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"Research Note"

TOPOLOGICAL RING-GROUPOIDS AND LIFTINGS*

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Abstract – We prove that the set of homotopy classes of the paths in a topological ring is a topological ring object (called topological ring-groupoid). Let $p:\tilde{X}\to X$ be a covering map and let X be a topological ring. We define a category *UTRCov(X)* of coverings of X in which both X and $\tilde{X}$ have universal coverings, and a category *UTRGdCov(*$\pi_1 X$*)* of coverings of topological ring-groupoid $\pi_1 X$, in which X and $\tilde{R}_0=\tilde{X}$ have universal coverings, and then prove the equivalence of these categories. We also prove that the topological ring structure of a topological ring-groupoid lifts to a universal topological covering groupoid.

Keywords – Fundamental groupoids, topological coverings, topological ring-groupoids

1. INTRODUCTION

Let X be a connected topological group with zero element 0, and let $p:\tilde{X}\to X$ be the universal covering map of the underlying space of X. It follows easily from classical properties of lifting maps to covering spaces that for any point $\tilde{0}$ in $\tilde{X}$ with $p(\tilde{0})=0$, there is a structure of topological group on $\tilde{X}$ such that $\tilde{0}$ is the zero element and $p:\tilde{X}\to X$ is a morphism of topological groups. We say that the structure of the topological group on X lifts to $\tilde{X}$ [1]. It is less generally appreciated that this result fails for the non-connected case. R. L. Taylor [2] showed that the topological group X determines an obstruction class k_X in $H^3(\pi_0 X,\pi_1(X,0))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the topological group structure on X to the universal covering so that the projection is a morphism. This result was generalized in terms of group-groupoids and crossed modules [3], and then written in a revised version in [4]. A topological version of that was also given in [5].

The ring version of the above results was proved in [6]. Let X and $\tilde{X}$ be connected topological spaces and let $p:\tilde{X}\to X$ be a universal covering. If X is a topological ring with a zero element 0, and $\tilde{0}\in\tilde{X}$ such that $p(\tilde{0})=0$, then the ring structure of X lifts to $\tilde{X}$ [6]. That is, $\tilde{X}$ becomes a topological ring with zero element $\tilde{0}\in\tilde{X}$ such that $p:\tilde{X}\to X$ is a morphism of topological rings.

In [6] Mucuk defined the notion of a ring-groupoid. He also proved that if X is a topological ring, then the fundamental groupoid $\pi_1 X$, which is the set of all relative to end points homotopy classes of paths in the topological space X, becomes a ring-groupoid. In addition to this, he proved that if X is a topological ring whose underlying space has a universal covering, then the category *TRCov*(X) of topological ring coverings of X is equivalent to the category *RGdCov*($\pi_1 X$) of ring-groupoid coverings of $\pi_1 X$.

In this paper we present a similar result for a topological ring-groupoid. The topological ring-groupoid is a topological ring object in the category of topological groupoids. Let R be a topological ring-

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groupoid and let $p : \tilde{R} \to R$ be a universal covering on underlying groupoids such that both topological groupoids R and $\tilde{R}$ are transitive. Let 0 be the zero element of R_0 and $\tilde{0} \in \tilde{R}_0$ such that $p(\tilde{0}) = 0$. We prove that the topological ring-groupoid structure of R lifts to $\tilde{R}$ with zero element $\tilde{0}$.

Here we also prove that if X is a topological ring, whose underlying space has a universal covering, then the category *UTRCov*(X) of topological ring coverings in which $\tilde{X}$ has a universal covering, is equivalent to the category *UTRGdCov*($\pi_1 X$) of topological ring-groupoid coverings of $\pi_1 X$, in which $\tilde{R}_0 = \tilde{X}$ has a universal covering.

2. TOPOLOGICAL RING-GROUPOIDS

We call a subset U of X liftable if it is open, path connected and the inclusion $U \to X$ maps each fundamental group $\pi_1(U, x)$, $x \in X$, to the trivial subgroup of $\pi_1(X, x)$. Remark that if X has a universal covering, then each point $x \in X$ has a liftable neighborhood [3].

A groupoid consists of two sets R and R_0 called respectively the set of morphisms or elements and the set objects of the groupoid together with two maps $\alpha, \beta: R \to R_0$, called source and target maps respectively, a map $1_{()} : R_0 \to R, x \mapsto 1_x$ called the object map and a partial multiplication or composition $R_\alpha \times_\beta R \to R, (b,a) \mapsto b \circ a$ is defined on the pullback

$$R_\alpha \times_\beta R = \left\{(b,a) : \alpha(b) = \beta(a)\right\} \text{ [7]}.$$

These maps are subject to the following conditions:

1. $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$, for each $(b,a) \in R_\alpha \times_\beta R$,
2. $c \circ (b \circ a) = (c \circ b) \circ a$ for all $c,b,a \in R$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$,
3. $\alpha(1_x) = \beta(1_x) = x$ for each $x \in R_0$, where 1_x is the identity at x,
4. $a \circ 1_{\alpha(a)} = a$ and $1_{\beta(a)} \circ a = a$ for all $a \in R$, and
5. each element a has an inverse a^{-1} such that $\alpha(a^{-1}) = \beta(a)$, $\beta(a^{-1}) = \alpha(a)$ and $a^{-1} \circ a = 1_{\alpha(a)}$, $a \circ a^{-1} = 1_{\beta(a)}$.

Let R be a groupoid. For each $x,y \in R_0$ we write $R(x, y)$ as a set of all morphisms $a \in R$ such that $\alpha(a) = x$ and $\beta(a) = y$. We will write $St_R x$ for the set $\alpha^{-1}(x)$, and $CoSt_R x$ for the set $\beta^{-1}(x)$ for $x \in R_0$. The object or vertex group at x is $R(x) = R(x, x) = St_R x \cap CoSt_R x$. We say R is transitive (resp. 1-transitive, simply transitive) if for each x, $y \in R_0$, $R(x, y)$ is non-empty (resp. a singleton, has no more than one element).

Let R and H be two groupoids. A morphism from H to R is a pair of maps $f: H \to R$ and $f_0: H_0 \to R_0$ such that $\alpha_R \circ f = f_0 \circ \alpha_H$, $\beta_R \circ f = f_0 \circ \beta_H$ and $f(b \circ a) = f(b) \circ f(a)$ for all $(b,a) \in H_\alpha \times_\beta H$.

We refer to [8] and [9] for more details concerning the basic concepts.

Covering morphisms of groupoids are defined in [8] as follows:

A morphism $f: H \to R$ of groupoids is called a *covering morphism* if for each $x \in H_0$, the restriction of f mapping $f_x: St_R x \to St_R f(x)$ is bijective. Also, the following definition of pullback is given in [10].

Let $R_\alpha \times_{f_0} H_0$ be the pullback

$$R_\alpha \times_{f_0} H_0 = \{(a,x) \in R \times H_0 : \alpha(a) = f_0(x)\}.$$

If $f: H \to R$ is a covering morphism, then we have a lifting function $s_f : R_\alpha \times_{f_0} H_0 \to H$ assigning to the pair (a,x) in $R_\alpha \times_{f_0} H_0$ the unique element b of $St_H x$ such that $f(b) = a$. Clearly s_f is inverse to $(f, \alpha): H \to R_\alpha \times_{f_0} H_0$. So it is stated that $f: H \to R$ is a covering morphism if and only if $(f, \alpha): H \to R_\alpha \times_{f_0} H_0$ is bijective.

Let $f: H \to R$ be a morphism of groupoids. Then for an object $x \in H_0$ the subgroup $f[H(x)]$ of $R(f(x))$ is called the characteristic group of f at x. So if f is the covering morphism then f maps $H(x)$ isomorphically to $f[H(x)]$. We say that a covering morphism $f: H \to R$ is a universal covering morphism if H is 1-transitive.

A topological groupoid is a groupoid R such that the sets R and R_0 are topological spaces, and source, target, object, inverse and composition maps are continuous. Let R and H be two topological groupoids. A morphism of topological groupoids is a pair of maps $f:H\to R$ and $f_0:H_0\to R_0$ such that f and f_0 are continuous. A morphism $f:H\to R$ of topological groupoids is called a topological covering morphism if and only if $(f,\alpha):H\to R_\alpha \times_{f_0} H_0$ is a homeomorphism.

A topological ring is a ring R with a topology on the underlying set such that the ring structure maps (i.e., group multiplication, group inverse and ring multiplication) are continuous. A topological ring morphism (topological homomorphism) of a topological ring into another is an abstract ring homomorphism which is also a continuous map.

Definition 1. A topological ring-groupoid R is a topological groupoid endowed with a topological ring structure such that the following ring structure maps are morphisms of topological groupoids:

1. $m:R\times R\to R$, $(a,b)\mapsto a+b$, group multiplication,
2. $u:R\to R$, $a\mapsto -a$, group inverse map,
3. $0:(*)\to R$, where $(*)$ is a singleton.
4. $n:R\times R\to R$, $(a,b)\mapsto ab$, ring multiplication,

We write $a+b$ for the group multiplication, ab for the ring multiplication of a and b, and $b\circ a$ for the composition in the topological groupoid R. Also, by 3 if 0 is the zero element of R_0 then 1_0 is that of R.

Proposition 2. In a topological ring-groupoid R, we have the interchange laws

1. $(c\circ a)+(d\circ b)=(c+d)\circ(a+b)$ and
2. $(c\circ a)(d\circ b)=(cd)\circ(ab)$

whenever both $(c\circ a)$ and $(d\circ b)$ are defined.

Proof: Since m is a morphism of groupoids,

$$(c\circ a)+(d\circ b)=m[c\circ a,d\circ b]=m[(c,d)\circ(a,b)]=m(c,d)\circ m(a,b)=(c+d)\circ(a+b).$$

Similarly, since n is a morphism of groupoids we have

$$(c\circ a)(d\circ b)=n[c\circ a,d\circ b]=n[(c,d)\circ(a,b)]=n(c,d)\circ n(a,b)=(cd)\circ(ab).$$

Example 3. Let R be a topological ring. Then a topological ring-groupoid $R\times R$ with object set R is defined as follows: The morphisms are the pairs (y,x), the source and target maps are defined by $\alpha(y,x)=x$ and $\beta(y,x)=y$, the groupoid composition is defined by $(z,y)\circ(y,x)=(z,x)$, the group multiplication is defined by $(z,t)+(y,x)=(z+y,t+x)$ and ring multiplication is defined by $(z,t)(y,x)=(zy,tx)$. $R\times R$ has product topology. So all structure maps of ring-groupoid $R\times R$ becomes continuous. Then $R\times R$ is a topological ring-groupoid.

We know from [6] that if X is a topological ring, then the fundamental groupoid π_1X becomes a ring-groupoid. We will now give a similar result.

Proposition 4. Let X be a topological ring whose underlying space X has a universal covering. Then the fundamental groupoid π_1X becomes a topological ring-groupoid.

Proof: Let X be a topological ring with the structure maps

$$m:X\times X\to X,\ (a,b)\mapsto a+b$$
$$n:X\times X\to X,\ (a,b)\mapsto ab$$
$$0:(*)\to R$$

and the inverse map

$$u{:}X \to X,\ a \mapsto -a.$$

Then these maps give the following induced maps:

$$\pi_1 m{:}\ \pi_1X \times \pi_1X \to \pi_1X,\ ([a],[b]) \mapsto [a+b]$$
$$\pi_1 n{:}\ \pi_1X \times \pi_1X \to \pi_1X,\ ([a],[b]) \mapsto [ab]$$
$$\pi_1 u{:}\ \pi_1X \to \pi_1X,\ [a] \mapsto [-a]$$
$$\pi_1 0{:}\ \pi_1(*) \to \pi_1\mathrm{R}.$$

It is known from [6] that π_1X is a ring-groupoid. In addition, from [11], π_1X is a topological group-groupoid. Further, we will prove that the ring multiplication

$$\pi_1 n{:}\pi_1X \times \pi_1X \to \pi_1X,\ ([a],[b]) \mapsto [a][b]=[ab]$$

is continuous.

By assuming that X has a universal covering [12], each $x \in X$ has a liftable neighbourhood. Let U consist of such sets. Then π_1X has a lifted topology [8]. So the set $\tilde{\mathrm{U}}$, consisting of all liftings of the sets in U, forms a basis for the topology on π_1X. Let $\tilde{U}$ be an open neighbourhood of $\tilde{e}$ and a lifting of U in U. Since the multiplication

$$n{:}X \times X \to X,\ (a,b) \mapsto ab$$

is continuous, there is a neighborhood V of 0 in X such that $n(V \times V) \subseteq U$. Using the condition on X and choosing V small enough we can assume that V has a liftable neighbourhood. Let $\tilde{V}$ be the lifting of V. Then we have $\pi_1 n(\tilde{V} \times \tilde{V}) \subseteq \tilde{U}$. Hence

$$\pi_1\mathrm{n}{:}\ \pi_1X \times \pi_1X \to \pi_1X,\ ([a],[b]) \mapsto [a][b]=[ab]$$

becomes continuous. So π_1X is a topological ring-groupoid.

Proposition 5. Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring R_0. Then the transitive component $C_R(0)$ of 0 is a topological ring-groupoid.

Proof: In [6] it was proved that $C_R(0)$ is a ring-groupoid. Further, since $C_R(0)$ is a subset of R, $C_R(0)$ is a topological ring-groupoid with induced topology.

Proposition 6. Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring R_0. Then the star $St_R0=\{a \in R{:}\ \alpha(a)=0\}$ of 0 becomes a topological ring.

The proof is straightforward.

Let R and H be two topological ring-groupoids. A morphism $f{:}H \to R$ from H to R is a morphism of underlying topological groupoids preserving the topological ring structure, i.e., $f(a+b)=f(a)+f(b)$ and $f(ab)=f(a)f(b)$ for $a,b \in H$. A morphism $f{:}H \to R$ of topological ring-groupoids is called a *topological covering morphism* if it is a covering morphism on the underlying topological groupoids.

Definition 7. Let R be a topological ring-groupoid and let X be a topological ring. A topological action of the topological ring-groupoid R on X consists of a topological ring morphism $w{:}X \to R_0$ and a continuous action of the underlying topological groupoid of R on the underlying space of X via $w{:}X \to R_0$ such that the following interchange laws hold

1.$({}^{b}y)+({}^{a}x)={}^{b+a}(y+x)$

2.$({}^{b}y)({}^{a}x)={}^{ba}(yx)$

whenever both sides are defined.

Example 8. Let R be a topological ring-groupoid which acts on a topological ring X via $w{:}X\to R_0$. In [13] it is proved that $R\bowtie X$ is a topological groupoid with object set $(R\bowtie X)_0=X$ and morphism set $R\bowtie X=\{(a,x)\in R\times X : {}^a x=y\}$. Furthermore, the projection $p{:}R\bowtie X\to R$, $(a,x)\mapsto a$ becomes a covering morphism of topological groupoids. Also, in [14] it is shown that if a ring-groupoid R acts on a ring X via $w{:}X\to R_0$, then $R\bowtie X$ becomes a ring-groupoid and the projection $p{:}R\bowtie X\to R$, $(a,x)\mapsto a$ is a covering morphism of ring-groupoids. Clearly, the ring operations

$$(a,x)+(b,y)=(a+b,x+y) \text{ and}$$
$$(a,x)(b,y)=(ab,xy)$$

are also continuous since they are defined by the operations of the topological rings R and X. Thus $R\bowtie X$ becomes a topological ring-groupoid and the projection $p{:}R\bowtie X\to R$, $(a,x)\mapsto a$ is a covering morphism of topological ring-groupoids.

3. TOPOLOGICAL COVERINGS

Let X be a topological space. Then we have a category denoted by $TCov(X)$ whose objects are covering maps $p:\tilde{X}\to X$ and a morphism from $p:\tilde{X}\to X$ to $q:\tilde{Y}\to X$ is a map $f:\tilde{X}\to\tilde{Y}$ (hence f is a covering map) such that $p=qf$. Further, we have a groupoid π_1X called a fundamental groupoid [8] and have a category denoted by $GdCov(\pi_1X)$ whose objects are the groupoid coverings $p{:}\tilde{R}\to\pi_1X$ of π_1X and a morphism from $p{:}\tilde{R}\to\pi_1X$ to $q{:}\tilde{H}\to\pi_1X$ is a morphism $f{:}\tilde{R}\to\tilde{H}$ of groupoids (hence f is a covering morphism) such that $p=qf$.
We recall the following result from Brown [8].

Proposition 9. Let X be a topological space which has a universal covering. Then the category $TCov(X)$ of topological coverings of X and the category $GdCov(\pi_1X)$ of covering groupoids of fundamental groupoid π_1X are equivalent.

Let X and $\tilde{X}$ be topological rings. A map $p{:}\tilde{X}\to X$ is called a covering morphism of topological rings if p is a morphism of rings and p is a covering map on the underlying spaces. For a topological ring X, we have a category denoted by $TRCov(X)$ whose objects are covering morphisms of topological rings $p{:}\tilde{X}\to X$ and a morphism from $p{:}\tilde{X}\to X$ to q: $\tilde{Y}\to X$ is a map $f{:}\tilde{X}\to\tilde{Y}$ (hence f is a covering map) such that $p=qf$. For a topological ring X, the fundamental groupoid π_1X is a ring-groupoid and so we have a category denoted by $RGdCov(\pi_1X)$ whose objects are the ring-groupoid coverings $p{:}\tilde{R}\to\pi_1X$ of π_1X and a morphism from p: $\tilde{R}\to\pi_1X$ to q: $\tilde{H}\to\pi_1X$ is a morphism $f{:}\tilde{R}\to\tilde{H}$ of ring-groupoids (hence f is a covering morphism) such that $p=qf$.
Then the following result is given in [6].

Proposition 10. Let X be a topological ring whose underlying space has a universal covering. Then the category $TRCov(X)$ of the topological ring coverings of X is equivalent to the category $RGdCov(\pi_1X)$ of ring-groupoid coverings of the ring-groupoid π_1X.
In addition to these results, here we prove Theorem 11.

Let $UTRCov(X)$ be the full subcategory of $TRCov(X)$ on those objects $p{:}\tilde{X}\to X$ in which both $\tilde{X}$ and X have universal coverings. Let $UTRGdCov(\pi_1X)$ be the full subcategory of $TRGdCov(\pi_1X)$ on those objects $p{:}\tilde{R}\to\pi_1X$ in which X and $\tilde{R}_0=\tilde{X}$ have universal coverings. Then we prove the following result.

Theorem 11. The categories $UTRCov(X)$ and $UTRGdCov(\pi_1X)$ are equivalent.

Proof: Define a functor

$$\pi_1 : UTRCov(X) \rightarrow UTRGdCov(\pi_1 X)$$

as follows: Let $p: \tilde{X} \rightarrow X$ be a covering morphism of topological rings in which both underlying spaces $\tilde{X}$ and X have universal coverings. Then the induced morphism $\pi_1 p : \pi_1 \tilde{X} \rightarrow \pi_1 X$ is a covering morphism of ring-groupoids [6]. Further, $\pi_1 p$ is a morphism of topological group-groupoids [11]. So $\pi_1 p$ becomes a morphism of topological ring-groupoids. Since $\pi_1 p$ is a covering morphism of ring-groupoids, $(\pi_1 p, \alpha) : \pi_1 \tilde{X} \rightarrow \pi_1 X_\alpha \times_{(\pi_1 p)_0} (\pi_1 \tilde{X})_0$ is bijective. On the other hand, $\pi_1 p$ is a morphism of topological ring-groupoids and α is source map of topological ring-groupoid $\pi_1 X$, so $(\pi_1 p, \alpha)$ becomes continuous. We prove that $(\pi_1 p, \alpha)$ is an open mapping.

Let $[\tilde{a}]$ be a morphism of $\pi_1 \tilde{X}(\tilde{x}, \tilde{y})$. Since X and $\tilde{X}$ have universal coverings, π_1X and $\pi_1 \tilde{X}$ have lifting topology. So we can choose liftable neighbourhoods $\tilde{V}, \tilde{V}'$ of $\tilde{x}, \tilde{y}$, respectively such that $U=p(\tilde{V})$, $U'=p(\tilde{V}')$ are liftable neighbourhoods of $x=p(\tilde{x})$, $y=p(\tilde{y})$, respectively. If $W = \tilde{V}_{\tilde{x}}[\tilde{a}](\tilde{V}'_{\tilde{y}})^{-1}$, then $\pi_1 p(W)$ is a basic neighbourhood of $\pi_1 p([\tilde{a}])$, while $(\pi_1 p, \alpha)(W) = \pi_1 p(W)_\alpha \times_{(\pi_1 p)_0} V$, which is open in $\pi_1 X_\alpha \times_{(\pi_1 p)_0} \tilde{X}$. So $(\pi_1 p, \alpha)$ is a homeomorphism. Hence $\pi_1 p : \pi_1 \tilde{X} \rightarrow \pi_1$X becomes a covering morphism of topological ring-groupoids.

We now define a functor

$$\Gamma : UTRGdCov(\pi_1 X) \rightarrow UTRCov(X)$$

as follows: Let $q: \tilde{R} \rightarrow \pi_1$X be a covering morphism of topological ring-groupoids in which both $\tilde{R}_0 = \tilde{X}$ and X have universal coverings. Since X has a universal covering, $\tilde{X}$ has lifting topology. Hence we have a covering map $p: \tilde{X} \rightarrow X$ of topological spaces, where $p=q_0$ and $\tilde{R}_0 = \tilde{X}$ [8]. Further, since q is a covering morphism of topological ring-groupoids, q and $p=q_0$ are morphisms of topological rings. So p becomes a covering morphism of topological rings.

Since the category of topological ring coverings is equivalent to the category of ring-groupoid coverings, by Proposition 10 the proof is completed by the following diagram:

$$\begin{array}{ccc} UTRCov(X) & \xrightarrow{\pi_1} & UTRGdCov(\pi_1 X) \\ \downarrow & & \downarrow \\ TRCov(X) & \xrightarrow[\pi_1]{} & RGdCov(\pi_1 X) \end{array}$$

Before giving the main theorem we adopt the following definition:

Definition 12. Let $p: \tilde{R} \rightarrow R$ be a covering morphism of groupoids and $q: H \rightarrow R$ a morphism of groupoids. If there exists a unique morphism $\tilde{q} : H \rightarrow \tilde{R}$ such that $q=p\tilde{q}$ then we say that q lifts to $\tilde{q}$ by p.

We recall the following theorem from [8] which is an important result to have the lifting maps on covering groupoids.

Theorem 13. Let $p: \tilde{R} \rightarrow R$ be a covering morphism of groupoids, $x \in R_0$ and $\tilde{x} \in \tilde{R}_0$ such that $p_0(\tilde{x})=x$. Let $q: H \rightarrow R$ be a morphism of groupoids such that H is transitive and $\tilde{y} \in H_0$ such that $q_0(\tilde{y})=x$. Then the morphism $q: H \rightarrow R$ uniquely lifts to a morphism $\tilde{q} : H \rightarrow \tilde{R}$ such that $\tilde{q}_0(\tilde{y})=\tilde{x}$ if and only if $q[H(\tilde{y})] \subseteq p[\tilde{R}(\tilde{x})]$, where $H(\tilde{y})$ and $\tilde{R}(\tilde{x})$ are the object groups.

Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring R_0. Let $\tilde{R}$ be just a topological groupoid and let $p: \tilde{R} \rightarrow R$ be a covering morphism of topological groupoids $\tilde{0} \in \tilde{R}_0$, such that $p(\tilde{0})=0$. We say the topological ring structure of R lifts to $\tilde{R}$ if there exists a topological ring structure on $\tilde{R}$ with the zero element $\tilde{0} \in \tilde{R}_0$, such that $\tilde{R}$ is a topological ring-groupoid and $p: \tilde{R} \rightarrow R$ is a morphism of topological ring-groupoids.

Theorem 14. Let $\tilde{R}$ be a topological groupoid and let R be a topological ring-groupoid. Let p: $\tilde{R}\to R$ be a universal covering on the underlying groupoids such that both groupoids R and $\tilde{R}$ are transitive. Let 0 be the zero element in the ring R_0 and $\tilde{0}\in\tilde{R}_0$ such that $p(\tilde{0})=0$. Then the topological ring structure of R lifts to $\tilde{R}$ with zero element $\tilde{0}$.

Proof: Since R is a topological ring-groupoid, it has the following maps:

$$m{:}R\times R\to R,\ (a,b)\mapsto a+b$$
$$n{:}R\times R\to R,\ (a,b)\mapsto ab$$
$$u{:}R\to R,\ a\mapsto -a$$
$$0{:}(*)\to R.$$

Since $\tilde{R}$ is a universal covering, the object group $\tilde{R}(\tilde{0})$ has one element at most. So by Theorem 13 these maps respectively lift to the maps

$$\tilde{m}: \tilde{R}\times\tilde{R}\to\tilde{R},\ (\tilde{a},\tilde{b})\mapsto\tilde{a}+\tilde{b}$$
$$\tilde{n}: \tilde{R}\times\tilde{R}\to\tilde{R},\ (\tilde{a},\tilde{b})\mapsto\tilde{a}\ \tilde{b}$$
$$\tilde{u}:\ \tilde{R}\to\tilde{R},\ \tilde{a}\mapsto -\tilde{a}$$
$$\tilde{0}:(\tilde{*})\to\tilde{R}$$

by p: $\tilde{R}\to R$ such that

$$p(\tilde{a}+\tilde{b})=p(\tilde{a})+p(\tilde{b}),$$
$$p(\tilde{a}\ \tilde{b})=p(\tilde{a})p(\tilde{b}),$$
$$p(\tilde{u}(\tilde{a}))=-p(\tilde{a}).$$

Since the multiplication $m{:}R\times R\to R$, $(a,b)\mapsto a+b$ is associative, we have $m(m\times 1)=m(1\times m)$, where 1 denotes the identity map. Then again by Theorem 13 these maps $m(m\times 1)$ and $m(1\times m)$ respectively lift to

$$\tilde{m}(\tilde{m}\times 1), \tilde{m}(1\times\tilde{m}): \tilde{R}\times\tilde{R}\times\tilde{R}\to\tilde{R}$$

which coincide on $(\tilde{0},\tilde{0},\tilde{0})$. By the uniqueness of the lifting we have $\tilde{m}(\tilde{m}\times 1)=\tilde{m}(1\times\tilde{m})$, i.e., $\tilde{m}$ is associative. Similarly, $\tilde{n}$ is associative. In a similar way, we can show that $\tilde{0}$ is the zero element and $-\tilde{a}$ is the inverse element of $\tilde{a}$. Further, we will prove that the group multiplication

$$\tilde{m}: \tilde{R}\times\tilde{R}\to\tilde{R},\ (\tilde{a},\tilde{b})\mapsto\tilde{a}+\tilde{b}$$

is continuous.

By assuming that R has a universal covering, we can choose a cover U of liftable subsets of R. Since the topology on $\tilde{R}$ is the lifted topology, the set consisting of all liftings of the sets in U forms a basis for the topology on $\tilde{R}$. Let $\tilde{U}$ be an open neighbourhood of $\tilde{0}$ and a lifting of U in U. Since the multiplication

$$m{:}R\times R\to R,\ (a,b)\mapsto a+b$$

is continuous, there is a neighbourhood V of 0 in R such that $m(V\times V)\subseteq U$. Using the condition on R and choosing V small enough, we can assume that V is liftable. Let $\tilde{V}$ be the lifting of V. Then $p\tilde{m}(\tilde{V}\times\tilde{V})=m(V\times V)\subseteq U$ and so we have $\tilde{m}(\tilde{V}\times\tilde{V})\subseteq\tilde{U}$. Hence

$$\tilde{m}: \tilde{R}\times\tilde{R}\to\tilde{R},\ (\tilde{a},\tilde{b})\mapsto\tilde{a}+\tilde{b}$$

becomes continuous. Similarly, $\tilde{n}$ is continuous. Further, the distributive law is satisfied as follows:
Let $p_1,p_2{:}R\times R\times R\to R$ be the morphisms defined by

$$p_1(a,b,c)=ab,\ p_2(a,b,c)=bc$$

and

$$(p_1,p_2):R\times R\times R\rightarrow R\times R,\ (a,b,c)\mapsto(ab,bc)$$

for $a,b,c\in R$. Since the distributive law is satisfied in R, we have $n(1\times m)=m(p_1,p_2)$. The maps $n(1\times m)$ and $m(p_1,p_2)$ respectively lift to the maps

$$\tilde{n}(1\times\tilde{m}),\tilde{m}(\tilde{p}_1,\tilde{p}_2):\tilde{R}\times\tilde{R}\times\tilde{R}\rightarrow\tilde{R}$$

coinciding at $(\tilde{0},\tilde{0},\tilde{0})$. So by Theorem 13 we have $\tilde{n}(1\times\tilde{m})=\tilde{m}(\tilde{p}_1,\tilde{p}_2)$. That means the distribution law on $\tilde{R}$ is satisfied. Hence $\tilde{R}$ becomes a topological ring-groupoid and clearly p is a morphism of the topological ring-groupoid.

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