

“Research Note”

TOPOLOGICAL RING-GROUPOIDS AND LIFTINGS*

A. FATIH OZCAN, I. ICEN** AND M. HABIL GURSOY

Inonu University, Science and Art Faculty
Department of Mathematics, Malatya, Turkey
iicen@inonu.edu.tr

Abstract – We prove that the set of homotopy classes of the paths in a topological ring is a topological ring object (called topological ring-groupoid). Let $p : \tilde{X} \rightarrow X$ be a covering map and let X be a topological ring. We define a category $UTRCov(X)$ of coverings of X in which both X and \tilde{X} have universal coverings, and a category $UTRGdCov(\pi_1 X)$ of coverings of topological ring-groupoid $\pi_1 X$, in which X and $\tilde{R}_0 = \tilde{X}$ have universal coverings, and then prove the equivalence of these categories. We also prove that the topological ring structure of a topological ring-groupoid lifts to a universal topological covering groupoid.

Keywords – Fundamental groupoids, topological coverings, topological ring-groupoids

1. INTRODUCTION

Let X be a connected topological group with zero element 0 , and let $p : \tilde{X} \rightarrow X$ be the universal covering map of the underlying space of X . It follows easily from classical properties of lifting maps to covering spaces that for any point $\tilde{0}$ in \tilde{X} with $p(\tilde{0}) = 0$, there is a structure of topological group on \tilde{X} such that $\tilde{0}$ is the zero element and $p : \tilde{X} \rightarrow X$ is a morphism of topological groups. We say that the structure of the topological group on X lifts to \tilde{X} [1]. It is less generally appreciated that this result fails for the non-connected case. R. L. Taylor [2] showed that the topological group X determines an obstruction class k_X in $H^3(\pi_0 X, \pi_1(X, 0))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the topological group structure on X to the universal covering so that the projection is a morphism. This result was generalized in terms of group-groupoids and crossed modules [3], and then written in a revised version in [4]. A topological version of that was also given in [5].

The ring version of the above results was proved in [6]. Let X and \tilde{X} be connected topological spaces and let $p : \tilde{X} \rightarrow X$ be a universal covering. If X is a topological ring with a zero element 0 , and $\tilde{0} \in \tilde{X}$ such that $p(\tilde{0}) = 0$, then the ring structure of X lifts to \tilde{X} [6]. That is, \tilde{X} becomes a topological ring with zero element $\tilde{0} \in \tilde{X}$ such that $p : \tilde{X} \rightarrow X$ is a morphism of topological rings.

In [6] Mucuk defined the notion of a ring-groupoid. He also proved that if X is a topological ring, then the fundamental groupoid $\pi_1 X$, which is the set of all relative to end points homotopy classes of paths in the topological space X , becomes a ring-groupoid. In addition to this, he proved that if X is a topological ring whose underlying space has a universal covering, then the category $TRCov(X)$ of topological ring coverings of X is equivalent to the category $RGdCov(\pi_1 X)$ of ring-groupoid coverings of $\pi_1 X$.

In this paper we present a similar result for a topological ring-groupoid. The topological ring-groupoid is a topological ring object in the category of topological groupoids. Let R be a topological ring-

*Received by the editor June 25, 2004 and in final revised form October 31, 2006

**Corresponding author

groupoid and let $p : \tilde{R} \rightarrow R$ be a universal covering on underlying groupoids such that both topological groupoids R and \tilde{R} are transitive. Let 0 be the zero element of R_0 and $\tilde{0} \in \tilde{R}_0$ such that $p(\tilde{0}) = 0$. We prove that the topological ring-groupoid structure of R lifts to \tilde{R} with zero element $\tilde{0}$.

Here we also prove that if X is a topological ring, whose underlying space has a universal covering, then the category $UTRCov(X)$ of topological ring coverings in which \tilde{X} has a universal covering, is equivalent to the category $UTRGdCov(\pi_1 X)$ of topological ring-groupoid coverings of $\pi_1 X$, in which $\tilde{R}_0 = \tilde{X}$ has a universal covering.

2. TOPOLOGICAL RING-GROUPOIDS

We call a subset U of X liftable if it is open, path connected and the inclusion $U \rightarrow X$ maps each fundamental group $\pi_1(U, x)$, $x \in X$, to the trivial subgroup of $\pi_1(X, x)$. Remark that if X has a universal covering, then each point $x \in X$ has a liftable neighborhood [3].

A groupoid consists of two sets R and R_0 called respectively the set of morphisms or elements and the set objects of the groupoid together with two maps $\alpha, \beta: R \rightarrow R_0$, called source and target maps respectively, a map $1_{(\cdot)}: R_0 \rightarrow R, x \mapsto 1_x$ called the object map and a partial multiplication or composition $R_\alpha \times_\beta R \rightarrow R, (b, a) \mapsto b \circ a$ is defined on the pullback

$$R_\alpha \times_\beta R = \{(b, a) : \alpha(b) = \beta(a)\} [7].$$

These maps are subject to the following conditions:

1. $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$, for each $(b, a) \in R_\alpha \times_\beta R$,
2. $c \circ (b \circ a) = (c \circ b) \circ a$ for all $c, b, a \in R$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$,
3. $\alpha(1_x) = \beta(1_x) = x$ for each $x \in R_0$, where 1_x is the identity at x ,
4. $a \circ 1_{\alpha(a)} = a$ and $1_{\beta(a)} \circ a = a$ for all $a \in R$, and
5. each element a has an inverse a^{-1} such that $\alpha(a^{-1}) = \beta(a)$, $\beta(a^{-1}) = \alpha(a)$ and $a^{-1} \circ a = 1_{\alpha(a)}$, $a \circ a^{-1} = 1_{\beta(a)}$.

Let R be a groupoid. For each $x, y \in R_0$ we write $R(x, y)$ as a set of all morphisms $a \in R$ such that $\alpha(a) = x$ and $\beta(a) = y$. We will write St_{Rx} for the set $\alpha^{-1}(x)$, and $CoSt_{Rx}$ for the set $\beta^{-1}(x)$ for $x \in R_0$. The object or vertex group at x is $R(x) = R(x, x) = St_{Rx} \cap CoSt_{Rx}$. We say R is transitive (resp. 1-transitive, simply transitive) if for each $x, y \in R_0$, $R(x, y)$ is non-empty (resp. a singleton, has no more than one element).

Let R and H be two groupoids. A morphism from H to R is a pair of maps $f: H \rightarrow R$ and $f_0: H_0 \rightarrow R_0$ such that $\alpha_R \circ f = f_0 \circ \alpha_H$, $\beta_R \circ f = f_0 \circ \beta_H$ and $f(b \circ a) = f(b) \circ f(a)$ for all $(b, a) \in H_\alpha \times_\beta H$.

We refer to [8] and [9] for more details concerning the basic concepts.

Covering morphisms of groupoids are defined in [8] as follows:

A morphism $f: H \rightarrow R$ of groupoids is called a *covering morphism* if for each $x \in H_0$, the restriction of f mapping $f_x: St_{Rx} \rightarrow St_{Rf(x)}$ is bijective. Also, the following definition of pullback is given in [10].

Let $R_\alpha \times_{f_0} H_0$ be the pullback

$$R_\alpha \times_{f_0} H_0 = \{(a, x) \in R \times H_0 : \alpha(a) = f_0(x)\}.$$

If $f: H \rightarrow R$ is a covering morphism, then we have a lifting function $s_f: R_\alpha \times_{f_0} H_0 \rightarrow H$ assigning to the pair (a, x) in $R_\alpha \times_{f_0} H_0$ the unique element b of St_{Hx} such that $f(b) = a$. Clearly s_f is inverse to $(f, \alpha): H \rightarrow R_\alpha \times_{f_0} H_0$. So it is stated that $f: H \rightarrow R$ is a covering morphism if and only if $(f, \alpha): H \rightarrow R_\alpha \times_{f_0} H_0$ is bijective.

Let $f: H \rightarrow R$ be a morphism of groupoids. Then for an object $x \in H_0$ the subgroup $f[H(x)]$ of $R(f(x))$ is called the characteristic group of f at x . So if f is the covering morphism then f maps $H(x)$ isomorphically to $f[H(x)]$. We say that a covering morphism $f: H \rightarrow R$ is a universal covering morphism if H is 1-transitive.

A topological groupoid is a groupoid R such that the sets R and R_0 are topological spaces, and source, target, object, inverse and composition maps are continuous. Let R and H be two topological groupoids. A morphism of topological groupoids is a pair of maps $f:H \rightarrow R$ and $f_0:H_0 \rightarrow R_0$ such that f and f_0 are continuous. A morphism $f:H \rightarrow R$ of topological groupoids is called a topological covering morphism if and only if $(f, \alpha):H \rightarrow R_\alpha \times_{f_0} H_0$ is a homeomorphism.

A topological ring is a ring R with a topology on the underlying set such that the ring structure maps (i.e., group multiplication, group inverse and ring multiplication) are continuous. A topological ring morphism (topological homomorphism) of a topological ring into another is an abstract ring homomorphism which is also a continuous map.

Definition 1. A topological ring-groupoid R is a topological groupoid endowed with a topological ring structure such that the following ring structure maps are morphisms of topological groupoids:

1. $m:R \times R \rightarrow R, (a,b) \mapsto a+b$, group multiplication,
2. $u:R \rightarrow R, a \mapsto -a$, group inverse map,
3. $0:(*) \rightarrow R$, where $(*)$ is a singleton.
4. $n:R \times R \rightarrow R, (a,b) \mapsto ab$, ring multiplication,

We write $a+b$ for the group multiplication, ab for the ring multiplication of a and b , and $b \circ a$ for the composition in the topological groupoid R . Also, by 3 if 0 is the zero element of R_0 then 1_0 is that of R .

Proposition 2. In a topological ring-groupoid R , we have the interchange laws

1. $(c \circ a) + (d \circ b) = (c+d) \circ (a+b)$ and
2. $(c \circ a)(d \circ b) = (cd) \circ (ab)$

whenever both $(c \circ a)$ and $(d \circ b)$ are defined.

Proof: Since m is a morphism of groupoids,

$$(c \circ a) + (d \circ b) = m[c \circ a, d \circ b] = m[(c, d) \circ (a, b)] = m(c, d) \circ m(a, b) = (c+d) \circ (a+b).$$

Similarly, since n is a morphism of groupoids we have

$$(c \circ a)(d \circ b) = n[c \circ a, d \circ b] = n[(c, d) \circ (a, b)] = n(c, d) \circ n(a, b) = (cd) \circ (ab).$$

Example 3. Let R be a topological ring. Then a topological ring-groupoid $R \times R$ with object set R is defined as follows: The morphisms are the pairs (y, x) , the source and target maps are defined by $\alpha(y, x) = x$ and $\beta(y, x) = y$, the groupoid composition is defined by $(z, y) \circ (y, x) = (z, x)$, the group multiplication is defined by $(z, t) + (y, x) = (z+y, t+x)$ and ring multiplication is defined by $(z, t)(y, x) = (zy, tx)$. $R \times R$ has product topology. So all structure maps of ring-groupoid $R \times R$ becomes continuous. Then $R \times R$ is a topological ring-groupoid.

We know from [6] that if X is a topological ring, then the fundamental groupoid $\pi_l X$ becomes a ring-groupoid. We will now give a similar result.

Proposition 4. Let X be a topological ring whose underlying space X has a universal covering. Then the fundamental groupoid $\pi_l X$ becomes a topological ring-groupoid.

Proof: Let X be a topological ring with the structure maps

$$\begin{aligned} m: X \times X &\rightarrow X, (a, b) \mapsto a+b \\ n: X \times X &\rightarrow X, (a, b) \mapsto ab \\ 0: (*) &\rightarrow R \end{aligned}$$

and the inverse map

$$u: X \rightarrow X, a \mapsto -a.$$

Then these maps give the following induced maps:

$$\begin{aligned}\pi_1 m: \pi_1 X \times \pi_1 X &\rightarrow \pi_1 X, ([a],[b]) \mapsto [a+b] \\ \pi_1 n: \pi_1 X \times \pi_1 X &\rightarrow \pi_1 X, ([a],[b]) \mapsto [ab] \\ \pi_1 u: \pi_1 X &\rightarrow \pi_1 X, [a] \mapsto [-a] \\ \pi_1 0: \pi_1(*) &\rightarrow \pi_1 R.\end{aligned}$$

It is known from [6] that $\pi_1 X$ is a ring-groupoid. In addition, from [11], $\pi_1 X$ is a topological ring-groupoid. Further, we will prove that the ring multiplication

$$\pi_1 n: \pi_1 X \times \pi_1 X \rightarrow \pi_1 X, ([a],[b]) \mapsto [a][b] = [ab]$$

is continuous.

By assuming that X has a universal covering [12], each $x \in X$ has a liftable neighbourhood. Let U consist of such sets. Then $\pi_1 X$ has a lifted topology [8]. So the set \tilde{U} , consisting of all liftings of the sets in U , forms a basis for the topology on $\pi_1 X$. Let \tilde{V} be an open neighbourhood of \tilde{e} and a lifting of U in U . Since the multiplication

$$n: X \times X \rightarrow X, (a,b) \mapsto ab$$

is continuous, there is a neighborhood V of 0 in X such that $n(V \times V) \subseteq U$. Using the condition on X and choosing V small enough we can assume that V has a liftable neighbourhood. Let \tilde{V} be the lifting of V . Then we have $\pi_1 n(\tilde{V} \times \tilde{V}) \subseteq \tilde{U}$. Hence

$$\pi_1 n: \pi_1 X \times \pi_1 X \rightarrow \pi_1 X, ([a],[b]) \mapsto [a][b] = [ab]$$

becomes continuous. So $\pi_1 X$ is a topological ring-groupoid.

Proposition 5. Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring R_0 . Then the transitive component $C_R(0)$ of 0 is a topological ring-groupoid.

Proof: In [6] it was proved that $C_R(0)$ is a ring-groupoid. Further, since $C_R(0)$ is a subset of R , $C_R(0)$ is a topological ring-groupoid with induced topology.

Proposition 6. Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring R_0 . Then the star $St_R 0 = \{a \in R: \alpha(a) = 0\}$ of 0 becomes a topological ring.

The proof is straightforward.

Let R and H be two topological ring-groupoids. A morphism $f: H \rightarrow R$ from H to R is a morphism of underlying topological groupoids preserving the topological ring structure, i.e., $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for $a, b \in H$. A morphism $f: H \rightarrow R$ of topological ring-groupoids is called a *topological covering morphism* if it is a covering morphism on the underlying topological groupoids.

Definition 7. Let R be a topological ring-groupoid and let X be a topological ring. A topological action of the topological ring-groupoid R on X consists of a topological ring morphism $w: X \rightarrow R_0$ and a continuous action of the underlying topological groupoid of R on the underlying space of X via $w: X \rightarrow R_0$ such that the following interchange laws hold

1. $({}^b y) + ({}^a x) = {}^{b+a} (y+x)$
2. $({}^b y)({}^a x) = {}^{ba} (yx)$

whenever both sides are defined.

Example 8. Let R be a topological ring-groupoid which acts on a topological ring X via $w: X \rightarrow R_0$. In [13] it is proved that $R \bowtie X$ is a topological groupoid with object set $(R \bowtie X)_0 = X$ and morphism set $R \bowtie X = \{(a, x) \in R \times X : {}^a x = y\}$. Furthermore, the projection $p: R \bowtie X \rightarrow R, (a, x) \mapsto a$ becomes a covering morphism of topological groupoids. Also, in [14] it is shown that if a ring-groupoid R acts on a ring X via $w: X \rightarrow R_0$, then $R \bowtie X$ becomes a ring-groupoid and the projection $p: R \bowtie X \rightarrow R, (a, x) \mapsto a$ is a covering morphism of ring-groupoids. Clearly, the ring operations

$$(a, x) + (b, y) = (a + b, x + y) \text{ and} \\ (a, x)(b, y) = (ab, xy)$$

are also continuous since they are defined by the operations of the topological rings R and X . Thus $R \bowtie X$ becomes a topological ring-groupoid and the projection $p: R \bowtie X \rightarrow R, (a, x) \mapsto a$ is a covering morphism of topological ring-groupoids.

3. TOPOLOGICAL COVERINGS

Let X be a topological space. Then we have a category denoted by $TCov(X)$ whose objects are covering maps $p: \tilde{X} \rightarrow X$ and a morphism from $p: \tilde{X} \rightarrow X$ to $q: \tilde{Y} \rightarrow X$ is a map $f: \tilde{X} \rightarrow \tilde{Y}$ (hence f is a covering map) such that $p = qf$. Further, we have a groupoid $\pi_1 X$ called a fundamental groupoid [8] and have a category denoted by $GdCov(\pi_1 X)$ whose objects are the groupoid coverings $p: \tilde{R} \rightarrow \pi_1 X$ of $\pi_1 X$ and a morphism from $p: \tilde{R} \rightarrow \pi_1 X$ to $q: \tilde{H} \rightarrow \pi_1 X$ is a morphism $f: \tilde{R} \rightarrow \tilde{H}$ of groupoids (hence f is a covering morphism) such that $p = qf$.

We recall the following result from Brown [8].

Proposition 9. Let X be a topological space which has a universal covering. Then the category $TCov(X)$ of topological coverings of X and the category $GdCov(\pi_1 X)$ of covering groupoids of fundamental groupoid $\pi_1 X$ are equivalent.

Let X and \tilde{X} be topological rings. A map $p: \tilde{X} \rightarrow X$ is called a covering morphism of topological rings if p is a morphism of rings and p is a covering map on the underlying spaces. For a topological ring X , we have a category denoted by $TRCov(X)$ whose objects are covering morphisms of topological rings $p: \tilde{X} \rightarrow X$ and a morphism from $p: \tilde{X} \rightarrow X$ to $q: \tilde{Y} \rightarrow X$ is a map $f: \tilde{X} \rightarrow \tilde{Y}$ (hence f is a covering map) such that $p = qf$. For a topological ring X , the fundamental groupoid $\pi_1 X$ is a ring-groupoid and so we have a category denoted by $RGdCov(\pi_1 X)$ whose objects are the ring-groupoid coverings $p: \tilde{R} \rightarrow \pi_1 X$ of $\pi_1 X$ and a morphism from $p: \tilde{R} \rightarrow \pi_1 X$ to $q: \tilde{H} \rightarrow \pi_1 X$ is a morphism $f: \tilde{R} \rightarrow \tilde{H}$ of ring-groupoids (hence f is a covering morphism) such that $p = qf$.

Then the following result is given in [6].

Proposition 10. Let X be a topological ring whose underlying space has a universal covering. Then the category $TRCov(X)$ of the topological ring coverings of X is equivalent to the category $RGdCov(\pi_1 X)$ of ring-groupoid coverings of the ring-groupoid $\pi_1 X$.

In addition to these results, here we prove Theorem 11.

Let $UTRCov(X)$ be the full subcategory of $TRCov(X)$ on those objects $p: \tilde{X} \rightarrow X$ in which both \tilde{X} and X have universal coverings. Let $UTRGdCov(\pi_1 X)$ be the full subcategory of $TRGdCov(\pi_1 X)$ on those objects $p: \tilde{R} \rightarrow \pi_1 X$ in which X and $\tilde{R}_0 = \tilde{X}$ have universal coverings. Then we prove the following result.

Theorem 11. The categories $UTRCov(X)$ and $UTRGdCov(\pi_1 X)$ are equivalent.

Proof: Define a functor

$$\pi_1:UTRCov(X)\rightarrow UTRGdCov(\pi_1X)$$

as follows: Let $p: \tilde{X} \rightarrow X$ be a covering morphism of topological rings in which both underlying spaces \tilde{X} and X have universal coverings. Then the induced morphism $\pi_1 p: \pi_1 \tilde{X} \rightarrow \pi_1 X$ is a covering morphism of ring-groupoids [6]. Further, $\pi_1 p$ is a morphism of topological group-groupoids [11]. So $\pi_1 p$ becomes a morphism of topological ring-groupoids. Since $\pi_1 p$ is a covering morphism of ring-groupoids, $(\pi_1 p, \alpha): \pi_1 \tilde{X} \rightarrow \pi_1 X_\alpha \times_{(\pi_1 p)_0} (\pi_1 \tilde{X})_0$ is bijective. On the other hand, $\pi_1 p$ is a morphism of topological ring-groupoids and α is source map of topological ring-groupoid $\pi_1 X$, so $(\pi_1 p, \alpha)$ becomes continuous. We prove that $(\pi_1 p, \alpha)$ is an open mapping.

Let $[\tilde{a}]$ be a morphism of $\pi_1 \tilde{X}(\tilde{x}, \tilde{y})$. Since X and \tilde{X} have universal coverings, $\pi_1 X$ and $\pi_1 \tilde{X}$ have lifting topology. So we can choose liftable neighbourhoods \tilde{V}, \tilde{V}' of \tilde{x}, \tilde{y} , respectively such that $U=p(\tilde{V}), U'=p(\tilde{V}')$ are liftable neighbourhoods of $x=p(\tilde{x}), y=p(\tilde{y})$, respectively. If $W = \tilde{V}_{\tilde{x}}[\tilde{a}](\tilde{V}'_{\tilde{y}})^{-1}$, then $\pi_1 p(W)$ is a basic neighbourhood of $\pi_1 p([\tilde{a}])$, while $(\pi_1 p, \alpha)(W) = \pi_1 p(W)_\alpha \times_{(\pi_1 p)_0} V$, which is open in $\pi_1 X_\alpha \times_{(\pi_1 p)_0} \tilde{X}$. So $(\pi_1 p, \alpha)$ is a homeomorphism. Hence $\pi_1 p: \pi_1 \tilde{X} \rightarrow \pi_1 X$ becomes a covering morphism of topological ring-groupoids.

We now define a functor

$$\Gamma:UTRGdCov(\pi_1X)\rightarrow UTRCov(X)$$

as follows: Let $q: \tilde{R} \rightarrow \pi_1 X$ be a covering morphism of topological ring-groupoids in which both $\tilde{R}_0 = \tilde{X}$ and X have universal coverings. Since X has a universal covering, \tilde{X} has lifting topology. Hence we have a covering map $p: \tilde{X} \rightarrow X$ of topological spaces, where $p=q_0$ and $\tilde{R}_0 = \tilde{X}$ [8]. Further, since q is a covering morphism of topological ring-groupoids, q and $p=q_0$ are morphisms of topological rings. So p becomes a covering morphism of topological rings.

Since the category of topological ring coverings is equivalent to the category of ring-groupoid coverings, by Proposition 10 the proof is completed by the following diagram:

$$\begin{array}{ccc} UTRCov(X) & \xrightarrow{\pi_1} & UTRGdCov(\pi_1X) \\ \downarrow & & \downarrow \\ TRCov(X) & \xrightarrow{\pi_1} & RGdCov(\pi_1X) \end{array}$$

Before giving the main theorem we adopt the following definition:

Definition 12. Let $p: \tilde{R} \rightarrow R$ be a covering morphism of groupoids and $q: H \rightarrow R$ a morphism of groupoids. If there exists a unique morphism $\tilde{q}: H \rightarrow \tilde{R}$ such that $q=p\tilde{q}$ then we say that q lifts to \tilde{q} by p .

We recall the following theorem from [8] which is an important result to have the lifting maps on covering groupoids.

Theorem 13. Let $p: \tilde{R} \rightarrow R$ be a covering morphism of groupoids, $x \in R_0$ and $\tilde{x} \in \tilde{R}_0$ such that $p_0(\tilde{x})=x$. Let $q: H \rightarrow R$ be a morphism of groupoids such that H is transitive and $\tilde{y} \in H_0$ such that $q_0(\tilde{y})=x$. Then the morphism $q: H \rightarrow R$ uniquely lifts to a morphism $\tilde{q}: H \rightarrow \tilde{R}$ such that $\tilde{q}_0(\tilde{y})=\tilde{x}$ if and only if $q[H(\tilde{y})] \subseteq p[\tilde{R}(\tilde{x})]$, where $H(\tilde{y})$ and $\tilde{R}(\tilde{x})$ are the object groups.

Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring R_0 . Let \tilde{R} be just a topological groupoid and let $p: \tilde{R} \rightarrow R$ be a covering morphism of topological groupoids $\tilde{0} \in \tilde{R}_0$, such that $p(\tilde{0})=0$. We say the topological ring structure of R lifts to \tilde{R} if there exists a topological ring structure on \tilde{R} with the zero element $\tilde{0} \in \tilde{R}_0$, such that \tilde{R} is a topological ring-groupoid and $p: \tilde{R} \rightarrow R$ is a morphism of topological ring-groupoids.

Theorem 14. Let \tilde{R} be a topological groupoid and let R be a topological ring-groupoid. Let $p: \tilde{R} \rightarrow R$ be a universal covering on the underlying groupoids such that both groupoids R and \tilde{R} are transitive. Let 0 be the zero element in the ring R_0 and $\tilde{0} \in \tilde{R}_0$ such that $p(\tilde{0})=0$. Then the topological ring structure of R lifts to \tilde{R} with zero element $\tilde{0}$.

Proof: Since R is a topological ring-groupoid, it has the following maps:

$$\begin{aligned} m: R \times R &\rightarrow R, (a, b) \mapsto a + b \\ n: R \times R &\rightarrow R, (a, b) \mapsto ab \\ u: R &\rightarrow R, a \mapsto -a \\ 0: (*) &\rightarrow R. \end{aligned}$$

Since \tilde{R} is a universal covering, the object group $\tilde{R}(\tilde{0})$ has one element at most. So by Theorem 13 these maps respectively lift to the maps

$$\begin{aligned} \tilde{m}: \tilde{R} \times \tilde{R} &\rightarrow \tilde{R}, (\tilde{a}, \tilde{b}) \mapsto \tilde{a} + \tilde{b} \\ \tilde{n}: \tilde{R} \times \tilde{R} &\rightarrow \tilde{R}, (\tilde{a}, \tilde{b}) \mapsto \tilde{a} \tilde{b} \\ \tilde{u}: \tilde{R} &\rightarrow \tilde{R}, \tilde{a} \mapsto -\tilde{a} \\ \tilde{0}: (*) &\rightarrow \tilde{R} \end{aligned}$$

by $p: \tilde{R} \rightarrow R$ such that

$$\begin{aligned} p(\tilde{a} + \tilde{b}) &= p(\tilde{a}) + p(\tilde{b}), \\ p(\tilde{a} \tilde{b}) &= p(\tilde{a})p(\tilde{b}), \\ p(\tilde{u}(\tilde{a})) &= -p(\tilde{a}). \end{aligned}$$

Since the multiplication $m: R \times R \rightarrow R, (a, b) \mapsto a + b$ is associative, we have $m(m \times 1) = m(1 \times m)$, where 1 denotes the identity map. Then again by Theorem 13 these maps $m(m \times 1)$ and $m(1 \times m)$ respectively lift to

$$\tilde{m}(\tilde{m} \times 1), \tilde{m}(1 \times \tilde{m}): \tilde{R} \times \tilde{R} \times \tilde{R} \rightarrow \tilde{R}$$

which coincide on $(\tilde{0}, \tilde{0}, \tilde{0})$. By the uniqueness of the lifting we have $\tilde{m}(\tilde{m} \times 1) = \tilde{m}(1 \times \tilde{m})$, i.e., \tilde{m} is associative. Similarly, \tilde{n} is associative. In a similar way, we can show that $\tilde{0}$ is the zero element and $-\tilde{a}$ is the inverse element of \tilde{a} . Further, we will prove that the group multiplication

$$\tilde{m}: \tilde{R} \times \tilde{R} \rightarrow \tilde{R}, (\tilde{a}, \tilde{b}) \mapsto \tilde{a} + \tilde{b}$$

is continuous.

By assuming that R has a universal covering, we can choose a cover U of liftable subsets of R . Since the topology on \tilde{R} is the lifted topology, the set consisting of all liftings of the sets in U forms a basis for the topology on \tilde{R} . Let \tilde{U} be an open neighbourhood of $\tilde{0}$ and a lifting of U in U . Since the multiplication

$$m: R \times R \rightarrow R, (a, b) \mapsto a + b$$

is continuous, there is a neighbourhood V of 0 in R such that $m(V \times V) \subseteq U$. Using the condition on R and choosing V small enough, we can assume that V is liftable. Let \tilde{V} be the lifting of V . Then $p \tilde{m}(\tilde{V} \times \tilde{V}) = m(V \times V) \subseteq U$ and so we have $\tilde{m}(\tilde{V} \times \tilde{V}) \subseteq \tilde{U}$. Hence

$$\tilde{m}: \tilde{R} \times \tilde{R} \rightarrow \tilde{R}, (\tilde{a}, \tilde{b}) \mapsto \tilde{a} + \tilde{b}$$

becomes continuous. Similarly, \tilde{n} is continuous. Further, the distributive law is satisfied as follows:

Let $p_1, p_2: R \times R \times R \rightarrow R$ be the morphisms defined by

$$p_1(a,b,c)=ab, p_2(a,b,c)=bc$$

and

$$(p_1, p_2): R \times R \times R \rightarrow R \times R, (a, b, c) \mapsto (ab, bc)$$

for $a, b, c \in R$. Since the distributive law is satisfied in R , we have $n(1 \times m) = m(p_1, p_2)$. The maps $n(1 \times m)$ and $m(p_1, p_2)$ respectively lift to the maps

$$\tilde{n}(1 \times \tilde{m}), \tilde{m}(\tilde{p}_1, \tilde{p}_2) : \tilde{R} \times \tilde{R} \times \tilde{R} \rightarrow \tilde{R}$$

coinciding at $(\tilde{0}, \tilde{0}, \tilde{0})$. So by Theorem 13 we have $\tilde{n}(1 \times \tilde{m}) = \tilde{m}(\tilde{p}_1, \tilde{p}_2)$. That means the distribution law on \tilde{R} is satisfied. Hence \tilde{R} becomes a topological ring-groupoid and clearly p is a morphism of the topological ring-groupoid.

REFERENCES

1. Chevalley, C. (1946). Theory of Lie Groups. *Princeton University Press*.
2. Taylor, R. L. (1954). Covering groups of non-connected topological groups. *Proc. Amer. Math. Soc.*, 5, 753-768.
3. Mucuk, O. (1993). Covering groups of non-connected topological groups and the monodromy groupoid of a topological groupoid. PhD Thesis, University of Wales England.
4. Brown, R. & Mucuk, O. (1994). Covering groups of non-connected topological groups revisited. *Math. Proc. Camb. Phill. Soc.* (97-110).
5. Icen, I., Ozcan, A. F. & GURSOY, M. H. (2005). Topological Group-groupoids & Their Coverings. *Indian J. Pure Appl. Math.*, 36(9), 493-502.
6. Mucuk, O. (1998). Coverings & Ring-Groupoids. *Georgian Math. J.*, 5(5), 475-482.
7. Brown, R. & Icen, I. (2003). Homotopies & Automorphisms of Crossed Modules over groupoid. *Appl. Categ. Structure*, 11, 185-206.
8. Brown, R. (2006). Topology & Groupoids. *Booksurge LLC*.
9. Mackenzie, K. C. H. (2005). *General Theory of Lie Groupoids & Lie Algebroids*. Cambridge, New York: Cambridge University Press.
10. Hardy, J. P. L. (1974). Topological groupoids: Coverings and Universal Constructions. PhD Thesis, University College of North Wales.
11. Icen, I. & Ozcan, A. F. (2001). Topological Crossed Modules and G-groupoids. *Algebras Groups & Geom.* 18, 401-410.
12. May, J. P. (1999). A concise course in algebraic topology. *Chicago Lectures in Mathematics series*.
13. Brown, R., Danesh-Naruie, G. & Hardy, J. P. L. (1976). Topological groupoids II: Covering morphisms and G-spaces. *Math. Nachr.*, 74, 143-145.
14. Mucuk, O. & Icen, I. (2001). Coverings of Groupoids. *Hadronic J. Suppl.*, 16(2), 183-196.