

NUMERICAL SOLUTION OF THIRD-ORDER BOUNDARY VALUE PROBLEMS*

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Abstract – In this paper, we use a third degree B-spline function to construct an approximate solution for third order linear and nonlinear boundary value problems coupled with the least square method. Several examples are given to illustrate the efficiency of the proposed technique.

Keywords – B-spline, least square method, third order BVPs

1. INTRODUCTION

Boundary value problems manifest themselves in many branches of science. For example engineering, technology, control and optimization theory. Some references [1, 2] contain theorems which detail the conditions for existence and uniqueness of solutions of such BVPs.

E. H. Twizell, H. N. Caglar and S. H. Caglar used fourth degree B-spline functions and the collocation method to solve a third order linear BVP of the form $y'''(t) = g(t)y(t) + q(t)$, $a \leq t \leq b$ or a third order nonlinear BVP of the form $y'''(t) = f(t, y)$, $a \leq t \leq b$ with boundary conditions $y(a) = k_1$, $y'(a) = k_2$, and $y(b) = k_3$. In their method, as they claim, they have some "unexpected" results near the boundaries.

In the present paper a third degree B-spline is used to solve third order boundary value problems. Consider a third order BVP in general form

$$g(t, y(t), y'(t), y''(t), y'''(t)) = 0 \quad (1)$$

on $[a, b]$ with the separated boundary conditions

$$\begin{cases} c_{10}y(a_1) + c_{11}y'(a_1) + c_{12}y''(a_1) = A_1 \\ c_{20}y(a_2) + c_{21}y'(a_2) + c_{22}y''(a_2) = A_2 \\ c_{30}y(a_3) + c_{31}y'(a_3) + c_{32}y''(a_3) = A_3 \end{cases} \quad (2)$$

Where $a \leq a_1 \leq a_2 \leq a_3 \leq b$. Our presentation finds a sequence of functions $\{v_k\}$ of the form $v_k(t) = \sum_{i=-3}^{3 \cdot 2^k - 1} c_i B_{ki}(t)$ which satisfy the exact separated boundary conditions. Also, up to an error ε_k , the function v_k satisfies the differential equation, where $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$.

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2. SPLINE SOLUTION FOR THIRD ORDER BVPs

Let $g : \mathfrak{R}^5 \longrightarrow \mathfrak{R}$ be a continuous function. We convert the problem to an optimal control problem $\min_{(y,u)} \int_a^b (g(t, y(t), y'(t), u(t), u'(t)))^2 dt$ with constraint $u(t) = y''(t)$ and boundary conditions

$$\begin{cases} c_{10}y(a_1) + c_{11}y'(a_1) + c_{12}u(a_1) = A_1 \\ c_{20}y(a_2) + c_{21}y'(a_2) + c_{22}u(a_2) = A_2 \\ c_{30}y(a_3) + c_{31}y'(a_3) + c_{32}u(a_3) = A_3 \end{cases}$$

where $u : \mathfrak{R} \longrightarrow \mathfrak{R}$ is taken to be control functions as in [3, 4]. The actual solution of (1)-(2) is a function v such that

$$\|g(t, v(t), v'(t), v''(t), v'''(t))\|_{L^2([a,b])}^2 = 0$$

and v satisfies the boundary conditions (2). For all $\varepsilon > 0$, the method finds an approximate solution v_ε satisfying (2), and

$$\|g(t, v_\varepsilon(t), v'_\varepsilon(t), v''_\varepsilon(t), v'''_\varepsilon(t))\|_{L^2([a,b])}^2 < \varepsilon.$$

The sketch of the method is delineated as follows:

Consider cubic spline function

$$B(t) = \begin{cases} t^3 & 0 \leq t \leq 1 \\ -3t^3 + 12t^2 - 12t + 4 & 1 < t \leq 2 \\ 3t^3 - 24t^2 + 60t - 44 & 2 < t \leq 3 \\ (4-t)^3 & 3 < t \leq 4 \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

For a fix $k \in \mathbb{N}$, consider an equal partition $a < a + h < a + 2h < \dots < a + 3 \cdot 2^k h = b$ on $[a, b]$ where $h = \frac{b-a}{3 \cdot 2^k}$. Define

$$B_{ki}(t) = B\left(\frac{3 \cdot 2^k}{b-a}(t-a) - i\right), \quad (i = -3, -2, \dots, 3 \cdot 2^k - 1)$$

where B is a scaling function (cubic spline in [5, 6]) and B_{ki} , ($k \in \mathbb{N}$, $i = -3, -2, \dots, 3 \cdot 2^k - 1$) are translations and dilations of B as prescribed in [7]. Let $v_k(t) = \sum_{i=-3}^{3 \cdot 2^k - 1} c_i B_{ki}(t)$ where the coefficients $\{c_i\}$ are determined from the conditions

$$\begin{cases} c_{10}v_k(a_1) + c_{11}v'_k(a_1) + c_{12}v''_k(a_1) = A_1 \\ c_{20}v_k(a_2) + c_{21}v'_k(a_2) + c_{22}v''_k(a_2) = A_2 \\ c_{30}v_k(a_3) + c_{31}v'_k(a_3) + c_{32}v''_k(a_3) = A_3 \end{cases}$$

and the following least square problem:

$$\min_{c_i} \|g(t, v_k(t), v'_k(t), v''_k(t), v'''_k(t))\|_{L^2([a,b])}^2.$$

The minimization problem is equivalent to the following system:

$$\frac{\partial}{\partial c_i} \|g(t, v_k(t), v'_k(t), v''_k(t), v'''_k(t))\|_{L^2([a,b])}^2 = 0, \quad (i = -3, -2, \dots, 3 \cdot 2^k - 1)$$

3. JUSTIFICATION OF THE METHOD

To justify the method, we consider the special case

$$\min_y \int_0^1 f(t, y(t), y'(t)) dt \tag{4}$$

and y satisfies the boundary conditions

$$C_{10}y(a_1) = A_1 \tag{5}$$

Where $0 \leq a_1 \leq 1$.

Consider the linear spline function

$$B(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $y(t) = \sum c_i B_{ki}(t)$, where $B_{ki}(t) := B(2^k t - i + 1)$ ($i = 0, 1, \dots, 2^k$). Then

$$\int_0^1 f(t, y(t), y'(t)) dt = \int_0^1 f\left(t, \sum_{i=0}^{2^k} c_i B_{ki}(t), \sum_{i=0}^{2^k} c_i B'_{ki}(t)\right) dt = F(c_0, c_1, \dots, c_{2^k}) \tag{6}$$

Therefore, in view of (6), the optimal problem (4), (5) reduces to the problem

$$\min_{c_0, c_1, \dots, c_{2^k}} F(c_0, c_1, \dots, c_{2^k}) \tag{7}$$

subject to

$$g(c_0, c_1, \dots, c_{2^k}) = 0 \tag{8}$$

where g are derived from equation (5). For finding c_1, \dots, c_{2^k} , we have to solve the system $\frac{\partial F}{\partial c_j} = 0$,

($j = 1, \dots, 2^k$). Let $c_0^*, c_1^*, \dots, c_{2^k}^*$ be the solution of (7), (8) and set

$$y_k^*(t) = \sum_{i=0}^{2^k} c_i^* B_{ki}(t). \tag{9}$$

Let $L(y) := \int_0^1 f(t, y(t), y'(t)) dt$, and suppose there exists a solution $y^* \in C([0,1])$ of (4) satisfying (5) (assume $L(y^*) \neq -\infty$). The following theorem shows that under reasonable conditions, $L(y_k^*)$ converges to $L(y^*)$ as $k \rightarrow +\infty$.

Theorem 1. Let f have the property that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t, x, y) - f(t, x_1, y_1)| < \varepsilon$, whenever $|x - x_1| < \delta$ and $|y - y_1| < \delta$. Let $y^* \in C([0,1])$ be a solution of the problem: (4), (5). Assume $(y^*)'$ is piecewise continuous, left continuous on $[0,1]$. The following assertions are true.

- a) For all $\varepsilon > 0$ there exists $k \in \mathbb{N}$ and y_k such that $0 < L(y_k) - L(y^*) < \varepsilon$, and y_k satisfies (5).
 b) Let y_k^* be as in (9). Then $L(y^*) \leq L(y_k^*) \leq L(y_k)$ and $L(y_k^*) \rightarrow L(y_k)$ as $k \rightarrow +\infty$.

Proof: See [7].

The above theorem shows that for all $\varepsilon > 0$ there exists an approximate solution y_k^* for the optimal control problem (4), (5) such that the difference between the value of $L(y_k^*)$ and the value $L(y^*)$ is at most ε .

Corollary 1. If the problem involves the higher derivative y'' , we will use the quadratic spline function B . Here, B'' is a left continuous step function. In this method, B is used when the regularity of B is minimal. That is, if the problem involves y' only, then B' must be chosen to be a step function; if it involves y' and y'' , then B'' must be chosen to be a step function, etc.

4. NUMERICAL RESULTS

In this section, the method discussed in section 2 was tested on two problems from the literature [1, 8]. The least square errors (LSE) in the analytical solutions for test problem 1 and 2 are shown in Table 1.

Test problem 1. ([8, equation 5. 1]). Consider the boundary value problem

$$y''' = ty + (t^3 - 2t^2 - 5t - 3)e^t, \quad 0 \leq t \leq 1, \quad y(0) = y(1) = 0, \quad y'(0) = 1$$

with exact solution $y(t) = t(1-t)e^t$.

Test problem 2. ([1, equation 1. 2]). Consider the boundary value problem

$$y''' - l^2 y' + a = 0, \quad 0 \leq t \leq 1, \quad y'(0) = y'(1) = 0, \quad y\left(\frac{1}{2}\right) = 0$$

found by Krajinovic [1] for analysis of beams physical phenomena, with exact solution

$$y(t) = \frac{a}{l^3} \left[\left(\sinh\left(\frac{l}{2}\right) - \sinh(lt) \right) + l \left(t - \frac{1}{2} \right) + \tanh\left(\frac{l}{2}\right) \left(\cosh(lt) - \cosh\left(\frac{l}{2}\right) \right) \right].$$

For $l = 2$ and $a = -3$ we obtain these results.

Table 1. Least square errors for test problems 1, 2

Test problem 1	$k = 0$	$LSE = 2.135 \times 10^{(-5)}$
	$k = 2$	$LSE = 7.375 \times 10^{(-8)}$
Test problem 2	$k = 0$	$LSE = 1.335 \times 10^{(-6)}$
	$k = 2$	$LSE = 5.463 \times 10^{(-9)}$

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