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BRAIDING FROM 2-GROUPS TO 2-GROUPOIDS*

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Abstract – We give the concept of 'braiding' for 2-groupoids, and we show that this structure is equivalent to braided regular, crossed modules.

Keywords – Braided crossed modules, cat-groups, internal category, monoidal category

1. INTRODUCTION

Loday and Guin-Walery, [1], defined the notion of the cat^1-group which is equivalent to the crossed module of groups introduced by Whitehead [2]. Brown and Spencer, [3], have defined *G*-groupoids and they then proved that the category of *G*-groupoids is equivalent to the category of crossed modules of groups. 2-groups were introduced by Baez and Lauda in [4]. At the same time, Baez and Crans, [5], have defined the notion of semistrict Lie 2- algebras and then showed that it is 2-equivalent to the 2-category of 2-term L_∞-algebras in the sense of Stasheff [6]. (For a detailed investigation of semistrict Lie 2-algebras see Crans's thesis [7])

Joyal and Street, [8, 9], have defined the notion of braiding for a monoidal category and proved that braided monoidal categories are equivalent to crossed modules with a bracket operation on monoids Categorical groups are monoidal groupoids in which every object is invertible, up to isomorphism, with respect to the tensor product (Breen [10] and Joyal-Street [9]). Breen, [10], also suggested the notion of a crossed module in the context of categorical groups. Categorical groups are sometimes equipped with a braiding or a symmetry (Joyal-Tierney's unpublished work, and [8, 9, 11]). Garzon and Miranda, [11], gave the relation between categorical groups equipped with a braiding and reduced 2-crossed modules given by Conduché [12]. They also gave the homotopy properties of braided categorical groups. One link with homotopy theory and related areas is that the nerve of a braided monoidal category is, up to group completion, a double loop space (Berger, [13]).

The notion of braided regular crossed modules of groupoids was introduced by Brown and Gilbert in [14]. They have shown that the category of braided regular crossed modules is equivalent to that of 2-crossed modules. A braided crossed module of groups is a special case of a braided regular crossed module. Braided crossed modules are equivalent to reduced 2-crossed modules. Also, the categories of braided crossed modules and braided 2-groups are equivalent. We briefly explain this equivalence in section 2 to shed light on the next sections.

Porter, [15], has investigated the category of internal categories in the category of small categories, these are also called double categories. Porter defined C-structures between small categories by using split extensions, gave the crossed modules on small categories, and then showed that the category of internal categories in the category of small categories is equivalent to that of crossed modules within small categories. This is a generalisation of the Brown-Spencer theorem [3]. Indirectly, as mentioned above,

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there are many other generalisations of this type in the literature.

These results shed light on the notion that 'braiding' on 2-groups can be generalised to 2-groupoids. We will define the notion of 'braiding' on 2-groupoids and give the relations between this structure and other well-known algebraic models of 3-types, as 2-crossed modules [12], crossed squares [16], simplicial groups [17, 18], etc. In this work, we concentrate on the structure of braided regular crossed modules given by Brown and Gilbert in [14]. We shall construct the equivalence between the category of braided 2-groupoids and that of braided regular crossed modules by using the equivalence between the category of crossed modules and that of internal categories [3, 15, 16].

2. BRAIDED 2-GROUPS AND CROSSED MODULES

In the following, **Cat(Gp)** will denote the category of 2-groups. An object of **Cat(Gp)** will be represented by a diagram of groups and group morphisms

$$G : C_1 \underset{I}{\overset{s,t}{\rightleftarrows}} C_0$$

such that $sI = tI = idC_0$, and the composition of two morphisms $x, y \in C_1$ with $t(x) = s(y)$ will be denoted by $x \circ y$ and interchange law holds.

The notion of a crossed module was introduced by Whitehead in [2] as models for homotopy (connected) 2-types. A crossed module (M, P, ∂) is a group homomorphism $\partial : M \to P$ together with an action of P on M, written m^p, for $p \in P$ and $m \in M$ satisfying:
$\partial(m^p) = p^{-1}\partial(m)p$ and $m^{\partial m'} = (m')^{-1}mm'$ for all $m, m' \in M$ and $p \in P$. The second condition is called the *Peiffer identity*. The following definition is due to [11].

Definition 2. 1. A 'braiding' for a 2-group

$$G : C_1 \underset{I}{\overset{s,t}{\rightleftarrows}} C_0$$

is a map $\tau : C_0 \times C_0 \to C_1$, $(a,b) \mapsto \tau_{a,b}$, which satisfies

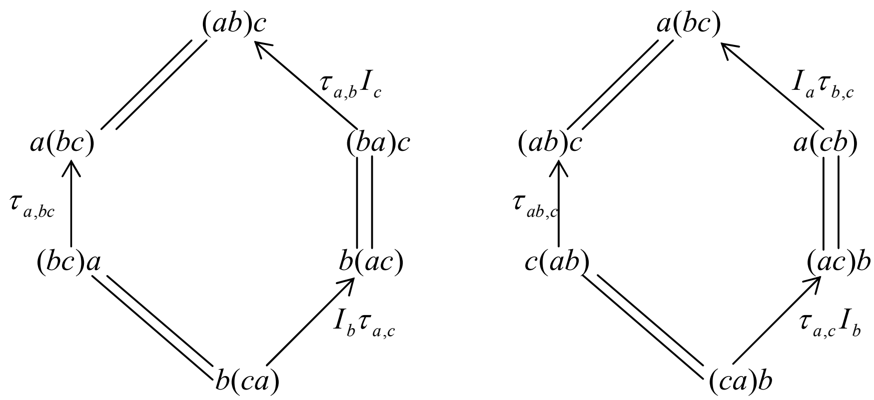
$a)$ $s\tau_{a,b} = ba$ and $t\tau_{a,b} = ab$;

$b)$ *Naturality*:

Given $x, y \in C_1$, $x : a \to a'$, $y : b \to b'$, the following diagram is commutative.

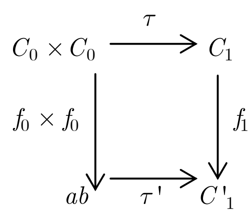
$$\begin{array}{ccc} ba & \xrightarrow{yx} & b'a' \\ {\scriptstyle \tau_{a,b}}\downarrow & & \downarrow{\scriptstyle \tau_{a',b'}} \\ ab & \xrightarrow[xy]{} & a'b' \end{array}$$

C) Hexagon axiom: For any $a, b, c \in C_0$, the following diagrams are commutative.



$d)\ \tau_{0,a} = \tau_{a,0} = I_a$.

A 2-group, together with a braiding map, is called a braided 2-group. Given braided 2-groups (G,τ), (G',τ'), a morphism between them is a morphism of 2-groups which is compatible with τ in the sense that the following square is commutative .



BCat(Gp) will denote the category of braided 2-groups.

Now, we recall the definition of a braided regular crossed module and braided crossed module from [14]. First of all, we shall mention some basic information about groupoids. Recall that a groupoid is a small category in which every arrow is an isomorphism. We write a groupoid as (C_1, C_0), where C_0 is the set of objects and C_1 is the set of arrows. The set of arrows $p \to q$ from p to q is written $C_1(p,q)$ and p,q are the source and target of such an arrow. The source and target maps are written respectively $s,t: C_1 \to C_0$. If $a \in C_1(p,q)$ and $b \in C_1(q,r)$, their composition is written by $a \circ b \in C_1(p,r)$. That is, $s(a \circ b) = s(a)$ and $t(a \circ b) = t(b)$. We write $C_1(p,p)$ as $C_1(p)$. For any arrow a there exists a (necessarily unique) arrow a^{-1} such that $a \circ a^{-1} = e_{s(a)}$ and $a^{-1} \circ a = e_{t(a)}$, where $e: C_0 \to C_1$ gives the identity arrow at an object, that is, $e_p \in C_1(p)$ is the identity arrow from p to p. For any groupoid C, if $C(p,q)$ is empty whenever p,q are distinct (that is, if $s = t$) then C is called totally disconnected. In any groupoid (C_1, C_0), for an element $p \in C_0$, the groupoid $C_1(p)$ becomes a group.

Now, we give the groupoid action and crossed module of groupoids from [14].

Definition 2. 2. Suppose C and D are two groupoids over the same object set O, and C is a totally disconnected groupoid. Then an action of $d \in D$ on $c \in C$, c^d, satisfies the following conditions.

1. c^d is defined if and only if $t(c) = s(d)$, and then $t(c^d) = t(d)$,
2. $(c_1 \circ c_2)^{d_1} = c_1^{d_1} \circ c_2^{d_1}$
3. $c_1^{d_1 \circ d_2} = (c_1^{d_1})^{d_2}$

for all $c_1, c_2 \in C(x,x)$ and $d_1 \in D(x,y), d_2 \in D(y,z)$.

Definition 2. 3. A crossed module of groupoids consists of a pair of groupoids, C_2 and C_1, over a common object set C_0, with C_2 totally disconnected, together with an action of C_1 on C_2 and a morphism

$$C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

which is the identity on C_0, satisfying the conditions

$$\textbf{CM1)}\ \delta(m^n) = n^{-1} \circ \delta(m) \circ n$$
$$\textbf{CM2)}\ m^{\delta(m')} = (m')^{-1} \circ m \circ m'$$

for all $m, m' \in C_2(p)$ and $n \in C_1(p,q)$.
Note in particular that for each $p \in C_0$, $C_2(p) \to C_1(p)$ is a crossed module of groups.
Let U be a monoid. A biaction of U on the crossed module

$$C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

consists of a pair of commuting left and right actions of U on the set C_0 and on the groupoids C_1 and C_2 compatible with the whole structure. Specifically we have functions $U \times C_i \to C_i$ and $C_i \times U \to C_i$ for $i = 0,1,2$, denoted by $(u,c) \to u \cdot c$ and $(c,u) \to c \cdot u$, such that
BA1: each function $U \times C_i \to C_i$ determines a left action of U and each function $C_i \times U \to C_i$ determines a right action of U and these actions commute;
BA2: each action of U preserves the groupoid structure of C_1 over C_0, and in particular the source and target maps $s,t : C_1 \to C_0$ are U-equivariant relative to each action;
BA3: each action of U preserves the group operation in C_2 and if $x \in C_2(p)$ and $u \in U$ then $u \cdot x \in C_2(u \cdot p)$ and $x \cdot u \in C_2(p \cdot u)$;
BA4: each action of U is compatible with the action of C_1 on C_2 so that if $x \in C_2(p)$, $a \in C_1(p,q)$, and $u \in U$ then

$$u \cdot (x^a) = (u \cdot x)^{u \cdot a} \in C_2(u \cdot q),$$
$$(x^a) \cdot u = (x \cdot u)^{a \cdot u} \in C_2(q \cdot u);$$

BA5: the boundary homomorphism $\delta : C_2 \to C_1$ is U-equivariant relative to each action.
The crossed module

$$C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

is *semiregular* if the object set C_0 is a monoid and there is a biaction of C_0 on C in which C_0 acts on itself in its left and right regular representations. A semiregular crossed module in which C_0 is a group is said to be *regular*. Note that every crossed module of groups is regular.
Now, we give the definition of a braided regular crossed module of groupoids from [14].

Definition 2. 4. ([14]) A braided regular crossed module

$$C_2 \xrightarrow{\delta} C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$$

is a regular crossed module with the map $\{-,-\}: C_1 \times C_1 \to C_2$, called a braiding map, satisfying the following axioms:

B1: $\{a,b\} \in C_2((ta)(tb))$ $\{1_e, b\} = 1_{tb}$ and $\{a, 1_e\} = 1_{ta}$, where $1_e \in C_1(e)$ is the identity arrow and e is the identity element of the group C_0;

B2: $\{a, b \circ b'\} = \{a,b\}^{ta \cdot b'} \circ \{a, b'\}$;

B3: $\{a \circ a', b\} = \{a', b\} \circ \{a,b\}^{a' \cdot tb}$;

B4: $\delta\{a,b\} = (ta \cdot b)^{-1} \circ (a^{-1} \cdot sb) \circ (sa \cdot b) \circ (a \cdot tb)$;

B5: $\{a, \delta y\} = (ta \cdot y)^{-1} \circ (sa \cdot y)^{a \cdot q}$ if $y \in C_2(q)$;

B6: $\{\delta x, b\} = ((x \cdot sb)^{p \cdot b})^{-1} \circ (x \cdot tb)$ if $x \in C_2(p)$;

B7: $p \cdot \{a,b\} = \{p \cdot a, b\}$, $\{a,b\} \cdot p = \{a, b \cdot p\}$, $\{a \cdot p, b\} = \{a, p \cdot b\}$

for $a, a', b, b' \in C_1, x, y \in C_2$ and $p, q \in C_0$.

A morphism between braided regular crossed modules is a morphism of crossed modules of groupoids which is compatible with the braiding map $\{-,-\}$. Let **BRCM** denote the category of braided regular crossed modules.

Example 2. 5. ([14]) A braiding on a crossed module of groups $\delta: C_2 \to C_1$ is a function

$$\{-,-\}: C_1 \times C_1 \to C_2$$

satisfying the following axioms.

BC1. $\{a, bb'\} = \{a,b\}^{b'}\{a,b'\}$,

BC2. $\{aa', b\} = \{a', b\}\{a,b\}^{a'}$,

BC3. $\delta\{a,b\} = [b,a]$,

BC4. $\{a, \delta x\} = x^{-1}x^{a}$,

BC5. $\{\delta y, a\} = (y^{-1})^{b} y$

where $a, a', b, b' \in C_1$ and $x, y \in C_2$.

This example leads us to define a new category. A crossed module together with a braiding map is called a braided crossed module. A morphism between braided crossed modules is a morphism of crossed modules which is compatible with the braiding map $\{-,-\}$. We show the category of braided crossed modules by **BCM**.

The following proposition, in some sense, can be considered well-known. However, we give its proof as it gives explicit formulae for the structures.

Proposition 2. 6. The category of braided crossed modules is equivalent to that of braided 2-groups.

Proof: We use the equivalence between crossed modules and internal categories in the category of groups. Let $\partial: L \to M$ be a braided crossed module of groups. We know from [3, 11, 15, 16] that, the diagram

$$G: M \tilde{\times} L \overset{s,t}{\rightrightarrows} C_0 \quad (I)$$

with $s(m,l)=m, t(m,l)=m\partial l$ and $I(m)=(m,0)$ is a 2-group, together with the composition

$$(m,l)\circ(m',l')=(m,ll')$$

$$\text{if } s(m',l')=m'=t(m,l)=m\partial l.$$

The group operation is given by

$$(m,l)(m',l')=(mm',l^{m'}l').$$

A braiding map on this 2-group can be defined by

$$\begin{aligned}\tau: M\times M &\to M\tilde{\times} L\\ (a,b) &\mapsto (ba,\{b,a\})\end{aligned}$$

where $\{-,-\}$ is the braiding map for crossed module ∂. Then, (G,τ) becomes a braided 2-group. For example, the equalities

$$\begin{aligned}s\tau_{a,b} &= s(ba,\{b,a\})\\ &= ba\delta\{b,a\}\\ &= baa^{-1}b^{-1}ab\\ &= ab\end{aligned}$$

and

$$\begin{aligned}t\tau_{a,b} &= t(ba,\{b,a\})\\ &= ba\end{aligned}$$

are Axiom a) of braided 2-groups. Furthermore, for $x=(a,l)$ and $y=(b,l')$ in $M\tilde{\times} L$, $s(x)=a, t(x)=a\partial l=a'$ and $s(y)=b, t(y)=b\partial l'=b'$, we show

$$\tau_{a,b}\circ xy = yx\circ\tau_{a',b'}.$$

The group operations in $M\tilde{\times} L$ are given by

$$xy=(ab,l^b l') \text{ and } yx=(ba,(l')^a l)$$

for the elements $x=(a,l)$ and $y=(b,l')\in M\tilde{\times} L$.
On the one hand we obtain

$$\begin{aligned}\tau_{a,b}\circ xy &= (ba,\{b,a\})\circ(ab,l^b l')\\ &= (ba,\{b,a\}l^b l'),\end{aligned}$$

while on the other hand we obtain

$$\begin{aligned}yx\circ\tau_{a',b'} &= (ba,(l')^a l)\circ(b\partial l' a\partial l,\{b\partial l',a\partial l\})\\ &= (ba,(l')^a l\{b\partial l',a\partial l\}).\end{aligned}$$

Therefore it suffices to show that

$$\{b,a\}l^b l' = (l')^a l\{b\partial l',a\partial l\}.$$

Now, we show this equality.

$$
\begin{aligned}
(l')^{a}l\{b\partial l',a\partial l\} &= (l')^{a}l\{\partial l',a\partial l\}\{b,a\partial l\}^{\partial l'} \quad \because BC2\\
&= (l')^{a}l\{\partial l',a\}^{\partial l}\{\partial l',\partial l\}(l')^{-1}\{b,a\partial l\}l' \quad \because BC3 \text{ and } \partial \text{ cross. mod.}\\
&= (l')^{a}l(l)^{-1}\{\partial l',a\}l[l,l'](l')^{-1}\{b,a\partial l\}l' \quad \because \partial \text{ cross. mod.}\\
&= (l')^{a}((l')^{-1})^{a}l'll^{-1}(l')^{-1}ll'(l')^{-1}\{b,a\partial l\}l' \quad \because BC5\\
&= l\{b,a\partial l\}l'\\
&= l\{b,a\}^{\partial l}\{b,\partial l\}l' \quad \because BC1\\
&= ll^{-1}\{b,a\}ll^{-1}l^{b}l' \quad \because BC4 \text{ and } \partial \text{ cross. mod.}\\
&= \{b,a\}l^{b}l'.
\end{aligned}
$$

Thus, we obtain

$$\tau_{a,b} \circ xy = yx \circ \tau_{a',b'}.$$

This is Axiom b) of braided 2-groups. We leave the other axioms as an exercise for the reader. Thus, we can define a functor

$$\Theta : \mathbf{BCM} \to \mathbf{BCat(Gp)}.$$

Conversely, let

$$\left(G : C_1 \underset{I}{\overset{s,t}{\rightrightarrows}} C_0 , \tau \right)$$

be a braided 2-group. We know from [3, 11, 15, 16, 19-24] that the homomorphism $t : \ker s \to C_0$ is a crossed module associated to the 2-group G together with the action given by $l^{x} = (Ix)^{-1}l(Ix)$. (For further work see [25]). The braiding map on this crossed module can be defined by

$$
\begin{aligned}
\{-,-\} : C_0 \times C_0 &\to \ker\, s\\
(a,b) &\mapsto (Ib)^{-1}(Ia)^{-1}\tau_{a,b}.
\end{aligned}
$$

For example, we obtain

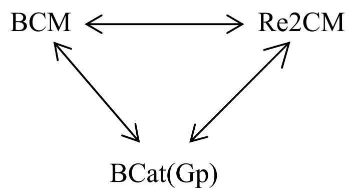
$$
\begin{aligned}
\{aa',b\} &= I(b)^{-1}I(a')^{-1}I(a)^{-1}\tau_{aa',b}\\
&= I(b)^{-1}I(a')^{-1}I(a)^{-1}(\tau_{a,b}I(a') \circ I(a)\tau_{a',b}) \because \text{hexagon axiom}\\
&= (I(b)^{-1}I(a')^{-1}\tau_{a',b}) \circ (I(a')^{-1}I(b)^{-1}I(a)^{-1}\tau_{a,b}I(a')) \because \text{interchange law}\\
&= \{a',b\} \circ (I(b)^{-1}I(a)^{-1}\tau_{a,b})^{a'}\\
&= \{a',b\}\{a,b\}^{a'}.
\end{aligned}
$$

for $a,a',b \in C_0$ and this is Axiom $BC2$. Then, this crossed module becomes a braided crossed module. This enables us to define a functor

$$\Delta : \mathbf{BCat(Gp)} \to \mathbf{BCM}.$$

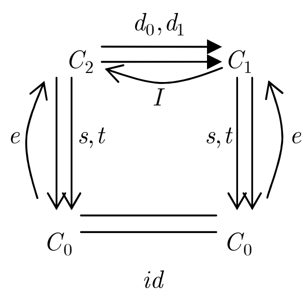
We easily obtain a natural equivalence $\Gamma : 1_{BCM} \to \Delta\Theta$, where $S = (M,L,\partial)$ is a braided crossed module, then Γ_S is identity on L and on M is given by $a \mapsto (a,e)$. To define the natural equivalence $\Phi : \Theta\Delta \to 1_{BCM}$, let G be a braided 2-group. A map $\Phi_G : \Phi\Delta(G) \to G$ is defined to be the identity on objects, and on morphisms is given by $(a,b) \mapsto 1_b a$. Clearly Φ_G is bijective on morphism so it only remains to check that Φ_G preserves composition, group operation and braiding. This is routine and so is omitted.

The category of braided 2-groups is equivalent to **Re2CM**, the category of reduced 2-crossed modules ([11]). Also the category of braided crossed modules is equivalent to the category reduced 2-crossed modules. Thus we can briefly give the following diagram of equivalences of categories.



3. BRAIDED 2-GROUPOIDS

Brown and Spencer [3] have defined the notion of G-groupoid and they have proved that the category of crossed modules is equivalent to that of G-groupoids. Of course, the category of crossed modules of groupoids is equivalent to the category of internal groupoids in the category of groupoids with the same object set. In this section, we will use this equivalence. We give the notion of 'braiding' for internal groupoids in the category of groupoids having the same objects set, and we define an equivalence between the category of internal groupoids within groupoids, together with a braiding map and that of braided regular crossed modules by using usual equivalence between crossed modules and internal groupoids ([3, 11, 15, 16, 19-25]).
Let C_2 and C_1 be the groupoids on a same object set C_0. A 2-groupoid in the category of groupoids having the same object set C_0;



consists of the following statements.
1. $(C_2, C_1, d_0, d_1, I, *)$ is a groupoid structure and, d_0 and d_1 are identities on C_0,
2. $(C_2, C_0, s, t, e, \circ)$ and $(C_1, C_0, s, t, e, \circ)$ are groupoid structures,
3. the morphisms d_0 and d_1 preserve the groupoid structure of C_2 on the objects set C_1 and C_0,
4. the interchange law holds for the operations $*$ and $\circ$;

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

whenever either side is defined.
We will show such a 2-groupoid by $(C_2, C_1, C_0, d_0, d_1, I, \circ, *)$.
Lets give the notion of biaction for 2-groupoids. Let U be a monoid. A biaction of U on the 2-groupoid $(C_2, C_1, C_0, d_0, d_1, I, \circ, *)$ consists of a pair of commuting left and right actions of U on the set C_0 and on the groupoids C_1 and C_2, compatible with the whole structure. Specifically, we have functions $U \times C_i \to C_i$ and $C_i \times U \to C_i$ for $i = 0,1,2,$ denoted by $(u,c) \to u \cdot c$ and $(c,u) \to c \cdot u$, such that
1. each function $U \times C_i \to C_i$ determines a left action of U and each function $C_i \times U \to C_i$ determines a right action of U and these actions commute;

2. each action of U preserves the groupoid structure of C_1 over C_0 and in particular the source and target maps $s,t : C_1 \to C_0$ are U-equivariant relative to each action;
3. each action of U preserves the groupoid structure of C_2 over C_0 and the source and target maps $s,t : C_2 \to C_0$ are U-equivariant relative to each action;
4. the morphisms d_0 and d_1 preserve the left and right action of U on C_2.
The 2-groupoid

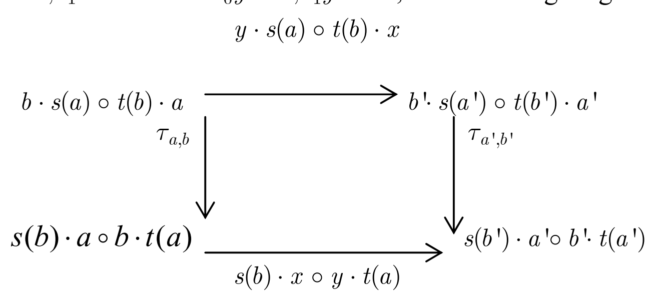
$$C := (C_2, C_1, C_0, d_0, d_1, I, \circ, *)$$

is *semiregular* if the object set C_0 is a monoid and there is a biaction of C_0 on C in which C_0 acts on itself in its left and right regular representations, a semiregular 2- groupoid in which C_0 is a group and is said to be regular. Note that for each $p \in C_0$ the 2-groupoid $C := (C_2(p), C_1(p), \{p\}, d_0, d_1, I, *)$ becomes a 2-group and this 2-group is always regular.

Definition 3. 1. Let $(C_2, C_1, C_0, d_0, d_1, I, \circ, *)$ be a regular 2-groupoid together with the biaction of the group C_0. A braiding map for this 2-groupoid is a map

$$\begin{aligned} \tau : C_1 \times C_1 &\to C_2 \\ (a,b) &\mapsto \tau_{a,b} \end{aligned}$$

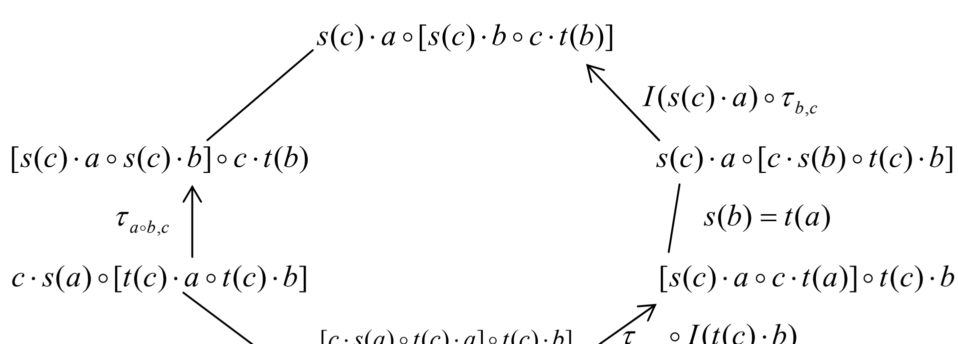
satisfying the following conditions.
1. For $a \in C_1(p,q)$ and $b \in C_1(l,m)$, $d_0\tau_{a,b} = b \cdot sa \circ tb \cdot a$ or equivalently
$d_0\tau_{a,b} = b \cdot p \circ m \cdot a,$
2. For $a \in C_1(p,q)$ and $b \in C_1(l,m)$, $d_1\tau_{a,b} = sb \cdot a \circ b \cdot ta$ or equivalently
$d_1\tau_{a,b} = l \cdot a \circ b \cdot q,$
3. For $x, y \in C_2$ with $d_0 x = a, d_1 x = a'$ and $d_0 y = b, d_1 y = b'$, the following diagram is commutative



or equivalently

$$(y \cdot s(a) \circ t(b) \cdot x) * \tau_{a',b'} = \tau_{a,b} * (s(b) \cdot x \circ y \cdot t(a)),$$

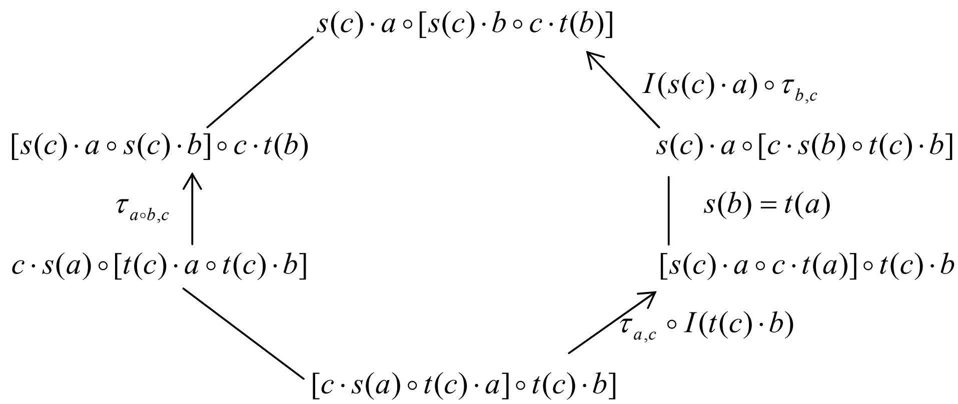
4. a) For $a \in C_1(d,f)$, $b \in C_1(f,g)$ and $c \in C_1(h,j)$ the following diagram is commutative;



or equivalently

$$\tau_{a\circ b,c} = (\tau_{a,c} \circ I(t(c)\cdot b)) * (I(s(c)\cdot a) \circ \tau_{b,c}),$$

b) $a \in C_1(k,l),\ b \in C_1(m,n)$ and $c \in C_1(n,p)$, the following diagram is commutative;



or equivalently

$$\tau_{a,b\circ c} = (I(b\cdot s(a)) \circ \tau_{a,c}) * (\tau_{a,b} \circ I(c\cdot t(a))),$$

5. For $a \in C_1(x,y)$ and $0_e \in C_1(e)$

$$\tau_{a,0_e} = I(a) = \tau_{0_e,a}.$$

A regular 2-groupoid together with a braiding map is called a braided regular 2-groupoid (also called a braided 2-groupoid shortly). **B2Grpoid** will denote the category of braided 2-groupoids.

Example 3. 2. Let $(C_2, C_1, C_0, d_0, d_1, I, \circ, *)$ be a braided regular 2-groupoid. If the objects set C_0 has exactly one object p, then $C_2(p)$ and $C_1(p)$ become totally disconnected groupoids. Then

$$G : C_2(p) \rightrightarrows C_1(p)$$

becomes a braided 2-group. The biaction of C_0 on C_1 and C_2, becomes trivial action. In fact, this is a special case of a braided 2-groupoid.

a) C-Structures and Semi-Direct Product Groupoids

In this section, we recall some notions about the semi-direct product of groupoids. The fundamental idea can be found in [15]. Porter has defined the notion of C-structure between small categories in [15], and by using this notion, gave the semi-direct product of small categories. By using these structures, he defined crossed modules in small categories. In fact, if we take the category of groupoids, instead of small categories, the C-structure becomes a groupoid action. In this section, we will briefly mention these ideas. We will use these statements in the next section.

Let $\mathbf{Cat}_O$ be the category of all small categories having the set O as their objects and all functors which are the identities on objects.
A sequence in $\mathbf{Cat}_O$

$$E : H \xrightarrow{i} K \xrightarrow{\pi} C$$

is an extension if i identifies H to a subcategory of K, and C is a quotient category of K, the projection

functor being π, such that for all k_1, k_2 in K,
$\pi(k_1) = \pi(k_2) \Leftrightarrow$ there is a unique $h \in H$ such that $k_2 = k_1 \circ i(h)$.
Thus:
a) $\pi i(H) = O$, that is, $\pi i(h)$ is an identity for all morphisms h in H,
b) If h, k are such that $i(h) \circ k$ is defined, then there is a unique $h_1 \in H$ such that

$$i(h) \circ k = k \circ i(h_1).$$

Proposition 3. 3. ([15]) If

$$E : H \xrightarrow{i} K \xrightarrow{\pi} C$$

is an extension of O-categories, then H is a disjoint union of an O-indexed family of groups.

The extension E is split if there is a functor $s : C \to K$ with $\pi s = id_C$. In this case, it is said that H has a C-structure. A C-structure on H is an action of C on H. In the category of groupoids, this C-structure, in fact, is a groupoid action.

Proposition 3. 4. Let C be a groupoid and let H be a totally disconnected groupoid.

$$E : H \xrightarrow{i} K \underset{s}{\overset{\pi}{\rightleftarrows}} C$$

is a C-structure on H, then for $h \in H(y), c \in C(x, y)$, there is a unique $h^c \in H(x)$ satisfying
i) $i(h^c) \circ s(c) = s(c) \circ i(h)$,
ii) $h \mapsto h^c$ is a morphism of groups.
We give the semi-direct product groupoid in the category of groupoids. Now, consider the category **Grpoid** of groupoids having the same objects set O. Let C and H be the objects of **Grpoid** and H be a totally disconnected groupoid. Suppose that H has C-structure, that is, there is a groupoid action of the groupoid C on the groupoid H. Then, we can define the semi-direct product as follows:
Let $h \in H(x)$ and $c \in C(x, y)$ and for $x, y \in O$

$$C \tilde{\times} H(x, y) = C(x, y) \times H(y)$$

becomes a groupoid together with the composition given by

$$(c, h) \circ (c', h') = (c \circ c', h^{c'} \circ h').$$

4. BRAIDED 2-GROUPOIDS AND REGULAR CROSSED MODULES

In this section, we will construct an equivalence between two categories: the category of braided 2-groupoids and the category of braided regular crossed modules of groupoids.

Theorem 4. 1. The category of braided 2-groupoids is equivalent to that of braided regular crossed modules.

Proof: Let

$$A \xrightarrow{\delta} C \underset{t}{\overset{s}{\rightrightarrows}} O$$

be the braided regular crossed module. We know that C has a groupoid action on A, which is a totally disconnected groupoid, and O has a biaction on the groupoids C and A. Therefore, we can define the semi-direct product groupoid

$$C \tilde{\ltimes} H(x,y) = C(x,y) \times H(y)$$

with the composition

$$(c,a) \circ (c',a') = (c \circ c', a^{c'} \circ a'),$$

for $c \in C(x,y), c' \in C(y,z)$ and $a \in A(y), a' \in A(z)$.

Now, consider the following diagram:

$$C \tilde{\ltimes} A \underset{I}{\overset{d_0, d_1}{\rightrightarrows}} C \quad (I : C \to C \tilde{\ltimes} A)$$

with the morphisms $d_0(c,a) = c, d_1(c,a) = c \circ \delta a$ and $I(c) = (c, e_y)$, for $c \in C(x,y)$ and $a \in A(y)$. Defining $*$ by

$$(c,a) * (c \circ \delta a, a') = (c, a \circ a')$$

gives an internal groupoid within groupoids. The inverse for $*$ of (c,a) is $(c \circ \delta a, a^{-1})$ where a^{-1} is the inverse in $A(y)$. Moreover, for $k = (c,a)$ and $l = (c \circ \delta a, a')$, $d_0(k * l) = d_0 k$ and $d_1(k * l) = d_1 l$. That is, $(C \tilde{\ltimes} A, C, d_0, d_1, I, *)$ is a groupoid with respect to the groupoid operation '$*$' defined as above. Now, we check the interchange law. Let $(c,a) \in C \tilde{\ltimes} A(x,y)$. Then $c \circ \delta a$ is also from x to y since δ is the identity on objects, so suppose $(c,a), (c \circ \delta a, a') : x \to y$ in $C \tilde{\ltimes} A$ and $(d,b), (d \circ \delta b, b') : y \to z$,

$$\begin{aligned}
[(c,a) \circ (d,b)] * [(c \circ \delta a, a') \circ (d \circ \delta b, b')] &= (c \circ d, a^d \circ b) * (c \circ \delta a \circ d \circ \delta b, (a')^{d \circ \delta b} \circ b') \\
&= (c \circ d, a^d \circ b \circ (a')^{d \circ \delta b} \circ b') \text{ since groupoid action} \\
&= (c \circ d, a^d \circ (a')^d \circ b \circ b') \text{ since } \delta \text{ cross. mod.} \\
&= (c \circ d, (a \circ a')^d \circ b \circ b') \\
&= (c, a \circ a') \circ (d, b \circ b') \\
&= [(c,a) * (c \circ \delta a, a')] \circ [(d,b) * (d \circ \delta b, b')]
\end{aligned}$$

That is, the interchange law holds. Furthermore, since

$$\begin{aligned}
d_0((c,a) \circ (d,b)) &= d_0(c \circ d, a^d \circ b) \\
&= c \circ d \\
&= d_0(c,a) \circ d_0(d,b)
\end{aligned}$$

and

$$\begin{aligned}
d_1((c,a) \circ (d,b)) &= d_1(c \circ d, a^d \circ b) \\
&= c \circ d \circ \delta(a^d \circ b) \\
&= c \circ d \circ d^{-1} \circ \delta a \circ d \circ \delta b \\
&= c \circ \delta a \circ d \circ \delta b \\
&= d_1(c,a) \circ d_1(d,b),
\end{aligned}$$

d_0 and d_1 preserve the operation $\circ$. Thus, we obtain a 2-groupoid

$$C\,\tilde{\times}\,A \underset{I}{\overset{d_0, d_1}{\rightrightarrows}} C$$

where, O has a biaction on the groupoids C and A. By using these actions, we can define the biaction of O on the groupoids $C\,\tilde{\times}\,A$ and C by

$$p\cdot(c,a)=(p\cdot c,a)$$

and

$$(c,a)\cdot p=(c\cdot p, I(ep^{-1})\circ a\circ I(ep)).$$

The morphisms d_0 and d_1 preserve these actions. For example

$$\begin{aligned} d_0(p\cdot(c,a)) &= d_0(p\cdot c,a)\\ &= p\cdot c\\ &= p\cdot d_0(c,a), \end{aligned}$$

And

$$\begin{aligned} d_1((c,a)\cdot p) &= d_1(c\cdot p, I(ep^{-1})\circ a\circ I(ep))\\ &= c\cdot p\circ e(p^{-1})\circ \delta a\circ e(p)\\ &= \ c\circ \delta a\circ e(p)\\ &= d_1(c,a)\circ e(p)\\ &= d_1(c,a)\cdot p. \end{aligned}$$

Thus we obtain a regular 2-groupoid $(C\,\tilde{\times}\,A, C, O, d_0, d_1, I, \circ, *)$ together with the biaction of O. We can define the braiding map on this regular 2-groupoid by

$$\begin{aligned} \tau: C\times C &\to C\,\tilde{\times}\,A\\ (a,b) &\mapsto \tau_{a,b}=(b\cdot s(a)\circ t(b)\cdot a, \{b,a\}), \end{aligned}$$

where $\{-,-\}$ is the braiding map on the regular crossed module δ. We will show that all the axioms of braided 2-groupoids are verified.

1.

$$\begin{aligned} d_0\tau_{a,b} &= d_0(b\cdot s(a)\circ t(b)\cdot a, \{b,a\})\\ &= b\cdot s(a)\circ t(b)\cdot a. \end{aligned}$$

2.

$$\begin{aligned} d_1\tau_{a,b} &= d_1(b\cdot s(a)\circ t(b)\cdot a, \{b,a\})\\ &= b\cdot s(a)\circ t(b)\cdot a\circ \delta\{b,a\}\\ &= b\cdot s(a)\circ t(b)\cdot a\circ (t(b)\cdot a)^{-1}\circ (b\cdot s(a))^{-1}\circ s(b)\cdot a\circ b\cdot t(a)\\ &= s(b)\cdot a\circ b\cdot t(a). \end{aligned}$$

3. From the axioms 1. and 2., we can write on the groupoid $(C\,\tilde{\times}\,A, C, d_0, d_1, I, \circ, *)$,

$$\tau_{a,b}\ : b\cdot s(a)\circ t(b)\cdot a \rightarrow s(b)\cdot a\circ b\cdot t(a).$$

Suppose that $x=(a,d):a\rightarrow a'=a\circ\delta d$ and $y=(b,d'):b\rightarrow b'=b\circ\delta d'$ are morphisms in the groupoid $(C\tilde{\times}A,C,d_0,d_1,I,\circ,*)$. Then, since δ is an identity on the objects, if $a\in C(k,l)$, we have $a'\in C(k,l)$. Similarly to b and b'. We thus obtain

$$y\cdot s(d_0x)\circ t(d_0y)\cdot x : b\cdot s(a)\circ t(b)\cdot a\rightarrow b'\cdot s(a')\circ t(b')\cdot a'$$

and

$$s(d_0y)\cdot x\circ y\cdot t(d_0x) : s(b)\cdot a\circ b\cdot t(a)\rightarrow s(b')\cdot a'\circ b'\cdot t(a').$$

Therefore, the following diagram is commutative.

$$\begin{array}{ccc} b\cdot s(a)\circ t(b)\cdot a & \xrightarrow{\ y\cdot s(a)\circ t(b)\cdot x\ } & b'\cdot s(a')\circ t(b')\cdot a' \\ \tau_{a,b}\big\downarrow & & \big\downarrow\tau_{a',b'} \\ s(b)\cdot a\circ b\cdot t(a) & \xrightarrow[\ s(b)\cdot x\circ y\cdot t(a)\ ]{} & s(b')\cdot a'\circ b'\cdot t(a'). \end{array}$$

4. For $a\in C(k,l), b\in C(l,m)$ and $c\in C(r,p)$, then we know

$$\tau_{a\circ b,c}\ : c\cdot s(a)\circ[t(c)\cdot a\circ t(c)\cdot b]\rightarrow[s(c)\cdot a\circ s(c)\cdot b]\circ c\cdot t(b).$$

Thus

$$\tau_{b,c}\ : c\cdot s(b)\circ t(c)\cdot b\rightarrow s(c)\cdot b\circ c\cdot t(b)$$

and $d_0I(s(c)\cdot a)=d_1I(s(c)\cdot a)=s(c)\cdot a$, then we obtain

$$I(s(c)\cdot a)\circ\tau_{b,c}\ : s(c)\cdot a\circ[c\cdot s(b)\circ t(c)\cdot b]\rightarrow s(c)\cdot a\circ[s(c)\cdot b\circ c\cdot t(b)].$$

We know that $s(c)\cdot a\circ[c\cdot s(b)\circ t(c)\cdot b]=[s(c)\cdot a\circ c\cdot s(b)]\circ t(c)\cdot b$ and from $s(b)=t(a)$, we obtain

$$[s(c)\cdot a\circ c\cdot s(b)]\circ t(c)\cdot b=[s(c)\cdot a\circ c\cdot t(a)]\circ t(c)\cdot b.$$

Moreover,

$$\tau_{a,c}\ : c\cdot s(a)\circ t(c)\cdot a\rightarrow s(c)\cdot a\circ c\cdot t(a)$$

and $d_0I(t(c)\cdot b)=d_1I(t(c)\cdot b)=t(c)\cdot b$, we obtain

$$\tau_{a,c}\circ I(t(c)\cdot b) : [c\cdot s(a)\circ t(c)\cdot a]\circ t(c)\cdot b\rightarrow[s(c)\cdot a\circ c\cdot t(a)]\circ t(c)\cdot b.$$

Therefore,

$$\tau_{a,c}\circ I(t(c)\cdot b)*(Is(c)\cdot a\circ\tau_{b,c}) : [c\cdot s(a)\circ t(c)\cdot a]\circ t(c)\cdot b\rightarrow s(c)\cdot a\circ[s(c)\cdot b\circ c\cdot t(b)]$$

and we obtain

$$\tau_{a\circ b,c}\ : \tau_{a,c}\circ I(tc\cdot b)*I(sc\cdot a)\circ\tau_{b,c}.$$

Consequently, the diagram a) in the hexagon axioms is commutative.
Similarly, we calculate

$$\tau_{a,b\circ c} : [b \cdot s(a) \circ c \cdot s(a)] \circ t(c) \cdot a \to s(b) \cdot a \circ [b \cdot t(a) \circ c \cdot t(a)]$$

for $a \in C(k,l)$ and $b \in C(m,n), c \in C(n,p)$.
Thus,

$$\tau_{a,b} : b \cdot s(a) \circ t(b) \cdot a \to s(b) \cdot a \circ b \cdot t(a)$$

and $d_0 I(c \cdot ta) = d_1 I(c \cdot ta) = c \cdot ta.$ Then,

$$\tau_{a,b} \circ I(c \cdot t(a)) : [b \cdot s(a) \circ t(b) \cdot a] \circ c \cdot t(a) \to [s(b) \cdot a \circ b \cdot t(a)] \circ c \cdot t(a).$$

Moreover, $[b \cdot s(a) \circ t(b) \cdot a] \circ c \cdot t(a) = b \cdot s(a) \circ [t(b) \cdot a \circ c \cdot t(a)]$ and from $t(b) = s(c)$, we obtain

$$[b \cdot s(a) \circ t(b) \cdot a] \circ c \cdot t(a) = b \cdot s(a) \circ [s(c) \cdot a \circ c \cdot t(a)].$$

Further,

$$\tau_{a,c} : c \cdot s(a) \circ t(c) \cdot a \to s(c) \cdot a \circ c \cdot t(a)$$

and $d_0 I(b \cdot sa) = d_1 I(b \cdot sa) = b \cdot sa$ we obtain

$$I(b \cdot sa) \circ \tau_{a,c} : b \cdot s(a) \circ [c \cdot s(a) \circ t(c) \cdot a] \to b \cdot s(a) \circ [s(c) \cdot a \circ c \cdot t(a)].$$

Therefore

$$I(b \cdot sa) \circ \tau_{a,c} * \tau_{a,b} \circ I(c \cdot t(a)) : b \cdot s(a) \circ [c \cdot s(a) \circ t(c) \cdot a] \to [s(b) \cdot a \circ b \cdot t(a)] \circ c \cdot t(a)$$

and then

$$\tau_{a,b\circ c} = I(b \cdot sa) \circ \tau_{a,c} * \tau_{a,b} \circ I(c \cdot t(a)).$$

Consequently, the diagram b) in the hexagon axioms is commutative.

5. For $a \in C(x,y)$ and $0_e \in C(e)$, then

$$\begin{aligned}\tau_{a,0_e} &= (0_e \cdot s(a) \circ e \cdot a, \{0_e, a\}) \\ &= (0_{s(a)} \circ a, 0_{s(a)}) \\ &= (a, 0) \\ &= I(a).\end{aligned}$$

Therefore, all the axioms of braided 2-groupoids are verified, and this enables us to define a functor from the category of braided regular crossed modules to the category of braided 2-groupoids

$$\Theta : \mathbf{BRCM} \to \mathbf{B2Grpoid}.$$

Conversely, suppose that $(D, C, O, d_0, d_1, I, \circ, *)$ is a braided 2-groupoid. We set

$$K(x) = \{d \in D : d_0(d) = e(x)\}$$

for each $x \in O$. Then, the set $K(x)$ is a group for both induced compositions $\circ$ and $*$, and for each $k \in K(x)$ the inverses for $\circ$ and $*$ are related by $k_*^{-1} = I(d_1 k) \circ k_\circ^{-1}$, and given $d \in D$ it can then be uniquely written as $d = Id_0(d) \circ k$ with $k \in K(s(d))$. Next, we can write the action of $c \in C$ on $k \in K$ by

$$k^c = I(c)^{-1} \circ k \circ I(c).$$

Therefore,

$$K \longrightarrow D \xrightarrow{d_0} C$$

is split by I and $\partial : K \to C$ is given by $d_1|K$. The verification that ∂ is a crossed module is long, but is a straightforward use of interchange law in the 2-groupoid, i.e.,

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

whenever one side is defined. This construction can be found in [15]. Then, we have a crossed module of groupoids $K \xrightarrow{\partial} C$ in which K is a totally disconnected groupoid. By using the biaction of O on the 2-groupoid $(D, C, d_0, d_1, I, \circ, *)$, we can similarly define the biaction of O on the groupoids K and C. Thus, we obtain a regular crossed module. Now, define the braiding map on this crossed module as follows:

$$\begin{aligned} \{-,-\} : C \times C &\to K \\ (b,a) &\mapsto (I(tb \cdot a))^{-1} \circ (I(b \cdot sa))^{-1} \circ \tau_{a,b}. \end{aligned}$$

We must show that all the axioms of braided regular crossed modules are verified. Since the morphism d_1 preserves the operation '$\circ$', we have

$$\begin{aligned} \partial\{b,a\} &= d_1((I(tb \cdot a))^{-1} \circ (I(b \cdot sa))^{-1} \circ \tau_{a,b}) \\ &= (tb \cdot a)^{-1} \circ (b \cdot s(a))^{-1} \circ d_1\tau_{a,b} \\ &= (tb \cdot a)^{-1} \circ (b \cdot s(a))^{-1} \circ s(b) \cdot a \circ b \cdot t(a). \end{aligned}$$

This is Axiom **B4** of braided regular crossed module. Similarly,

$$\begin{aligned} \{b, a \circ c\} &= I(tb \cdot (a \circ c))^{-1} \circ I(b \cdot s(a \circ c))^{-1} \circ \tau_{a \circ c, b} \\ &= I(tb \cdot c)^{-1} \circ I(tb \cdot a)^{-1} \circ I(b \cdot s(a))^{-1} \circ (\tau_{a,b} \circ I(tb \cdot c) * I(sb \cdot a) \circ \tau_{c,b}) \\ &= I(tb \cdot c)^{-1} \circ I(tb \cdot a)^{-1} \circ I(b \cdot s(a))^{-1} \circ (\tau_{a,b} * I(sb \cdot a)) \circ (I(tb \cdot c) * \tau_{c,b}) \\ &= I(tb \cdot c)^{-1} \circ I(tb \cdot a)^{-1} \circ I(b \cdot s(a))^{-1} \circ \tau_{a,b} \circ (Itb \cdot c * \tau_{c,b}) \\ &= \{b,a\}^{tb \cdot c} * \{b,c\} \end{aligned}$$

and this is Axiom **B2**. The other axioms can be similarly shown. Then, we can define a functor from the category of braided 2-groupoids to that of braided regular crossed modules,

$$\Delta : \textbf{B2Grpoid} \to \textbf{BRCM.}$$

One can define the natural equivalences $\Psi : 1_{BRCM} \to \Delta\Theta$ and $\Xi : \Theta\Delta \to 1_{BRCM}$ by using a similar way given in the proof of the Proposition 2.6.

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