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ON PROBLEMS REDUCED TO THE GOURSAT PROBLEM FOR A THIRD ORDER EQUATION*

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Abstract – The aim of the present paper is the investigation of some problems which can be reduced to the Goursat problem for a third order equation. Some results and theorems are given concerning the existence and uniqunce for solving the suggested problem.

Keywords – Third order partial differential equation, goursat problem

1. INTRODUCTION

In the domain $D = \left\{(x,y); \quad x_0 < x < x_1, \quad y_0 < y < y_1\right\}$ we consider the equation

$$L(u) = u_{xxy} + au_{xx} + bu_{xy} + cu_x + du_y + eu = f\,;$$
$$a\,,\ b,\ c,\ d,\ e, f \in C^{1+2}(D), \tag{1}$$

The special cases of (1) are encountered during the investigation of processes of moisture absorption by plants (see [1]), where the class C^{k+l} means the existence and continuity for all derivatives

$$\partial^{r+s} / \partial x^r \partial y^s \qquad (r = 0,...k; s = 0,...,l).$$

The Goursat problem for (1) consists of finding a solution in D satisfying the following conditions on the characteristics:

$$u(x_0,y) = \varphi(y), u_x(x_0,y) = \varphi_1(y),\ y \in p\,,\ \varphi,\varphi_1 \in C^1(p),$$
$$u(x,y_0) = \psi(x),\ x \in q\,,\ \psi \in C^2(q), \tag{2}$$

Where

$$y \in p = [y_0,y_1],\ x \in q = [x_0,x_1].$$

Here we consider the agreement (co-ordination) conditions of function coincidence from (2) as satisfied:

$$\varphi(y_0) = \psi(x_0).$$

The solution of the mentioned problem is obtained in [2]. In this paper we investigated characteristic problems for the equation (1), in which at least one of the Goursat conditions is changed by the value of the next normal derivative. As a result we obtain that, every time, for one characteristic that is a carrier of boundary conditions, the highest order given by a normal derivative is increased by a unit. In fact, we

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speak about the determination of conditions for coefficients of the equation that provide a definite level of the informative, of these problems and, naturally, about the acquisition of the decision itself. Our method develops the works ([3], [4]).

We decided to indicate only a class of the desired unknown function u in formulations of all problems suggested below, taking into account that coefficients of the equations (1) are also chosen from the same class.

2. Main Results:

Problem 1. To find the function

$$u \in C^{2+1}(D) \cap C^{1+0}(D \cup p) \cap C^{0+1}(D \cup q),$$

which in D is the solution of the equation (1), which satisfies the first two conditions (2) and the following condition:

$$u_y(x, y_0) = \psi_1(x), \qquad \psi_1 \in C^2(q). \tag{3}$$

This problem may be reduced to the Goursat problem with the help of an integral equation. We integrate equation (1) twice with respect to x in bounds from x_* to x_0, and y to y_0. Taking into account the boundary conditions we obtain the above-mentioned equation:

$$a(x, y_0)\psi(x) + \int_{x_0}^{x} [(x-t)A(t) + B(t)]\psi(t)dt = r(t), \tag{4}$$

Where

$$A(x) = a_{xx}(x, y_0) - c_x(x, y_0) + e(x, y_0), \quad B(x) = c(x, y_0) - 2a_x(x, y_0),$$

$$r(t) = \int_{x_0}^{x} \{(x-t)[b_t(t, y_0) - d(t, y_0)] - b(t, y_0)\}\psi_1(t)dt - \psi_1(x) +$$

$$+(x - x_0)[a(x_0, y_0)\varphi_1(y_0) + b(x_0, y_0)\varphi'(y_0) + c(x_0, y_0)\varphi(y_0) +$$

$$+\varphi'(y_0) - a_x(x_0, y_0)\varphi(y_0)] + \varphi'(y_0) + a(x_0, y_0)\varphi(y_0).$$

Here, $\psi(x)$ is the function of the third condition (2). For its membership to the class $C^2(q)$ when $a(x, y_0) \neq 0$, in addition to already available conditions of coefficient smoothness, it is sufficient to assume that

$$a \in C^{2+0}(D \cup q), \qquad b, c \in C^{1+0}(D \cup q)$$

Here, we show that $\psi(x)$ is uniquely defined from (4).

The first one arises when $A(x) \equiv 0$. In this case the equation (4) takes the form

$$a(x, y_0)\psi(x) + \int_{x_0}^{x} B(t)\psi(t)dt = r(x).$$

Then

$$\psi(x) = \frac{r(x)}{a(x, y_0)} - \frac{1}{a(x, y_0)} \int_{x_0}^{x} \frac{B(\xi)r(\xi)}{a(\xi, y_0)} \left[\exp \int_{x}^{\xi} \frac{B(t)}{a(t, y_0)} dt \right] d\xi.$$

The second possibility is that $\psi(x)$ is defined by the condition $B(x) - xA(x) \equiv 0$. Arguments that are analogous to those given in the first case reduced to the solution (4) as in the form:

$$\psi(x)=\frac{r(x)}{a(x,y_0)}-\frac{x}{a(x,y_0)}\int\limits_{x_0}^{x}\frac{A(\xi)r(\xi)}{a(\xi,y_0)}\left[\exp\int\limits_{x}^{\xi}\frac{tA(t)}{a(t,y_0)}dt\right]d\xi.$$

Now let $a(x,y_0)\equiv 0.$ Here two other possibilities of the explicit solution (4) are also available. When $c(x,y_0)\neq 0$ and

$$d\in C^{1+0}(D\cup q),\quad b,c\in C^{2+0}(D\cup q),\qquad \psi_1\in C^3(q)$$

then:

$$\psi(x)=\frac{1}{a(x,y_0)}\left\{r'(x)-\int\limits_{x_0}^{x}T(\xi,y_0)r'(\xi)\left[\exp\int\limits_{x}^{\xi}T(t,y_0)dt\right]d\xi\right\},$$

Where

$$T(x,y_0)=\frac{-c_x(x,y_0)+e(x,y_0)}{c(x,y_0)},$$

while at $c(x,y_0)\equiv 0,\quad e(x,y_0)\neq 0$ and

$$b\in C^{3+0}(D\cup q),\quad e,d\in C^{2+0}(D\cup q),\qquad \psi_1\in C^4(q)$$

We have

$$\psi(x)=r''(x)[e(x,y_0)]^{-1}.$$

Obviously, the character of the solvability of the problem 1 is defined by the corresponding situation with the definition at the function $\psi(x)$. As a result of these arguments the following theorem can be formulated.

Theorem 1. Problem 1 is uniquely solved during the satisfaction of the inequality $a(x,y_0)\neq 0.$ In the general case the function $\psi(x)$ is written through the resolution of the equation (4). The function $\psi(x)$ can be written in explicit form in two different cases:
1) $a(x,y_0)\neq 0.$ and at least one of the identities
$A(x)\equiv 0,\quad B(x)-xA(x)\equiv 0$ is satisfied:
Or

$$a(x,y_0)\equiv 0,\quad c^2\ (x,y_0)+e^2(x,y_0)\neq 0.$$

Problem 2. To find the function

$$u\in C^{2+1}(D)\cap C^{2+0}(D\cup p)\cap C^{0+0}(D\cup q),$$

which in D is the solution of the equation (1), and satisfies all the conditions (2), where the first one is replaced by the following condition

$$u_{xx}(x_0,y)=\varphi_2(y),\qquad \varphi_2(y)\in C^1(p). \tag{5}$$

In this case we integrate (1) with respect to y, then we direct x to x_0. Thus, we obtain the integral equation for $\varphi(y)$:

$$d(x_0, y)\varphi(y) + \int_{y_0}^{y} [e(x_0, \tau) - d_\tau(x_0, \tau)]\varphi(\tau)d\tau = r_1(y), \tag{6}$$

Where

$$r_1(y) = \varphi_2(y_0) - \varphi_2(y) + \int_{y_0}^{y} \{[b_\tau(x_0, \tau) - c(x_0, \tau)]\varphi_1(\tau) - a(x_0, \tau)\varphi_2(\tau)\}d\tau -$$

$$-b(x_0, y)\varphi_1(y) + b(x_0, y_0)\varphi_1(y_0) + d(x_0, y_0)\psi(x_0).$$

is completely known. Arguments analogous to those given in the problem 1 reduced to the following results.

Theorem 2. Problem 2 is uniquely and explicitly reduced to the Goursat problem when any one of the following group conditions is satisfied:
1) $d(x_0, y) \neq 0.$ and $b, d \in C^{0+1}(D \cup p);$
2) $d(x_0, y) \equiv 0, \quad e(x_0, y) \neq 0$ and

$$b \in C^{0+2}(D \cup p),\; a, e \in C^{0+1}(D \cup p),\; \varphi_1, \varphi_2 \in C^2(p).$$

The function $\varphi(y)$ can be written in the following forms

$$\varphi(y) = \frac{1}{d(x_0, y)}\left\{r_1(y) - \int_{y_0}^{y} T_1(x_0, \eta)r_1(\eta)\left[\exp\int_{y}^{\eta} T_1(x_0, \tau)d\tau\right]d\eta\right\},$$

Where

$$T_1(x_0, y) = \frac{-d_y(x_0, y) + e(x_0, y)}{d(x_0, y)};$$

Or

$$\varphi(y) = r_1'(y)[e(x_0, y)]^{-1}.$$

According to the group of condition, (1) or (2), respectively.

Problem 3. is different from the previous one the way that (5) is introduced instead of the second one from the conditions (2). The class of solutions is the same as the problem 2.
In this case the relation (6) should be written as the equation for definition $\varphi_1(y)$:

$$b(x_0, y)\varphi_1(y) + \int_{y_0}^{y} [c(x_0, \tau) - d_\tau(x_0, \tau)]\varphi_1(\tau)d\tau = r_2(y), \tag{7}$$

Where

$$r_2(y) = \int_{y_0}^{y} \{[d_\tau(x_0, \tau) - e(x_0, \tau)]\varphi(\tau) - a(x_0, \tau)\varphi_2(\tau)\}d\tau - d(x_0, \tau)\varphi(\tau) +$$

$$+d(x_0, y_0)\varphi(y_0) - \varphi_2(y) + \varphi_2(y_0) + b(x_0, y_0)\psi'(x_0).$$

Hence, we have the following theorem

Theorem 3. Problem 3 is uniquely and explicitly reduced to the Goursat problem during the satisfaction of any of the following groups of conditions:

1) $b(x_0, y\ ) \neq 0.$ and $b, d \in C^{0+1}(D \cup p);$

2) $b(x_0, y) \equiv 0, \quad c\ (x_0, y) \neq 0.$ and

$$d \in C^{0+2}(D \cup p),\ \ a, c \in C^{0+1}(D \cup p),\ \ \varphi_1, \varphi_2 \in C^2(p).$$

Here the solution can be written in the following forms:

$$\varphi_1(y) = \frac{1}{b(x_0, y)}\left\{ r_2(y) - \int_{y_0}^{y} T_2(x_0, \eta) r_2(\eta) \left[\exp \int_{y}^{\eta} T_2(x_0, \tau) d\tau \right] d\eta \right\},$$

Where

$$T_2(x_0, y) = \frac{c(x_0, y) - b_y(x_0, y)}{b(x_0, y)};$$

Or

$$\varphi_1(y) = r_2'(y)[c(x_0, y)]^{-1}.$$

according to the group of condition (1) or (2) respectively.

The theorem is verified by arguments similar to those presented for Theorems 1 and 2.

Now let us consider problems in which two Goursat conditions are changed.

Problem 4. To find the solution of (1) of the class

$$C^{2+1}\,(\,D\,) \cap C^{2+0}\,(\,D \cup p\,) \cap C^{0+1}\,(\,D \cup q\,),$$

which satisfies the conditions (5), (3) and second from (2).

Functions $\varphi(y)\ and\ \psi(x)$ in this case are defined from the equations (4) and (6). From (4) it follows that

$$\varphi'(y_0) = \psi_1(x_0).$$

If we differentiate (6) and put $y = y_0$, then we have the relation

$$d(x_0, y_0)\varphi'(y_0) + e(x_0, y_0)\varphi(y_0) = r_1'(y_0) = -\varphi_2'(y_0) - \\ -c(x_0, y_0)\varphi_1(y_0) - a(x_0, y_0)\varphi_2(y_0) - b(x_0, y_0)\varphi_1'(y_0)$$

the right part will be completely known only when $e(x_0, y_0) \neq 0.$ For $e(x_0, y_0) = 0,\ \ \varphi(y_0)$ is to be considered as an arbitrary constant.

For the convenience of writing the reduction conditions for the Goursat problem we will introduce the following notations:

Let A [0] matrix- row from k (k-type) elements, B [0] matrix- row from *l* (l-type) elements. We will understand A [0] x B [0] as matrix- row of which are obtained as follows:

The first *l* elements are the union of the first element A [0] with each element of the row B [0], next *l* elements are the union of the second element A [0] with each element of the row B[0], etc.

Let us indicate

$$\text{A}[0]=\left\{\begin{bmatrix} a(x,y_0)\neq 0, \\ a\in C^{2+0}(D\cup q), \\ b,c\in C^{1+0}(D\cup q) \end{bmatrix}\right\}$$

This matrix consists of one element in which all mentioned conditions are included. Later we will call these elements blocks.

$$\text{A}[1]=\left\{\begin{bmatrix} a(x_0,y)\neq 0 \\ and\,at\;\, least\, one \\ identity\, is\, satisfied \\ A(x)\equiv 0, B(x)-xA(x)\equiv 0 \\ a\in C^{2+0}(D\cup q), \\ b,c\in C^{1+0}(D\cup q) \end{bmatrix};\begin{bmatrix} a(x,y_0)\equiv 0 \\ c(x,y_0)\neq 0 \\ d\in C^{1+0}(D\cup q), \\ b,c\in C^{2+0}(D\cup q) \\ \psi_1\in C^3(q) \end{bmatrix};\begin{bmatrix} a(x,y_0)\equiv \\ \equiv c(x,y_0)\equiv 0 \\ e(x,y_0)\neq 0 \\ b\in C^{2+0}(D\cup q), \\ e,d\in C^{2+0}(D\cup q) \\ \psi_1\in C^4(q) \end{bmatrix}\right\}$$

this matrix consists of three block-columns.

$$\text{B}_1[1]=\left\{\begin{bmatrix} d(x_0,y)\neq 0 \\ b,d\in C^{0+1}(D\cup p) \end{bmatrix};\begin{bmatrix} d(x_0,y)\equiv 0, e(x_0,y)\neq 0 \\ b\in C^{0+2}(D\cup p),\; a,e\in C^{0+1}(D\cup p) \\ \varphi_1,\varphi_2\in C^2(p) \end{bmatrix}\right\} \tag{8}$$

Matrix $\text{B}_1[1]$ consists of two block-columns. Then the following takes place:

Theorem 4. Each block of the product A [0] x B_1 [1] is the sufficient condition for the reduction of the problem 4 to the Goursat problem in resolved (resolution) terms of integral equations, while each block of the A [1] x B_1[1] provides the writing of the Goursat conditions in explicit form.

In the block of conditions (8), if we take the condition $e(x_0,y)\neq 0$ or $e(x,y_0)\neq 0$, it is easy to see that the solvability is unique. If so, then when $e(x_0,y_0)\equiv 0$, the appearance of an arbitrary constant is possible.

For example, the block obtained by the mentioned multiplication of the first block A [1] by the first block B_1 [1] represents the following group of conditions:
$a(x_0,y)\neq 0$, at least one of the identities is satisfied, $A(x)\equiv 0, B(x)-xA(x)\equiv 0$
And

$$a\in C^{2+0}(D\cup q); b,c\in C^{1+0}(D\cup q); d(x_0,y)\neq 0 \text{ and } b,d\in C^{0+1}(D\cup p).$$

The functions $\varphi(y)$ and $\psi(x)$ are defined here explicitly. Since for the condition of inequality to the zero the coefficient $e(x,y)$ is not written here, then when $e(x_0,y_0)=0,$ the function $\varphi(y_0)$ should be considered as an arbitrary constant.
Let us consider an example of the unique solvability. To do this, we will multiply A [0] by the second block B_1 [1]. We obtain the block that consists of the following conditions:

$$\begin{aligned} &a(x,y_0)\neq 0, \quad a\in C^{2+0}(D\cup q), \quad b,c\in C^{1+0}(D\cup q), \quad d(x_0,y)\equiv 0, \\ &e(x_0,y)\neq 0, \quad b\in C^{0+2}(D\cup p), \quad a,e\in C^{0+1}(D\cup p), \quad \varphi_1,\varphi_2\in C^2(p). \end{aligned}$$

Here, the existence of the condition $e(x_0,y)\neq 0$ exactly speaks about the uniqueness of the definition $\varphi(y)$ and $\psi(x)$.

Problem 5. Is different from the previous ones in the way that on (5), not the first, but the second condition is changed (2).

Here it is necessary to find $\varphi_1(y), \psi(x)$ from (7) and (4). The right hand side of (7) and (4) depends on $\varphi_1(y_0)$ and $\varphi_1'(y_0)$ respectively. From (4) it follows that

$$\varphi_1'(y_0) = \psi_1'(x_0).$$

While from (7) we have

$$b(x_0,y_0)\varphi_1'(y_0) + c(x_0,y_0)\varphi_1(y_0) = r_2'(y_0) = -e(x_0,y_0)\varphi\ (y_0) -$$
$$- a(x_0,y_0)\varphi_2(y_0) - d(x_0,y_0)\varphi'(y_0) - \varphi_2'(y_0).$$

The right hand side of the latter relation is a known number only when $c(x_0,y_0) \neq 0$. For $c(x_0,y_0) \equiv 0, \varphi_1(y_0)$ should be considered as an arbitrary constant. Here, we introduce the next additional matrix-row.

$$B_2[1]=\left\{\begin{bmatrix} b(x_0,y) \neq 0 \\ b,d \in C^{0+1}(D \cup p) \end{bmatrix};\begin{bmatrix} b(x_0,y) \equiv 0,\, c(x_0,y) \neq 0 \\ d \in C^{0+2}(D \cup p),\ a,c \in C^{0+1}(D \cup p) \\ \varphi_1,\, \varphi_2\ \in C^2(p) \end{bmatrix}\right\} \tag{9}$$

Theorem 5. Each block of the product A [0] x B_2[1] provides for the reduction of the problem 5 to the Goursat problem in resolving (resolution) terms of integral equations, while each block of the A[1] x B_1[1]- is computed explicitly.

In the block of conditions (9), if we take the condition $c(x_0,y) \neq 0$ or $c(x,y_0) \neq 0$, it is easy to see that the solvability is unique. Otherwise the appearance of an arbitrary constant is possible. If the conditions $c(x,y_0) \equiv 0$, and $c(x_0,y\ ) \neq 0$ are written, the latter should be supplemented (complemented) by the following:

The zero $y = y_0$ of the function $c(x_0,y)$ has an order which does not exceed the orders of the zeros of the functions $\varphi(y), d(x_0.y), a(x_0,y), \varphi_2'(y)$ in the same point $y = y_0$.

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