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DUAL SPLIT QUATERNIONS AND SCREW MOTION IN MINKOWSKI 3-SPACE*

L. KULA AND Y. YAYLI**

Ankara University, Faculty of Science, Department of Mathematics
06100 Tandogan, Ankara, Turkey
Email: kula@science.ankara.edu.tr, yayli@science.ankara.edu.tr

Abstract – In this paper, two new Hamilton operators are defined and the algebra of dual split quaternions is developed using these operators. It is shown that finite screw motions in Minkowski 3-space can be expressed by dual-number (3×3) matrices in dual Lorentzian space. Moreover, by means of Hamilton operators, screw motion is obtained in 3-dimensional Minkowski space $\mathbb{R}_1^3$.

Keywords – Dual split quaternions, semi-orthogonal dual matrix, screw-motion

1. INTRODUCTION

Quaternion algebra, enunciated by Hamilton, has played a significant role recently in several areas of physical science; namely, in differential geometry, in analysis and synthesis of mechanism and machines, simulation of particle motion in molecular physics and quaternionic formulation of equation of motion in the theory of relativity. Agrawal [1] gave some algebraic properties of Hamilton operators. Also, quaternions have been expressed in terms of 4×4 matrices by means of these operators.

The purpose of this paper is to develop the screw motion of a line in $\mathbb{R}_1^3$. Firstly, we defined dual split quaternions and gave some algebraic properties of dual split quaternions. Also, we discussed Hamilton operators and their properties. Moreover, by means of dual-number (3×3) matrices, screw motion is expressed in Minkowski 3-space. The last section is devoted to give the screw motion, in $\mathbb{R}_1^3$, using the properties of the Hamilton operators defined in this paper.

2. DUAL NUMBERS AND DUAL SPLIT QUATERNIONS

A brief summary of dual numbers and dual split quaternions is presented in this section for easy reference and to provide the necessary background for the mathematical formulations to be developed in this paper. In this paper, a dual number A has the form $a + \epsilon a^*$, where *a* and a^* are real numbers and ϵ is the dual symbol subjected to the rules

$$\epsilon \neq 0,\ 0\epsilon = \epsilon 0 = 0,\ 1\epsilon = \epsilon 1 = \epsilon,\ \epsilon^2 = 0.$$

A split quaternion q is an expression of the form

$$q = a_0 + a_1 i + a_2 j + a_3 k,$$

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**Corresponding author

where a_0, a_1, a_2 and a_3 are real numbers, and $i,\ j,\ k$ are split quaternionic units which satisfy the non-commutative multiplication rules

$$\begin{aligned} &i^2 = -1, \quad j^2 = k^2 = 1 \\ &ij = -ji = k, \quad jk = -kj = -i, \quad ki = -ik = j \end{aligned} \tag{1}$$

(See [2] for split quaternions).

Similarly, a dual split quaternion Q is written as

$$Q = A_0 + A_1 i + A_2 j + A_3 k.$$

As a consequence of this definition, a dual split quaternion Q can also be written as

$$Q = q + \epsilon q^*$$

where $q = a_0 + a_1 i + a_2 j + a_3 k$ and $q^* = a_0^* + a_1^* i + a_2^* j + a_3^* k$ are, respectively, real and dual split quaternion components. The multiplication of split quaternionic units with a dual symbol is commutative; i.e. $\epsilon i = i\epsilon$, and so on. Therefore, it is a matter of indifference whether one writes $A_1 i$ or iA_1, and so on. Owing to the properties of the eight units equality, additions and subtraction of dual split quaternion are governed by the rules of ordinary algebra.

The three dual split quaternionic units ($i,\ j$ and k) are orthogonal unit vector with respect to the scalar product defined below. Further, under a proper semi-orthogonal transformation, these units preserve the definition of split quaternionic units given in equations (1) and the definition of the scalar product. For this reason $i,\ j$ and k are identified as an orthogonal triad of unit vectors in Minkowski 3-space (R^3 with the metric tensor $g(u,v) = -u_1v_1 + u_2v_2 + u_3v_3$, $u, v \in \mathrm{R}^3$ is called Minkowski 3-space and denoted by $\mathbb{R}_1^3$ [3]). It is useful, therefore, to define the following terms:

Dual number part of Q:

$$S_Q = A_0.$$

Dual vector part of Q:

$$V_Q = A_1 i + A_2 j + A_3 k\ .$$

Hamiltonian conjugate of Q:

$$\begin{aligned} \overline{Q} &= A_0 - (A_1 i + A_2 j + A_3 k) \\ &= S_Q - V_Q \\ &= \overline{q} + \epsilon \overline{q}^*. \end{aligned}$$

The split quaternion multiplication is, in general, not commutative. If Q and P are the two dual split quaternions and let $R = QP$, then R is given by

$$\begin{aligned} R &= (A_0B_0 - A_1B_1 + A_2B_2 + A_3B_3) + (A_0B_1 + A_1B_0 + A_3B_2 - A_2B_3)i \\ &\quad + (A_0B_2 + A_2B_0 + A_3B_1 - A_1B_3)j + (A_0B_3 + A_3B_0 + A_1B_2 - A_2B_1)k \\ &= qp + \epsilon(qp^* + q^*p), \end{aligned} \tag{2}$$

where $P = B_0 + B_1 i + B_2 j + B_3 k = p + \epsilon p^*$, p and p^* are split quaternions.

Norm of Q:

$$\begin{aligned} N_Q = \; & Q\bar{Q} = \bar{Q}Q = A_0^2 + A_1^2 - A_2^2 - A_3^2 \\ & = (a_0^2 + a_1^2 - a_2^2 - a_3^2) + 2\epsilon(a_0a_0^* + a_1a_1^* - a_2a_2^* - a_3a_3^*) \\ & = q\bar{q} + \epsilon(q\bar{q}^* + q^*\bar{q}). \end{aligned}$$

Reciprocal of Q :

$$\begin{aligned} Q^{-1} &= \frac{\bar{Q}}{N_Q} \\ &= q^{-1} - \epsilon(q^{-1}q^*q^{-1}), \end{aligned}$$

where $(N_q)^2 \neq 0$.

Unit quaternion:

$$N_Q = 1.$$

Scalar product of quaternion Q and P

$$\begin{aligned} g(Q,P) &= g(P,Q) \\ &= -A_0B_0 - A_1B_1 + A_2B_2 + A_3B_3 \\ &= \frac{1}{2}(\bar{Q}P + \bar{P}Q) = \frac{1}{2}(Q\bar{P} + P\bar{Q}) \\ &= g(q,p) + \epsilon(g(q,p^*) + g(q^*,p)). \end{aligned}$$

Cross product of dual split vectors *R* and *U*:

$$R \wedge U = \frac{1}{2}\ (RU - UR).$$

From the computation point of view, the above operations are expressed in terms of matrix operations. It is clear that a dual split quaternion is essentially a tetrad of dual numbers. In this paper, these tetrad of dual numbers are also written in a column matrix form as

$$Q = [A_0\ A_1\ A_2\ A_3]^T .$$

Thus it follows that

$$Q = q + \epsilon q^*,$$

where q and q^* are 4×1 column vectors.

3. HAMILTON OPERATORS AND THEIR PROPERTIES

In this section, two new operators $\overset{+}{H}$ and $\overset{-}{H}$, called Hamilton's operators, are defined and their properties are discussed. If q is a split quaternion, then Hamilton operators $\overset{+}{H}$ and $\bar{H}$ are, respectively, defined as

$$\overset{+}{H}(q) = \begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \qquad (3)$$

and

$$\bar{H}(q) = \begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}. \qquad (4)$$

A direct consequence of the above operators is the following identities:

$$\overset{+}{H}(1) = \bar{H}(1) = I\,,$$

$$\overset{+}{H}(i) = E_1; \quad \overset{+}{H}(j) = E_2; \quad \overset{+}{H}(k) = E_3\,,$$

$$\bar{H}(i) = F_1; \quad \bar{H}(j) = F_2; \quad \bar{H}(k) = F_3\,.$$

where I is a 4×4 identity matrix. Note that the properties of E_n and F_n $(n = 1,2,3)$ are identical to that of split quaternionic units $i,\ j,\ k$.
Since $\overset{+}{H}$ and $\bar{H}$ are linear in their elements, it follows that

$$\begin{aligned} \overset{+}{H}(q) &= a_0\,\overset{+}{H}(1) + a_1\,\overset{+}{H}(i) + a_2\,\overset{+}{H}(j) + a_3\,\overset{+}{H}(k) \\ &= a_o I + a_1 E_1 + a_2 E_2 + a_3 E_3, \end{aligned} \qquad (5)$$

$$\begin{aligned} \bar{H}(q) &= a_0\,\bar{H}(1) + a_1\,\bar{H}(i) + a_2\,\bar{H}(j) + a_3\,\bar{H}(k) \\ &= a_o I + a_1 F_1 + a_2 F_2 + a_3 F_3, \end{aligned} \qquad (6)$$

$$\overset{+}{H}(Q) = \overset{+}{H}(q) + \epsilon\,\overset{+}{H}(q^*)\,, \qquad (7)$$

$$\bar{H}(Q) = \bar{H}(q) + \epsilon\,\bar{H}(q^*)\,. \qquad (8)$$

Using the definitions of $\overset{+}{H}$ and $\bar{H}$, the multiplication of the two dual split quaternions Q and P is given by

$$R = \overset{+}{H}(Q)P = \bar{H}(P)Q\,. \qquad (9)$$

(See [1] for Hamilton operators in $\mathbb{R}^4$). Equation (9) is of central importance in proving several identities. In the discussion to follow some theorems associated with these operators ($\overset{+}{H}$ and $\bar{H}$) are presented.

Theorem 3. 1. If Q and P are dual split quaternions, λ is a real number and $\overset{+}{H}$ and $\bar{H}$ are operators as defined in equations (3) and (4), respectively, then the following identities hold:

i. $Q = P \Leftrightarrow \overset{+}{H}(Q) = \overset{+}{H}(P) \Leftrightarrow \bar{H}(Q) = \bar{H}(P).$

ii. $\overset{+}{H}(Q+P) = \overset{+}{H}(Q) + \overset{+}{H}(P),\ \bar{H}(Q+P) = \bar{H}(Q) + \bar{H}(P).$

iii. $\overset{+}{H}(\lambda Q) = \lambda\,\overset{+}{H}(Q),\ \bar{H}(\lambda Q) = \lambda\,\bar{H}(Q).$

iv. $\overset{+}{H}(QP) = \overset{+}{H}(Q)\,\overset{+}{H}(P),\ \bar{H}(QP) = \bar{H}(P)\,\bar{H}(Q).$

v. $\overset{+}{H}(\overline{Q}) = \varepsilon\,\overset{+}{H}(Q)^T \varepsilon,\ \bar{H}(\overline{Q}) = \varepsilon\,\bar{H}(Q)^T \varepsilon,\ \varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}.$

vi. $\overset{+}{H}(Q^{-1}) = \overset{+}{H}(Q)^{-1},\ \bar{H}(Q^{-1}) = \bar{H}(Q)^{-1},\ (N_Q)^2 \neq 0.$

vii. $\det\left[\overset{+}{H}(Q)\right]=(N_Q)^2=g(Q,Q)^2,\ \det\left[\bar{H}(Q)\right]=(N_Q)^2=g(Q,Q)^2.$

viii. $\operatorname{tr}\left[\overset{+}{H}(Q)\right]=4A_0,\ \operatorname{tr}\left[\bar{H}(Q)\right]=4A_0.$

ix. $\overset{+}{H}(Q)=L\,\bar{H}(Q)^T L,\ L=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ L^{-1}=L^T=L,\ L^2=I$

Proof: Identities (i), (ii) and (iii) follow from equations (7) and (8). Using the associative property of quaternion's multiplication it is clear that the following identities hold:

$(QP)R=Q(PR)=QPR.$

In terms of operator $\overset{+}{H}$, the above identities can be written as

$$\begin{aligned}\overset{+}{H}(QP)R&=\overset{+}{H}(\overset{+}{H}(Q)P)R\\&=\overset{+}{H}(Q)(\overset{+}{H}(P)R)=\overset{+}{H}(Q)\,\overset{+}{H}(P)R.\end{aligned}\tag{10}$$

And similarly,

$$\begin{aligned}\bar{H}(QP)R&=\bar{H}(\bar{H}(Q)P)R\\&=\bar{H}(P)(\bar{H}(Q)R)=\bar{H}(P)\,\bar{H}(Q)R.\end{aligned}\tag{11}$$

Since column R is arbitrary, the above relation employs equation (iv). Identities (v), (vi), (vii), (viii) and (ix) can be proved (7), (8) and (9).

Theorem 3. 2. Matrices generated by operators $\overset{+}{H}$ and $\bar{H}$ commute, or mathematically this can be stated as

$$\overset{+}{H}(Q)\,\bar{H}(P)=\bar{H}(P)\,\overset{+}{H}(Q).$$

Proof: This follows from equations (9), (10) and (11).

From equations (7) and (8), it is clear that the matrices $\overset{+}{H}(Q)$ and $\bar{H}(Q)$ are 4×4 dual matrices. Following the usual matrix nomenclature, a matrix $\hat{A}^T$ is called a semi-orthogonal dual matrix (sogdm) if $\hat{A}^T\varepsilon\hat{A}\,\varepsilon=\hat{A}\,\varepsilon\hat{A}^T\varepsilon=AI$, where $A\neq 0+\epsilon a^*$ is a dual number and I is an identity matrix and $\varepsilon=\begin{bmatrix}-I_2&0\\0&I_2\end{bmatrix}$. A matrix $\hat{A}$ is called semi-orthonormal dual matrix (sondm) if $A=1$.

Theorem 3. 3. Matrices generated by operators $\overset{+}{H}$ and $\bar{H}$ are semi-orthogonal matrices, i.e.

i) $\overset{+}{H}(Q)^T\varepsilon\,\overset{+}{H}(Q)\varepsilon=\overset{+}{H}(Q)\varepsilon\,\overset{+}{H}(Q)^T\varepsilon=N_QI,\ \varepsilon=\begin{bmatrix}-I_2&0\\0&I_2\end{bmatrix}.$

ii) $\bar{H}(P)^T\varepsilon\,\bar{H}(P)\varepsilon=\bar{H}(P)\varepsilon\,\bar{H}(P)^T\varepsilon=N_PI,\ \varepsilon=\begin{bmatrix}-I_2&0\\0&I_2\end{bmatrix}.$

$\overset{+}{H}(Q)$ and $\bar{H}(Q)$ are semi-orthonormal dual matrices if and only if Q is a unit dual split quaternion.

Proof: Equation (i) follows from Theorem 3. 1. Equation (ii) is proved in the same way. If Q is a unit dual split quaternion, then $N_Q = 1$. This implies that $\overset{+}{H}(Q)$ is a semi-orthonormal dual matrix. The same arguments apply for matrix $\bar{H}(Q)$.

Theorem 3. 4. Let $\hat{A}$ be a 4×4 semi-orthonormal dual matrix with $A_{\bullet k} = a_{\bullet k} + \epsilon a^*_{\bullet k}$,$(k = 1,2,3,4)$ as its k' th column vector. Then there exists two vectors *u* and *v* (i.e. $u_o = v_o = 0$) such that

$$a^*_{\bullet k} = ua_{\bullet k} + a_{\bullet k}\, v,\ (k = 1,2,3,4) \tag{12}$$

and vectors *u* and *v* are unique.

Proof: Semi-orthonormal dual matrix $\hat{A}$ can be written as

$$\hat{A} = A + \epsilon A^*.$$

Semi-orthonormality condition implies

$$A\varepsilon A^T\varepsilon = A^T\varepsilon A\varepsilon = I\,, \tag{13}$$

$$A\varepsilon A^{*T}\varepsilon + A^*\varepsilon A^T\varepsilon = 0. \tag{14}$$

Define

$$A^*\varepsilon A^T\varepsilon = B. \tag{15}$$

Equation (14) implies that matrix B is a skew-symmetric matrix ($B^T = -\varepsilon B\varepsilon$). Multiplying equation (15) by A from the right and using equation (13), one finds

$$A^* = BA\,. \tag{16}$$

Matrix *B* in equation (16) is unique. If not, then let D be the other matrix that implies equation (16). Then it follows that

$$A^* = BA = DA$$

or

$$(D - B)A = 0.$$

Since *A* is a semi-orthonormal matrix it follows that $B = D$, i. e. *B* is unique. Matrix *B* can be written uniquely as

$$B = \overset{+}{H}(u) + \bar{H}(v)\,, \tag{17}$$

where *u* and *v* are two vectors, i.e. $u_0 = v_0 = 0.$ Equation (12) follows using equations (9), (16) and (17). The following theorem is now obvious.

Theorem 3. 5. Any 4×4 semi-orthonormal dual matrix can be represented as

$$\hat{A} = [a_{.0} + \epsilon(ua_{.0} + a_{.0}v) \;\; a_{.1} + \epsilon(ua_{.1} + a_{.1}v) \;\; a_{.2} + \epsilon(ua_{.2} + a_{.2}v) \;\; a_{.3} + \epsilon(ua_{.3} + a_{.3}v)]$$

where $A = (a_{ik})$ is a real semi-orthogonal matrix and *u* and *v* are two vectors. *A*, *u* and *v* are unambiguously determined by $\hat{A}$.

For a dual vector *U*, the matrices $\overset{+}{H}(U)$ and $\bar{H}(U)$ are skew-symmetric matrices. These two matrices are denoted, respectively, as $\overset{+}{V}(U)$ and $\bar{V}(U)$ to emphasize the fact that $\overset{+}{V}(U)$ and $\bar{V}(U)$ are generated using the vector component of a quaternion only. Matrices $\overset{+}{V}(U)$ and $\bar{V}(U)$ are written, respectively as

$$\overset{+}{V}(U) = \begin{bmatrix} 0 & -U_1 & U_2 & U_3 \\ U_1 & 0 & U_3 & -U_2 \\ U_2 & U_3 & 0 & -U_1 \\ U_3 & -U_2 & U_1 & 0 \end{bmatrix} = \overset{+}{H}(V(U)), \tag{18}$$

$$\bar{V}(U) = \begin{bmatrix} 0 & -U_1 & U_2 & U_3 \\ U_1 & 0 & -U_3 & U_2 \\ U_2 & -U_3 & 0 & U_1 \\ U_3 & U_2 & -U_1 & 0 \end{bmatrix} = \bar{H}(V(U)) \,. \tag{19}$$

Corollary 3. 6. Matrices $\overset{+}{V}(U)$ and $\bar{V}(U)$ satisfy the following identities:

i) $\overset{+}{V}(U)^T = -\varepsilon \overset{+}{V}(U)\varepsilon, \quad \varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix},$

and

ii) $(\bar{V}(U))^T = -\varepsilon \bar{V}(U)\varepsilon, \quad \varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}.$

iii) $\overset{+}{V}(U)^T \varepsilon \overset{+}{V}(U)\varepsilon = \overset{+}{V}(U)\varepsilon \overset{+}{V}(U)^T \varepsilon = N_{V(U)} I, \quad \varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$

and

iv) $\bar{V}(U)^T \varepsilon \bar{V}(U)\varepsilon = \bar{V}(U)\varepsilon \bar{V}(U)^T \varepsilon = N_{V(U)} I, \quad \varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}.$

v) $(\overset{+}{V}(U))^{2n} = (-N_{V(U)})^n I = (\bar{V}(U))^{2n}.$

vi) $(\overset{+}{V}(U))^{2n+1} = (-N_{V(U)})^n \overset{+}{V}(U)$

and

$(\bar{V}(U))^{2n+1} = (-N_{V(U)})^n \bar{V}(U).$

4. THE SCREW MOTION OF A LINE IN MINKOWSKI 3-SPACE

Theorem 4. 1. (E. Study Theorem)
There exist one-to-one correspondence between directed timelike (resp. spacelike) lines of $\mathbb{R}^3_1$ and an ordered pair of vectors (a, a^*) such that $\langle a, a \rangle = -1$ (resp. $\langle a, a \rangle = 1$) and $\langle a, a^* \rangle = 0$ [4].

Definition 4. 2. Let us consider a dual split vector

$$V_Q = a + \epsilon a^* = (a_1 i + a_2 j + a_3 k) + \epsilon(a_1^* i + a_2^* j + a_3^* k).$$

The dual split vector is said to be timelike if $\langle a, a \rangle < 0$, spacelike if $\langle a, a \rangle > 0$ or $a = 0$, and null if $\langle a, a \rangle = 0$ and $a \neq 0$, where $\langle \, , \rangle$ is a Minkowskian scalar product with signature $(-, +, +)$.

Lemma 4. 3. Let $Q = A_0 + A_1 i + A_2 j + A_3 k$ be a unit dual split quaternion.

$$\mathsf{A} = \begin{bmatrix} A_0^2 + A_1^2 + A_2^2 + A_3^2 & 2(A_0A_3 - A_1A_2) & -2(A_0A_2 + A_1A_3) \\ 2(A_0A_3 + A_1A_2) & A_0^2 - A_1^2 - A_2^2 + A_3^2 & -2(A_0A_1 + A_2A_3) \\ 2(-A_0A_2 + A_1A_3) & 2(A_0A_1 - A_2A_3) & A_0^2 - A_1^2 + A_2^2 - A_3^2 \end{bmatrix} \quad (20)$$

is a semi-orthonormal dual matrix i.e. $\mathsf{A}\,\varepsilon\mathsf{A}^{T}\varepsilon = \mathsf{A}^{T}\varepsilon\mathsf{A}\,\varepsilon = I_3$, where $\varepsilon = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
By the matrix in (20) we have the following theorem.

Theorem 4. 4. Let Q be a unit dual split quaternion.

If V_Q is a spacelike dual split vector, then that the matrix A defined in Lemma 4.3 can be written as

$$\mathsf{A}_{(\Phi,\mathbb{S}^*)} = I_3 + \sinh\Phi\mathbb{S}^* + (-1+\cosh\Phi)(\mathbb{S}^*)^2 \quad (21)$$

where $\Phi = \varphi + \epsilon\varphi^*$ is a dual hyperbolic angle, $\cosh\frac{\Phi}{2} = A_0$, $\sinh\frac{\Phi}{2} = \sqrt{-A_1^2 + A_2^2 + A_3^2}$,

$\mathbb{S}^* = \begin{bmatrix} 0 & S_3 & -S_2 \\ S_3 & 0 & -S_1 \\ -S_2 & S_1 & 0 \end{bmatrix}$, $S_n = \dfrac{A_n}{\sinh\frac{\Phi}{2}}$, $1 \le n \le 3$, and $\mathbb{S}^*$ is a skew-symmetric matrix.

If V_Q is a timelike dual split vector, then the matrix A defined in Lemma 4.3 can be written as

$$\mathsf{A}_{(\Theta,\mathbb{T}^*)} = I_3 + \sin\Theta\mathbb{T}^* + (1-\cos\Theta)(\mathbb{T}^*)^2, \quad (22)$$

where $\Theta = \theta + \epsilon\theta^*$ is a dual angle, $\cos\frac{\Theta}{2} = A_0$, $\sin\frac{\Theta}{2} = \sqrt{A_1^2 - A_2^2 - A_3^2}$, $\mathbb{T}^* = \begin{bmatrix} 0 & T_3 & -T_2 \\ T_3 & 0 & -T_1 \\ -T_2 & T_1 & 0 \end{bmatrix}$, $T_n = \dfrac{A_n}{\sin\frac{\Theta}{2}}$, $1 \le n \le 3$,

and $\mathbb{T}^*$ is a skew-symmetric matrix.

Proof: Let V_Q be a spacelike dual split vector. Then, we have

$$Q = \cosh\frac{\Phi}{2} + \sinh\frac{\Phi}{2}\mathbb{S}^*$$

establishing (21), where $\mathbb{S}^* = S_1 i + S_2 j + S_3 k$ and $S_n = \dfrac{A_n}{\sinh\frac{\Phi}{2}}$, $1 \le n \le 3$.

If V_Q is a timelike dual split vector, then

$$Q = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\mathbb{T}^*$$

establishing (22), where $\mathbb{T}^* = T_1 i + T_2 j + T_3 k$ and $T_n = \dfrac{A_n}{\sin\frac{\Theta}{2}}$, $1 \le n \le 3$.

The matrices $\mathsf{A}_{(\Phi,\mathbb{S}^*)}$ and $\mathsf{A}_{(\Theta,\mathbb{T}^*)}$ satisfy the following:

For a unit dual split vector $X = x + \epsilon x^* = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$, we write

$$\mathsf{A}_{(\Phi,\mathbb{S}^*)}X = \cosh\Phi X + \sinh\Phi\left(\mathbb{S}^* \wedge X\right) + \left(1-\cosh\Phi\right)\left\langle \mathbb{S}^*, X\right\rangle \mathbb{S}^*, \tag{23}$$

and

$$\mathsf{A}_{(\Theta,\mathbb{T}^*)}X = \cos\Theta X + \sin\Theta\left(\mathbb{T}^* \wedge X\right) - \left(1-\cos\Theta\right)\left\langle \mathbb{T}^*, X\right\rangle \mathbb{T}^*. \tag{24}$$

If $\left\langle X, \mathbb{S}^* \right\rangle = 0,$ we have

$$\mathsf{A}_{(\Phi,\mathbb{S}^*)}X = \cosh\Phi X + \sinh\Phi\left(\mathbb{S}^* \wedge X\right).$$

If $\left\langle X, \mathbb{T}^* \right\rangle = 0,$ we have

$$\mathsf{A}_{(\Theta,\mathbb{T}^*)}X = \cos\Theta X + \sin\Theta\left(\mathbb{T}^* \wedge X\right).$$

Theorem 4. 5.

i. The matrix

$$\mathsf{R}_{\left(\varphi,\mathbb{S}^*\right)} = I_3 + \sinh\varphi \mathbb{S}^* + (-1+\cosh\varphi)(\mathbb{S}^*)^2 \tag{25}$$

yield a finite rotation of a line about the spacelike line $\mathbb{S}^*$ with the hyperbolic angle φ in Minkowski 3-space.

ii. The matrix

$$\mathsf{R}_{\left(\varphi,\mathbb{S}^*\right)} = I_3 + \sinh\varphi \mathbb{S}^* + (-1+\cosh\varphi)(\mathbb{S}^*)^2 \tag{26}$$

yield a finite rotation of a line about the timelike line $\mathbb{T}^*$ with the angle θ in Minkowski 3-space.

Proof:

i. In equation (25) the real part and dual part may be interpreted separately.

The real part of $\mathsf{R}_{\left(\varphi,\mathbb{S}^*\right)}$ is

$$\mathsf{A}\,d_{(\varphi,S_0)} = I_3 + \sinh\varphi S_0 + (-1+\cosh\varphi)(S_0)^2 \tag{27}$$

and the dual part of $\mathsf{R}_{\left(\varphi,\mathbb{S}^*\right)}$ is

$$\mathsf{J}_{\left(\varphi,\mathbb{S}^*\right)} = \sinh\varphi S_0^* + \left(-1+\cosh\varphi\right)\left(S_0 S_0^* + S_0^* S_0\right), \tag{28}$$

where S_0 is a real part of $\mathbb{S}^*$ and S_0^* is a dual part of $\mathbb{S}^*$. Thus, equation (25) may be written

$$\mathsf{R}_{(\varphi,\mathbb{S}^*)} = \mathsf{A}\,d_{(\varphi,S_0)} + \epsilon \mathsf{J}_{\left(\varphi,\mathbb{S}^*\right)}. \tag{29}$$

Let $X = x + \epsilon x^* = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ be a unit dual split vector. In this case,

$$\mathsf{R}_{(\varphi,\mathbb{S}^*)}X = \mathsf{A}\,d_{(\varphi,S_0)}x + \epsilon\left(\mathsf{A}\,d_{(\varphi,S_0)}x^* + \mathsf{J}_{\left(\varphi,\mathbb{S}^*\right)}x\right), \tag{30}$$

where the vector $\mathrm{A}d_{(\varphi,S_0)}x$ is a finite rotation of a vector x about the spacelike axis S_0 with the hyperbolic angle φ.

ii. By a similar calculation, the real part of $\mathrm{R}_{(\theta,\mathbb{T}^*)}$ is

$$\mathrm{A}d_{(\theta,T_0)} = I_3 + \sin\theta T_0 + (1-\cos\theta)(T_0)^2 \tag{31}$$

and the dual part of $\mathrm{R}_{(\theta,\mathbb{T}^*)}$ is

$$\mathrm{J}_{(\theta,\mathbb{T}^*)} = \sin\theta T_0^* + (1-\cos\theta)(T_0 T_0^* + T_0^* T_0), \tag{32}$$

where T_0 is a real part of $\mathbb{T}^*$ and T_0^* is a dual part of $\mathbb{T}^*$. Thus, equation (26) may be written

$$\mathrm{R}_{(\theta,\mathbb{T}^*)} = \mathrm{A}d_{(\theta,T_0)} + \epsilon \mathrm{J}_{(\theta,\mathbb{T}^*)}. \tag{33}$$

For a unit dual split vector $X = x + \epsilon x^* = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$, we write

$$\mathrm{R}_{(\theta,\mathbb{T}^*)}X = \mathrm{A}d_{(\theta,T_0)}x + \epsilon\left(\mathrm{A}d_{(\theta,T_0)}x^* + \mathrm{J}_{(\theta,\mathbb{T}^*)}x\right), \tag{34}$$

where $\mathrm{A}d_{(\theta,T_0)}x$ is a finite rotation of a vector x about timelike axis T_0 with the angle θ.

Theorem 4. 6.

i. The matrix

$$\mathrm{T}_{(\varphi^*,\mathbb{S}^*)} = I_3 + \epsilon\varphi^*\mathbb{S}^* \tag{35}$$

yield a translation along the spacelike line $\mathbb{S}^*$ in Minkowski 3-space.

ii. The matrix

$$\mathrm{T}_{(\theta^*,\mathbb{T}^*)} = I_3 + \epsilon\theta^*\mathbb{T}^* \tag{36}$$

yield a translation along the timelike line $\mathbb{T}^*$ in Minkowski 3-space.

Proof:

i. In the equation (35), the real part of $\mathrm{T}_{(\varphi^*,\mathbb{S}^*)}$ is I_3 and the dual part of $\mathrm{T}_{(\varphi^*,\mathbb{S}^*)}$ is $\varphi^* S_0$.

For a unit dual split vector $X = x + \epsilon x^* = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$, we write

$$\mathrm{T}_{(\varphi^*,\mathbb{S}^*)}X = x + \epsilon\left(x^* + \varphi^*\left(S_0 \wedge x\right)\right). \tag{37}$$

Therefore, the unit dual split vector $\mathsf{T}_{(\varphi^*,\mathbb{S}^*)}X$ is parallel to directed spacelike lines X of Minkowski 3-space.

ii. By a similar calculation, from equation (36) we write

$$\mathsf{T}_{(\theta^*,\mathbb{T}^*)}X = x + \epsilon\left(x^* + \theta^*\left(T_0 \wedge x\right)\right). \tag{38}$$

Thus, the unit dual split vector $\mathsf{T}_{(\theta^*,\mathbb{T}^*)}X$ is parallel to directed timelike lines X of Minkowski 3-space.

The following theorem is now obvious.

Theorem 4. 7. The dual matrix $\mathsf{A}_{(\Phi,\mathbb{S}^*)}$ in equation (21) is written

$$\mathsf{A}_{(\Phi,\mathbb{S}^*)} = \mathsf{R}_{(\varphi,\mathbb{S}^*)}.\mathsf{T}_{(\varphi^*,\mathbb{S}^*)}, \tag{39}$$

or

$$\mathsf{A}_{(\Phi,\mathbb{S}^*)} = \mathsf{T}_{(\varphi^*,\mathbb{S}^*)}.\mathsf{R}_{(\varphi,\mathbb{S}^*)}$$

and the dual matrix $\mathsf{A}_{(\Theta,\mathbb{T}^*)}$ in equation (22) is written

$$\mathsf{A}_{(\Theta,\mathbb{T}^*)} = \mathsf{R}_{(\theta,\mathbb{T}^*)}.\mathsf{T}_{(\theta^*,\mathbb{T}^*)}, \tag{40}$$

or

$$\mathsf{A}_{(\Theta,\mathbb{T}^*)} = \mathsf{T}_{(\theta^*,\mathbb{T}^*)}.\mathsf{R}_{(\theta,\mathbb{T}^*)}.$$

Consequently, the screw motion of a line may be considered a combination of a finite rotation about the spacelike line (resp. timelike line) and translation along the spacelike line (resp. timelike line) in Minkowski 3-space.

Special cases:

i. The case that $\varphi^* = 0$:

$$\mathsf{A}_{(\Phi,\mathbb{S}^*)} = \mathsf{R}_{(\varphi,\mathbb{S}^*)}.$$

ii. The case that $\varphi = 0$:

$$\mathsf{A}_{(\Phi,\mathbb{S}^*)} = \mathsf{T}_{(\varphi^*,\mathbb{S}^*)}.$$

iii. The case that $\theta^* = 0$:

$$\mathsf{A}_{(\Theta,\mathbb{T}^*)} = \mathsf{R}_{(\theta,\mathbb{T}^*)}.$$

iv. The case that $\theta = 0$:

$$\mathsf{A}_{(\Theta,\mathbb{T}^*)} = \mathsf{T}_{(\theta^*,\mathbb{T}^*)}.$$

Corollary 4. 8.

i. Let $\mathsf{A}_{(\Phi_1,\mathbb{S}^*)}$ and $\mathsf{A}_{(\Phi_2,\mathbb{S}^*)}$ be a two screw motion about a spacelike axis $\mathbb{S}^*$ with two dual hyperbolic angle Φ_1 and Φ_2. In this case

$$\mathsf{A}_{(\Phi_1,\mathbb{S}^*)}.\mathsf{A}_{(\Phi_2,\mathbb{S}^*)} = \mathsf{A}_{(\Phi_1+\Phi_2,\mathbb{S}^*)}.$$

ii. Let $\mathsf{A}_{(\Theta_1,\mathbb{T}^*)}$ and $\mathsf{A}_{(\Theta_2,\mathbb{T}^*)}$ be a two screw motion about a timelike axis $\mathbb{T}^*$ with two dual angles Θ_1 and Θ_2. In this case

$$\mathsf{A}_{(\Theta_1,\mathbb{T}^*)}.\mathsf{A}_{(\Theta_2,\mathbb{T}^*)} = \mathsf{A}_{(\Theta_1+\Theta_2,\mathbb{T}^*)}.$$

5. THE SCREW MOTION IN MINKOWSKI 3-SPACE

Let us consider the transformation

$$R = QR_f\overline{Q}, \tag{41}$$

where R_f is a unit dual vector and $Q = A_0 + A_1 i + A_2 j + A_3 k$ is a unit dual split quaternion. Since R_f is a unit dual vector, R is also a unit dual vector.

Using the definition of $\overset{+}{H}$ and $\bar{H}$, equation (41) can be written as

$$R = \overset{+}{H}(Q)\,\bar{H}(\overline{Q})R_f = \bar{H}(\overline{Q})\,\overset{+}{H}(Q)R_f$$

or

$$R = \begin{bmatrix} 1 & 0 \\ 0 & \mathsf{A} \end{bmatrix} R_f \tag{42}$$

where the matrix A is a 3×3 semi-orthonormal matrix defined by equation (20).

The time derivative of R (denoted by a dot ".") is

$$\dot{R} = \overset{+}{H}(\dot{Q})\,\bar{H}(\overline{Q})R_f + \overset{+}{H}(Q)\,\bar{H}(\dot{\overline{Q}})R_f \quad . \tag{43}$$

Using equations (41) and (42), from Theorem 3.3, equation (43) can be written as

$$\dot{R} = \left(\overset{+}{H}(\dot{Q}\overline{Q}) - \bar{H}(\dot{Q}\overline{Q})\right)R$$

or

$$\dot{R} = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} R \tag{44}$$

where W is a 3×3 skew symmetric matrix. In terms of dual split quaternions, we write

$$\dot{R} = (\dot{Q}\overline{Q})R - R(\dot{Q}\overline{Q}). \tag{45}$$

Owing to the fact that Q is a unit dual split quaternion, it is clear that $\dot{Q}\overline{Q}$ is a dual vector. Therefore, equation (45) can be written as a cross product form

$$\dot{R} = 2(\dot{Q}\overline{Q}) \wedge R\,. \tag{46}$$

Also, the time derivative of R is given by

$$\dot{R} = W \wedge R, \tag{47}$$

where W is the dual split vector of the velocity screw of the line. Since R is an arbitrary vector, equations (46) and (47) lead to

$$W = 2\dot{Q}\overline{Q} \tag{48}$$

or equivalently,

$$W = 2\overset{+}{H}(\dot{Q})\overline{Q} = 2\bar{H}(\overline{Q})\dot{Q}\,. \tag{49}$$

From the equation (49), if $g\left(\dot{Q},\dot{Q}\right) > 0$ or $\dot{Q} = 0$ (i.e. $g(\dot{q},\dot{q}) > 0$ or $\dot{q} = 0$), then the dual split vector W is a spacelike dual split vector. If $g\left(\dot{Q},\dot{Q}\right) < 0$ (i.e. $g(\dot{q},\dot{q}) < 0$), then the dual split vector W is a timelike dual split vector. In other words, the skew symmetric matrix W in equation (44) can be written as the form $\mathbb{S}^*$ or $\mathbb{T}^*$ in equations (21), (22).

Also

$$\dot{Q} = \frac{1}{2}WQ$$

or equivalently,

$$\dot{Q} = \frac{1}{2}\overset{+}{H}(W)Q = \frac{1}{2}\bar{H}(Q)W\,.$$

Using equations (47) and (48), the two components of W can be written as

$$w = 2\dot{q}\,\overline{q}$$

or

$$w = 2\,\bar{H}(\overline{q})\dot{q}$$

and

$$v_0 = 2(\dot{q}\overline{q}^* + \dot{q}^*\overline{q})$$

or

$$v_0 = 2\left[\overset{+}{H}(\dot{q})\overline{q}^* + \overset{+}{H}(\dot{q}^*)\overline{q}\right].$$

Since $\overset{+}{H}$ and $\bar{H}$ are semi-orthogonal operators, the formulation presented here provides a singularity free relation between W, Q and its time derivative.

6. CONCLUSIONS

Split quaternion algebra of dual numbers has been formulated in terms of two operators. Properties of these two operators are presented that establish a relationship between dual split quaternions and their equivalent matrix operations. These properties are applied to develop kinematics equations of the screw-motion of a line and a point in Minkowski 3-space.

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