilde{X} \) lifts to \( \tilde{X} \) [6]. That is, \( \tilde{X} \) becomes a topological ring with zero element \( \tilde{0} \in \tilde{X} \) such that \( p : \tilde{X} \rightarrow X \) is a morphism of topological rings.

In [6] Mucuk defined the notion of a ring-groupoid. He also proved that if \( X \) is a topological ring, then the fundamental groupoid \( \pi_1 X \), which is the set of all relative to end points homotopy classes of paths in the topological space \( X \), becomes a ring-groupoid. In addition to this, he proved that if \( X \) is a topological ring whose underlying space has a universal covering, then the category \( TRCov(X) \) of topological ring coverings of \( X \) is equivalent to the category \( RGCov(\pi_1 X) \) of ring-groupoid coverings of \( \pi_1 X \).

In this paper we present a similar result for a topological ring-groupoid. The topological ring-groupoid is a topological ring object in the category of topological groupoids. Let \( R \) be a topological ring-
We call a subset \( U \) of \( X \) liftable if it is open, path connected and the inclusion \( U \rightarrow X \) maps each fundamental group \( \pi_1(U, x), x \in X \), to the trivial subgroup of \( \pi_1(X, x) \). Remark that if \( X \) has a universal covering, then each point \( x \in X \) has a liftable neighborhood [3].

A groupoid consists of two sets \( R \) and \( R_0 \) called respectively the set of morphisms or elements and the set objects of the groupoid together with two maps \( f, R \rightarrow R_0 \), called source and target maps respectively, a map \( 1_x : R_0 \rightarrow R, x \mapsto x \) called the object map and a partial multiplication or composition \( R_0 \times_R R \rightarrow R, (b, a) \mapsto b \circ a \) is defined on the pullback

\[
R_a \times_R R = \{(b, a) : \alpha(b) = \beta(a)\} [7].
\]

These maps are subject to the following conditions:
1. \( \alpha(b \circ a) = \alpha(a) \) and \( \beta(b \circ a) = \beta(b) \), for each \( (b, a) \in R_0 \times R \),
2. \( c \circ (b \circ a) = (c \circ b) \circ a \) for all \( c, b, a \in R \) such that \( \alpha(b) = \beta(c) \) and \( \alpha(a) = \beta(b) \),
3. \( \alpha(1_x) = \beta(1_x) = x \) for each \( x \in R_0 \), where \( 1_x \) is the identity at \( x \),
4. \( a \circ 1_{a(a)} = a \) and \( 1_{\beta(a)} \circ a = a \) for all \( a \in R \), and
5. each element \( a \) has an inverse \( a^{-1} \) such that \( \alpha(a^{-1}) = \beta(a) \), \( \beta(a^{-1}) = \alpha(a) \) and \( a^{-1} \circ a = 1_{a(a)}, a \circ a^{-1} = 1_{\beta(a)} \).

Let \( R \) be a groupoid. For each \( x, y \in R_0 \) we write \( R(x, y) \) as a set of all morphisms \( a \in R \) such that \( \alpha(a) = x \) and \( \beta(a) = y \). We will write \( St_{R_0}x \) for the set \( \alpha^{-1}(x) \), and \( CoSt_{R_0}x \) for the set \( \beta^{-1}(x) \) for \( x \in R_0 \). The object or vertex group at \( x \) is \( R(x) = St_{R_0}x \cap CoSt_{R_0}x \). We say \( R \) is transitive (resp. 1-transitive, simply transitive) if for each \( x, y \in R_0 \), \( R(x, y) \) is non-empty (resp. a singleton, has no more than one element).

Let \( R \) and \( H \) be two groupoids. A morphism from \( H \) to \( R \) is a pair of maps \( f:H \rightarrow R \) and \( f_0:H_0 \rightarrow R_0 \) such that \( \alpha \circ f = f_0 \circ \alpha \), \( \beta \circ f = f_0 \circ \beta \) and \( f(b \circ a) = f(b) \circ f(a) \) for all \( (b, a) \in H_0 \times_R H \).

We refer to [8] and [9] for more details concerning the basic concepts.

Covering morphisms of groupoids are defined in [8] as follows:

A morphism \( f:H \rightarrow R \) of groupoids is called a covering morphism if for each \( x \in H_0 \), the restriction of \( f \) mapping \( f_\#:St_{H_0}x \rightarrow St_{R_0}(f(x)) \) is bijective. Also, the following definition of pullback is given in [10].

Let \( R_\alpha \times_R H_0 \) be the pullback

\[
R_\alpha \times_R H_0 = \{(a, x) \in R \times H_0 : \alpha(a) = f_\#(x)\}.
\]

If \( f:H \rightarrow R \) is a covering morphism, then we have a lifting function \( s_f : R_\alpha \times_R H_0 \rightarrow H \) assigning to the pair \( (a, x) \) in \( R_\alpha \times_R H_0 \) the unique element \( b \) of \( St_{R_0}x \) such that \( f(b) = a \). Clearly \( s_f \) is inverse to \( (f, \alpha) : H \rightarrow R_\alpha \times_R H_0 \). So it is stated that \( f:H \rightarrow R \) is a covering morphism if and only if \( (f, \alpha) : H \rightarrow R_\alpha \times_R H_0 \) is bijective.

Let \( f:H \rightarrow R \) be a morphism of groupoids. Then for an object \( x \in H_0 \) the subgroup \( f[H(x)] \) of \( R(f(x)) \) is called the characteristic group of \( f \) at \( x \). So if \( f \) is the covering morphism then \( f \) maps \( H(x) \) isomorphically to \( f[H(x)] \). We say that a covering morphism \( f:H \rightarrow R \) is a universal covering morphism if \( H \) is 1-transitive.
A topological groupoid is a groupoid \( R \) such that the sets \( R \) and \( R_0 \) are topological spaces, and source, target, object, inverse and composition maps are continuous. Let \( R \) and \( H \) be two topological groupoids. A morphism of topological groupoids is a pair of maps \( f:H\rightarrow R \) and \( f_0:H_0\rightarrow R_0 \) such that \( f \) and \( f_0 \) are continuous. A morphism \( f:H\rightarrow R \) of topological groupoids is called a topological covering morphism if and only if \( (f,\alpha) : H \rightarrow R_0 \times_{H_0} H_0 \) is a homeomorphism.

A topological ring is a ring \( R \) with a topology on the underlying set such that the ring structure maps (i.e., group multiplication, group inverse and ring multiplication) are continuous. A topological ring morphism (topological homomorphism) of a topological ring into another is an abstract ring homomorphism which is also a continuous map.

**Definition 1.** A topological ring-groupoid \( R \) is a topological groupoid endowed with a topological ring structure such that the following ring structure maps are morphisms of topological groupoids:

1. \( m:R\times R\rightarrow R, (a,b)\mapsto a+b \), group multiplication,
2. \( u:R\rightarrow R, a\mapsto -a \), group inverse map,
3. \( 0:(\ast)\rightarrow R \), where \( (\ast) \) is a singleton.
4. \( n:R\times R\rightarrow R, (a,b)\mapsto ab \), ring multiplication,

We write \( a+b \) for the group multiplication, \( ab \) for the ring multiplication of \( a \) and \( b \), and \( b\circ a \) for the composition in the topological groupoid \( R \). Also, by 3 if \( 0 \) is the zero element of \( R_0 \) then \( 1_0 \) is that of \( R \).

**Proposition 2.** In a topological ring-groupoid \( R \), we have the interchange laws

1. \( (c\circ a)+(d\circ b)=(c+d)\circ (a+b) \) and
2. \( (c\circ a)(d\circ b)=(cd)\circ (ab) \)

whenever both \( (c\circ a) \) and \( (d\circ b) \) are defined.

**Proof:** Since \( m \) is a morphism of groupoids,

\[
(c\circ a)+(d\circ b)=\left[(c\circ a,d\circ b)\right]=m[(c,d)\circ (a,b)]=m(c,d)\circ m(a,b)\equiv (c+d)\circ (a+b).
\]

Similarly, since \( n \) is a morphism of groupoids we have

\[
(c\circ a)(d\circ b)=\left[(c\circ a,d\circ b)\right]=n[(c,d)\circ (a,b)]=n(c,d)\circ n(a,b)\equiv (cd)\circ (ab).
\]

**Example 3.** Let \( R \) be a topological ring. Then a topological ring-groupoid \( R\times R \) with object set \( R \) is defined as follows: The morphisms are the pairs \( (y,x) \), the source and target maps are defined by \( \alpha(y,x)=x \) and \( \beta(y,x)=y \), the groupoid composition is defined by \( (z,y)\circ (y,x)=(z,x) \), the group multiplication is defined by \( (z,t)+(y,x)=(z+y,t+x) \) and ring multiplication is defined by \( (z,t)(y,x)=(zy,tx) \). \( R\times R \) has product topology. So all structure maps of ring-groupoid \( R\times R \) becomes continuous. Then \( R\times R \) is a topological ring-groupoid.

We know from [6] that if \( X \) is a topological ring, then the fundamental groupoid \( \pi_1 X \) becomes a ring-groupoid. We will now give a similar result.

**Proposition 4.** Let \( X \) be a topological ring whose underlying space \( X \) has a universal covering. Then the fundamental groupoid \( \pi_1 X \) becomes a topological ring-groupoid.

**Proof:** Let \( X \) be a topological ring with the structure maps

\[
m:X\times X\rightarrow X, (a,b)\mapsto a+b
\]

\[
n:X\times X\rightarrow X, (a,b)\mapsto ab
\]

\[
0:(\ast)\rightarrow R
\]

and the inverse map
$u: X \to X$, $a \mapsto -a$.

Then these maps give the following induced maps:

\[ \pi_1 m: \pi_1 X \times \pi_1 X \to \pi_1 X, ([a],[b]) \mapsto [a+b] \]
\[ \pi_1 n: \pi_1 X \times \pi_1 X \to \pi_1 X, ([a],[b]) \mapsto [ab] \]
\[ \pi_1 u: \pi_1 X \to \pi_1 X, [a] \mapsto [-a] \]
\[ \pi_1 0: \pi_1 (\ast) \to \pi_1 R. \]

It is known from [6] that $\pi_1 X$ is a ring-groupoid. In addition, from [11], $\pi_1 X$ is a topological groupoid. Further, we will prove that the ring multiplication

\[ \pi_1 n: \pi_1 X \times \pi_1 X \to \pi_1 X, ([a],[b]) \mapsto [a][b]=[ab] \]

is continuous.

By assuming that $X$ has a universal covering [12], each $x \in X$ has a liftable neighbourhood. Let $U$ consist of such sets. Then $\pi_1 X$ has a lifted topology [8]. So the set $U_x$ consisting of all liftings of the sets in $U$, forms a basis for the topology on $\pi_1 X$. Let $\hat{U}$ be an open neighbourhood of $\hat{e}$ and a lifting of $U$ in $U$. Since the multiplication

\[ n: X \times X \to X, (a,b) \mapsto ab \]

is continuous, there is a neighborhood $V$ of 0 in $X$ such that $n(V \times V) \subseteq U$. Using the condition on $X$ and choosing $V$ small enough we can assume that $V$ has a liftable neighbourhood. Let $\hat{V}$ be the lifting of $V$. Then we have $\pi_1 n(\hat{V} \times \hat{V}) \subseteq \hat{U}$. Hence

\[ \pi_1 n: \pi_1 X \times \pi_1 X \to \pi_1 X, ([a],[b]) \mapsto [a][b]=[ab] \]

becomes continuous. So $\pi_1 X$ is a topological ring-groupoid.

**Proposition 5.** Let $R$ be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring $R_0$. Then the transitive component $C_R(0)$ of 0 is a topological ring-groupoid.

**Proof:** In [6] it was proved that $C_R(0)$ is a ring-groupoid. Further, since $C_R(0)$ is a subset of $R$, $C_R(0)$ is a topological ring-groupoid with induced topology.

**Proposition 6.** Let $R$ be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring $R_0$. Then the star $St_R 0 = \{ a \in R : \alpha (a) = 0 \}$ of 0 becomes a topological ring.

The proof is straightforward.

Let $R$ and $H$ be two topological ring-groupoids. A morphism $f: H \to R$ is a morphism of underlying topological groupoids preserving the topological ring structure, i.e., $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for $a, b \in H$. A morphism $f: H \to R$ of topological ring-groupoids is called a topological covering morphism if it is a covering morphism on the underlying topological groupoids.

**Definition 7.** Let $R$ be a topological ring-groupoid and let $X$ be a topological ring. A topological action of the topological ring-groupoid $R$ on $X$ consists of a topological ring morphism $w: X \to R_0$ and a continuous action of the underlying topological groupoid of $R$ on the underlying space of $X$ via $w: X \to R_0$ such that the following interchange laws hold

\[ 1. (\alpha y) + \iota x = b + a \quad (y + x) \]
\[ 2. \iota (\alpha y) \iota x = b \quad (y \iota x) \]

\[ \iota (\alpha y) \iota x = b \quad (y \iota x) \]
whenever both sides are defined.

**Example 8.** Let $R$ be a topological ring-groupoid which acts on a topological ring $X$ via $w: X \rightarrow R_0$. In [13] it is proved that $R \times \mathbb{X}$ is a topological groupoid with object set $(R \times \mathbb{X})_0 = X$ and morphism set $R \times \mathbb{X} = \{(a, x) \in R \times X : x = y\}$. Furthermore, the projection $p: R \times \mathbb{X} \rightarrow R$, $(a, x) \mapsto a$ becomes a covering morphism of topological groupoids. Also, in [14] it is shown that if a ring-groupoid $R$ acts on a ring $X$ via $w: X \rightarrow R_0$, then $R \times \mathbb{X}$ becomes a ring-groupoid and the projection $p: R \times \mathbb{X} \rightarrow R$, $(a, x) \mapsto a$ is a covering morphism of ring-groupoids. Clearly, the ring operations

$$(a, x) + (b, y) = (a + b, x + y)$$

and

$$(a, x)(b, y) = (ab, xy)$$

are also continuous since they are defined by the operations of the topological rings $R$ and $X$. Thus $R \times \mathbb{X}$ becomes a topological ring-groupoid and the projection $p: R \times \mathbb{X} \rightarrow R$, $(a, x) \mapsto a$ is a covering morphism of topological ring-groupoids.

### 3. TOPOLOGICAL COVERINGS

Let $X$ be a topological space. Then we have a category denoted by $TCov(X)$ whose objects are covering maps $p: \hat{X} \rightarrow X$ and a morphism from $p: \hat{X} \rightarrow X$ to $q: \hat{Y} \rightarrow X$ is a map $f: \hat{X} \rightarrow \hat{Y}$ (hence $f$ is a covering map) such that $p = qf$. Further, we have a groupoid $\pi_1 X$ called a fundamental groupoid [8] and have a category denoted by $GdCov(\pi_1 X)$ whose objects are the groupoid coverings $p: \hat{R} \rightarrow \pi_1 X$ of $\pi_1 X$ and a morphism from $p: \hat{R} \rightarrow \pi_1 X$ to $q: \hat{H} \rightarrow \pi_1 X$ is a morphism $f: \hat{R} \rightarrow \hat{H}$ of groupoids (hence $f$ is a covering morphism) such that $p = qf$.

We recall the following result from Brown [8].

**Proposition 9.** Let $X$ be a topological space which has a universal covering. Then the category $TCov(X)$ of topological coverings of $X$ and the category $GdCov(\pi_1 X)$ of covering groupoids of fundamental groupoid $\pi_1 X$ are equivalent.

Let $X$ and $\hat{X}$ be topological rings. A map $p: \hat{X} \rightarrow X$ is called a covering morphism of topological rings if $p$ is a morphism of rings and $p$ is a covering map on the underlying spaces. For a topological ring $X$, we have a category denoted by $TRCov(X)$ whose objects are covering morphisms of topological rings $p: \hat{X} \rightarrow X$ and a morphism from $p: \hat{X} \rightarrow X$ to $q: \hat{Y} \rightarrow X$ is a map $f: \hat{X} \rightarrow \hat{Y}$ (hence $f$ is a covering map) such that $p = qf$. For a topological ring $X$, the fundamental groupoid $\pi_1 X$ is a ring-groupoid and so we have a category denoted by $RGdCov(\pi_1 X)$ whose objects are the ring-groupoid coverings $p: \hat{R} \rightarrow \pi_1 X$ of $\pi_1 X$ and a morphism from $p: \hat{R} \rightarrow \pi_1 X$ to $q: \hat{H} \rightarrow \pi_1 X$ is a morphism $f: \hat{R} \rightarrow \hat{H}$ of ring-groupoids (hence $f$ is a covering morphism) such that $p = qf$.

Then the following result is given in [6].

**Proposition 10.** Let $X$ be a topological ring whose underlying space has a universal covering. Then the category $TRCov(X)$ of the topological ring coverings of $X$ is equivalent to the category $RGdCov(\pi_1 X)$ of ring-groupoid coverings of the ring-groupoid $\pi_1 X$.

In addition to these results, here we prove Theorem 11.

Let $UTRCov(X)$ be the full subcategory of $TRCov(X)$ on those objects $p: \hat{X} \rightarrow X$ in which both $\hat{X}$ and $X$ have universal coverings. Let $UTRGdCov(\pi_1 X)$ be the full subcategory of $TRGdCov(\pi_1 X)$ on those objects $p: \hat{R} \rightarrow \pi_1 X$ in which $X$ and $\hat{R}_0 = \hat{X}$ have universal coverings. Then we prove the following result.

**Theorem 11.** The categories $UTRCov(X)$ and $UTRGdCov(\pi_1 X)$ are equivalent.
Proof: Define a functor

\[ \pi_1: UTRCov(X) \to UTRGdCov(\pi_1X) \]

as follows: Let \( p: \tilde{X} \to X \) be a covering morphism of topological rings in which both underlying spaces \( \tilde{X} \) and \( X \) have universal coverings. Then the induced morphism \( \pi_1p: \pi_1\tilde{X} \to \pi_1X \) is a covering morphism of ring-groupoids [6]. Further, \( \pi_1p \) is a morphism of topological group-groupoids [11]. So \( \pi_1p \) becomes a morphism of topological ring-groupoids. Since \( \pi_1p \) is a covering morphism of ring-groupoids, \( (\pi_1p, \alpha): \pi_1\tilde{X} \to \pi_1X_{\alpha} \times_{(\pi_1p)_0} (\pi_1\tilde{X})_0 \) is bijective. On the other hand, \( \pi_1p \) is a morphism of topological ring-groupoids and \( \alpha \) is source map of topological ring-groupoid \( \pi_1X \), so \( (\pi_1p, \alpha) \) becomes continuous. We prove that \((\pi_1p, \alpha)\) is an open mapping.

Let \([\tilde{a}]\) be a morphism of \( \pi_1\tilde{X} (\tilde{x}, \tilde{y}) \). Since \( X \) and \( \tilde{X} \) have universal coverings, \( \pi_1X \) and \( \pi_1\tilde{X} \) have lifting topology. So we can choose liftable neighbourhoods \( \tilde{V}, \tilde{V}' \) of \( \tilde{x}, \tilde{y} \), respectively such that \( U=\pi(\tilde{V}) \), \( \tilde{U}'=\pi(\tilde{V}') \) are liftable neighbourhoods of \( x=p(\tilde{x}), y=p(\tilde{y}) \), respectively. If \( W = V_\epsilon(\tilde{y})(\tilde{V}')^{-1} \), then \( \pi_1p(W) \) is a basic neighbourhood of \( \pi_1p([\tilde{a}]) \), while \( (\pi_1p, \alpha)(W)=\pi_1p(W)_\alpha \times_{(\pi_1p)_0} V \), which is open in \( \pi_1X_{\alpha} \times_{(\pi_1p)_0} \tilde{X} \). So \((\pi_1p, \alpha)\) is a homeomorphism. Hence \( \pi_1p: \pi_1\tilde{X} \to \pi_1X \) becomes a covering morphism of topological ring-groupoids.

We now define a functor

\[ \Gamma: UTRGdCov(\pi_1X) \to UTRCov(X) \]

as follows: Let \( q: \tilde{R} \to \pi_1X \) be a covering morphism of topological ring-groupoids in which both \( \tilde{R}_0 = \tilde{X} \) and \( X \) have universal coverings. Since \( X \) has a universal covering, \( \tilde{X} \) has lifting topology. Hence we have a covering map \( p: \tilde{X} \to X \) of topological spaces, where \( p=q_0 \) and \( \tilde{R}_0 = \tilde{X} \) [8]. Further, since \( q \) is a covering morphism of topological ring-groupoids, \( q \) and \( p=q_0 \) are morphisms of topological rings. So \( p \) becomes a covering morphism of topological rings.

Since the category of topological ring coverings is equivalent to the category of ring-groupoid coverings, by Proposition 10 the proof is completed by the following diagram:

\[ \begin{array}{ccc}
UTRCov(X) & \xrightarrow{\pi_1} & UTRGdCov(\pi_1X) \\
\downarrow & & \downarrow \\
TRCov(X) & \xrightarrow{\pi_1} & RGdCov(\pi_1X)
\end{array} \]

Before giving the main theorem we adopt the following definition:

**Definition 12.** Let \( p: \tilde{R} \to R \) be a covering morphism of groupoids and \( q: H \to R \) a morphism of groupoids. If there exists a unique morphism \( \tilde{q}: H \to \tilde{R} \) such that \( q=p \tilde{q} \) then we say that \( q \) lifts to \( \tilde{q} \) by \( p \).

We recall the following theorem from [8] which is an important result to have the lifting maps on covering groupoids.

**Theorem 13.** Let \( p: \tilde{R} \to R \) be a covering morphism of groupoids, \( x \in R_0 \) and \( \tilde{x} \in \tilde{R}_0 \) such that \( p_0(\tilde{x})=x \). Let \( q: H \to R \) be a morphism of groupoids such that \( H \) is transitive and \( \tilde{y} \in H_0 \) such that \( q_0(\tilde{y})=x \). Then the morphism \( q: H \to R \) uniquely lifts to a morphism \( \tilde{q}: H \to \tilde{R} \) such that \( \tilde{q}_0(\tilde{y})=\tilde{x} \) if and only if \( q[H(\tilde{y})] \subseteq p[\tilde{R}(\tilde{x})] \), where \( H(\tilde{y}) \) and \( \tilde{R}(\tilde{x}) \) are the object groups.

Let \( R \) be a topological ring-groupoid and let \( 0 \in R_0 \) be the zero element in the ring \( R_0 \). Let \( \tilde{R} \) be just a topological groupoid and let \( p: \tilde{R} \to R \) be a covering morphism of topological groupoids \( \tilde{0} \in \tilde{R}_0 \) such that \( p(\tilde{0})=0 \). We say the topological ring structure of \( R \) lifts to \( \tilde{R} \) if there exists a topological ring structure on \( \tilde{R} \) with the zero element \( \tilde{0} \in \tilde{R}_0 \), such that \( \tilde{R} \) is a topological ring-groupoid and \( p: \tilde{R} \to R \) is a morphism of topological ring-groupoids.
**Theorem 14.** Let \( \hat{R} \) be a topological groupoid and let \( R \) be a topological ring-groupoid. Let \( p: \hat{R} \to R \) be a universal covering on the underlying groupoids such that both groupoids \( R \) and \( \hat{R} \) are transitive. Let \( 0 \) be the zero element in the ring \( R_0 \) and \( \hat{0} \in \hat{R}_0 \) such that \( p(\hat{0}) = 0 \). Then the topological ring structure of \( R \) lifts to \( \hat{R} \) with zero element \( \hat{0} \).

**Proof:** Since \( R \) is a topological ring-groupoid, it has the following maps:

\[
\begin{align*}
m &: R \times R \to R, \quad (a, b) \mapsto a + b \\
n &: R \times R \to R, \quad (a, b) \mapsto ab \\
u &: R \to R, \quad a \mapsto -a \\
o &: \{ * \} \to R.
\end{align*}
\]

Since \( \hat{R} \) is a universal covering, the object group \( \hat{R}(\hat{0}) \) has one element at most. So by Theorem 13 these maps respectively lift to the maps

\[
\begin{align*}
\hat{m} &: \hat{R} \times \hat{R} \to \hat{R}, \quad (\hat{a}, \hat{b}) \mapsto \hat{a} + \hat{b} \\
\hat{n} &: \hat{R} \times \hat{R} \to \hat{R}, \quad (\hat{a}, \hat{b}) \mapsto \hat{a} \hat{b} \\
\hat{u} &: \hat{R} \to \hat{R}, \quad \hat{a} \mapsto -\hat{a} \\
\hat{o} &: \{ \hat{*} \} \to \hat{R}.
\end{align*}
\]

by \( p: \hat{R} \to R \) such that

\[
\begin{align*}
p(\hat{a} + \hat{b}) &= p(\hat{a}) + p(\hat{b}), \\
p(\hat{a} \hat{b}) &= p(\hat{a}) p(\hat{b}), \\
p(\hat{u}(\hat{a})) &= -p(\hat{a}).
\end{align*}
\]

Since the multiplication \( m: R \times R \to R, \quad (a, b) \mapsto a + b \) is associative, we have \( m(m \times 1) = m(1 \times m) \), where \( 1 \) denotes the identity map. Then again by Theorem 13 these maps \( m(m \times 1) \) and \( m(1 \times m) \) respectively lift to

\[
\begin{align*}
\hat{m}(\hat{m} \times 1), \hat{m}(1 \times \hat{m}): \hat{R} \times \hat{R} \times \hat{R} \to \hat{R}
\end{align*}
\]

which coincide on \( (\hat{0}, \hat{0}, \hat{0}) \). By the uniqueness of the lifting we have \( \hat{m}(\hat{m} \times 1) = \hat{m}(1 \times \hat{m}) \), i.e., \( \hat{m} \) is associative. Similarly, \( \hat{n} \) is associative. In a similar way, we can show that \( \hat{0} \) is the zero element and \(-\hat{a}\) is the inverse element of \( \hat{a} \). Further, we will prove that the group multiplication

\[
\hat{m}: \hat{R} \times \hat{R} \to \hat{R}, \quad (\hat{a}, \hat{b}) \mapsto \hat{a} + \hat{b}
\]

is continuous.

By assuming that \( R \) has a universal covering, we can choose a cover \( U \) of liftable subsets of \( R \). Since the topology on \( \hat{R} \) is the lifted topology, the set consisting of all liftings of the sets in \( U \) forms a basis for the topology on \( \hat{R} \). Let \( \hat{U} \) be an open neighbourhood of \( \hat{0} \) and a lifting of \( U \) in \( \hat{U} \). Since the multiplication

\[
\begin{align*}
m &: R \times R \to R, \quad (a, b) \mapsto a + b
\end{align*}
\]

is continuous, there is a neighbourhood \( V \) of \( 0 \) in \( R \) such that \( m(V \times V) \subseteq U \). Using the condition on \( R \) and choosing \( V \) small enough, we can assume that \( V \) is liftable. Let \( \hat{V} \) be the lifting of \( V \). Then \( p \hat{m}(\hat{V} \times \hat{V}) = m(V \times V) \subseteq U \) and so we have \( \hat{m}(\hat{V} \times \hat{V}) \subseteq \hat{U} \). Hence

\[
\hat{m}: \hat{R} \times \hat{R} \to \hat{R}, \quad (\hat{a}, \hat{b}) \mapsto \hat{a} + \hat{b}
\]

becomes continuous. Similarly, \( \hat{n} \) is continuous. Further, the distributive law is satisfied as follows:

Let \( p_1, p_2: R \times R \times R \to R \) be the morphisms defined by
\[ p_1(a,b,c) = ab, \quad p_2(a,b,c) = bc \]

and

\[ (p_1, p_2): R \times R \times R \rightarrow R \times R, \quad (a, b, c) \mapsto (ab, bc) \]

for \( a, b, c \in R \). Since the distributive law is satisfied in \( R \), we have \( n(1 \times m) = m(p_1, p_2) \). The maps \( n(1 \times m) \) and \( m(p_1, p_2) \) respectively lift to the maps

\[ \tilde{n}(1 \times \tilde{m}), \tilde{m}(\tilde{p}_1, \tilde{p}_2) : \tilde{R} \times \tilde{R} \times \tilde{R} \rightarrow \tilde{R} \]

coinciding at \((\tilde{0}, \tilde{0}, \tilde{0})\). So by Theorem 13 we have \( \tilde{n}(1 \times \tilde{m}) = \tilde{m}(\tilde{p}_1, \tilde{p}_2) \). That means the distribution law on \( \tilde{R} \) is satisfied. Hence \( \tilde{R} \) becomes a topological ring-groupoid and clearly \( p \) is a morphism of the topological ring-groupoid.

REFERENCES