Iranian Journal of Science & Technology, Transaction A, Vol. 30, No. A2
Printed in The Islamic Republic of Iran, 2006
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ON STRONGLY Δ^n-SUMMABLE SEQUENCE SPACES*

A. ESI** AND H. POLAT

Adiyaman University, Science and Art Faculty in Adiyaman,
02040, Adiyaman, Turkey
Emails: aesi23@hotmail.com and hpolat@inonu.edu.tr

Abstract – In the present paper we define strongly Δ^n-summable sequences which generalize A-summable sequences and prove such spaces to be complete paranormed spaces under certain conditions, some topological results have also been discussed.

Keywords – Difference sequence, paranorm

1. INTRODUCTION

Let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ respectively, normed by $\|x\| = \sup_k |x_k|$.

The notion of difference sequence space was introduced by Kizmaz [1] as follows:

$$Z(\Delta) = \left\{ x = (x_k) : \quad (\Delta x_k) \in Z \right\}$$

for $Z = l_\infty,\ c$, or c_0 where $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$.

Later, the difference sequence spaces were generalized by Et and Çolak [2] as follows:

Let $n \in N$ be fixed, then

$$Z(\Delta^n) = \left\{ x = (x_k) : \quad (\Delta^n x_k) \in Z \right\} \text{ for } Z = l_\infty,\ c, \text{ or } c_0,$$

where $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$, and so $\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}$. They showed that the above spaces are Banach spaces, normed by

$$\|(x_k)\|_{\Delta^n} = \sum_{i=1}^{n} |x_i| + \sup_k \|\Delta^n x_k\|.$$

Recently, difference sequence spaces have been discussed in Esi [3, 4], Tripathy [5] and many others.

Let $A = (a_{nk})$ be an infinite matrix of non-negative real numbers and $p = (p_k)$ be a sequence such that $0 < p_k \le \sup_k p_k = H < \infty$. We write $Ax = \{(A_n x)\}$ if $A_n(x) = \sum_k a_{nk} |x_k|^{p_k}$ converges for each n. Maddox [6] define

$$[A, p]_0 = \left\{ x : A_n(x) \to 0 \quad as \quad n \to \infty \right\},$$

$$[A, p]_0 = \left\{ x : A_n(x - L) \to 0 \quad as \quad n \to \infty, for\ some\ L \right\},$$

*Received by the editor February 13, 2005 and in final revised form April 25, 2006
**Corresponding author

And

$$[A,p]_\infty = \{x : \sup_n A_n(x) < \infty\}.$$

The spaces $[A,p]_0$, $[A,p]$ and $[A,p]_\infty$ are called the spaces of strongly summable to zero, strongly summable and strongly bounded sequences, respectively.

The purpose of this paper is to introduce the spaces of strongly Δ^n – summable sequences, which generalize well-known strongly A-summable sequences $[A,p]_0$, $[A,p]$ and $[A,p]_\infty$, [6] and [7].

We now generalize these spaces by means of a given matrix $B = (b_{mk})$. We write

$$T_m(x) = \sum_k a(k,m)|\Delta^n x_k|^{p_k}$$

where $a(k,m) = \sum_j b_{mj}a_{jk}$ and $b_{mj}a_{jk}$ is of the same sign for each m, j and k.

We now write

$$[B_{\Delta^n}, p]_0 = \left\{x : T_m(x) \to 0 \quad as \quad m \to \infty\right\},$$

$$[B_{\Delta^n}, p] = \left\{x : T_m(x-L) \to 0 \quad as \quad m \to \infty, \, for\ some\ L\right\},$$

And

$$[B_{\Delta^n}, p]_0 = \{x : \sup_m T_m(x) < \infty\}.$$

These are the spaces of strongly Δ^n – summable to zero, strongly Δ^n – summable and strongly Δ^n – bounded sequences, respectively, and these spaces of strongly Δ^n – summable sequences depend on the fixed chosen matrix B. In case B=I (unit matrix) and replacing x_k in the place of $\Delta^n x_k$ in the above definition we get the sequence spaces $[A,p]_0$, $[A,p]$ and $[A,p]_\infty$, respectively [6].

2. MAIN RESULTS

First we establish a number of lemmas.

Lemma 1. If $p = (p_k) \in l_\infty$, then $[B_{\Delta^n}, p]_0$, $[B_{\Delta^n}, p]$ and $[B_{\Delta^n}, p]_\infty$ are linear spaces over the complex field C.

Proof: We consider only $[B_{\Delta^n}, p]$. If $\sup_k p_k = H < \infty$ and $K = \max\left(1, 2^{H-1}\right)$, we have Maddox ([6], p.346)

$$|x_k + y_k|^{p_k} \le K\left(|x_k|^{p_k} + |y_k|^{p_k}\right) \tag{1}$$

and for $\lambda \in C$

$$|\lambda|^{p_k} \le \max\left(1, |\lambda|^H\right) \tag{2}$$

Now the linearity follows from (1) and (2).

Lemma 2. $[B_{\Delta^n}, p] \subset [B_{\Delta^n}, p]_\infty$ if

$$\|A\| = \sup_m \sum_k a(k,m) < \infty \tag{3}$$

Proof: Suppose that $x \in [B_{\Delta^n}, p]$ and (3) holds. Now by inequality (1)

$$T_m(x) = T_m(x - L + L)$$
$$\leq KT_m(x - L) + K\sum_k a(k,m)|L|^{p_k}$$
$$\leq KT_m(x - L) + K(\sup|L|^{p_k})\sum_k a(k,m). \quad (4)$$

Therefore, $x \in [B_{\Delta^n}, p]_\infty$ and this completes the proof.

Lemma 3. Let $0 < \Theta = \inf p_k \leq \sup_k p_k = H < \infty$, then $[B_{\Delta^n}, p]_\infty$ are linear topological spaces paranormed by G defined by

$$G(x) = \sum_{i=1}^{n}|x_i| + \sup_m |T_m(x)|^{1/M}$$

where $M = \max\left(1, \sup_k p_k = H\right)$. If (3) holds, then $[B_{\Delta^n}, p]$ has the same paranorm.

Proof: Clearly G(0)=0 and G(x)=G(-x). Since $M \geq 1$, then G(x+y)$\leq$ G(x)+G(y). Further, from (2) it follows that

$$G(\lambda x) \leq \begin{matrix} |\lambda|^{\Theta/M} G(x) \ \ if \ \ |\lambda| \leq 1 \\ |\lambda| G(x) \quad if \quad |\lambda| \geq 1 \end{matrix}$$

where $\Theta = \inf p_k > 0$. Therefore, $x \to 0, \lambda \to 0 \Rightarrow \lambda x \to 0$ and $x \to 0, \lambda \ fixed \Rightarrow \lambda x \to 0$ and also $\lambda \to 0 \Rightarrow \lambda x \to 0$, x is fixed. This completes the proof for $[B_{\Delta^n}, p]_0$. If $\inf p_k = \Theta > 0$ and $0 < |\lambda| < 1$, then for each $x \in [B_{\Delta^n}, p]_\infty$,

$$G^M\left(\lambda x\right) \leq |\lambda|^{\Theta} G(x).$$

Therefore, $[B_{\Delta^n}, p]_\infty$ has the paranorm G. If (3) holds it is clear from Lemma 2 that G(x) exists for each $x \in [B_{\Delta^n}, p]$. Hence the proof is complete.

Lemma 4. $[B_{\Delta^n}, p]_0$ and $[B_{\Delta^n}, p]_\infty$ are complete with respect to their paranorm topologies. $[B_{\Delta^n}, p]$ is complete if (3) holds and

$$\sum_k a(k,m) \to 0 \ \ as \ \ m \to \infty \quad (5)$$

Proof: (x^s) is a Cauchy sequence in $[B_{\Delta^n}, p]_\infty$, where $x^s = (x_k^s)_{k=1}^\infty$ for all $s \in N$. Then we have

$$G\left(x^s - x^t\right) = \sum_{i=1}^{n}\left|x_i^s - x_i^t\right| + \sup_m \left|T_m\left(x_k^s - x_k^t\right)\right|^{1/M} \to 0 \text{ as } s,t \to \infty.$$

Hence we obtain $\left|x_i^s - x_i^t\right| \to 0$ as $s,t \to \infty$ for each $i \in N$. Therefore $(x_i^s)_{s=1}^\infty$ is a Cauchy sequence in C, the set of complex numbers. Since C is complete, it is convergent. Let $\lim_s x_i^s = x_i$ say, for each $i \in N$. Since (x^s) is a Cauchy sequence, for each $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $G\left(x^s - x^t\right) < \varepsilon$ for all $s,t \geq N$.

Hence

$$\sum_{i=1}^{n}\left|x_i^s - x_i^t\right| \leq \varepsilon \text{ and } \sup_m \left|T_m\left(x_k^s - x_k^t\right)\right|^{1/M} \leq \varepsilon$$

for all $i \in N$ and for all $s,t \geq N$. So we have

$$\lim_t \sum_{i=1}^{n} \left| x_i^s - x_i^t \right| = \sum_{i=1}^{n} \left| x_i^s - x_i \right| \leq \varepsilon$$

and

$$\lim_t \left| T_m \left(x_k^s - x_k^t \right) \right|^{1/M} = \left| T_m \left(x_k^s - x_k^t \right) \right|^{1/M} \leq \varepsilon \text{ for all } s \geq N .$$

This implies that $G(x^s - x) < 2\varepsilon$ for all $s \geq N$, that is $x^s \to x$ as $s \to \infty$, where $x = (x_i)$. Without loss of generality, let $s \geq N$, then $x^s \in [B_{\Delta^n}, p]_\infty$ and $x - x^s \in [B_{\Delta^n}, p]_\infty$ imply that $x = x^s + (x - x^s) \in [B_{\Delta^n}, p]_\infty$, since $[B_{\Delta^n}, p]_\infty$ is linear. Thus $[B_{\Delta^n}, p]_\infty$ is complete.

We now consider $[B_{\Delta^n}, p]$. If (3) holds and (x^s) be a Cauchy sequence in $[B_{\Delta^n}, p]$, then there exists $x = (x_k)$ such that $G(x^s - x) \to 0$ as $s \to \infty$. If (5) holds, then from inequality (4) it is clear that $[B_{\Delta^n}, p] = [B_{\Delta^n}, p]_0$. This completes the proof.

Now, combining the above lemmas, we obtain the following result:

Theorem 1. Let $0 < \Theta = \inf_k p_k \leq \sup_k p_k = H < \infty$, then $[B_{\Delta^n}, p]_0$ and $[B_{\Delta^n}, p]_\infty$ are complete linear topological spaces paranormed by G. If (3) and (5) hold, then $[B_{\Delta^n}, p]$ has the same property. Further, if $p_k = p$ for all k, they are Banach spaces for $1 \leq p < \infty$ and p-normed spaces for $0 < p < 1$.

3. SOME TOPOLOGICAL RESULTS

We now study locally boundedness and r-convexity for the spaces of strongly Δ^n – summable sequences. We start with some definitions.

For $r > 0$ a non-void subset Ψ of a linear space is said to be absolutely r-convex if $x,y \in \Psi$ and $|\lambda|^\gamma + |\mu|^\gamma \leq 1$ together imply that $\lambda x + \mu y \in \Psi$ [8]. A linear topological space X is said to be r-convex if every neighbourhood of zero contains an absolutely r-convex neighbourhood of zero. A subset B of X is said to be bounded if for each neighbourhood U of $0 \in X$ $\exists$ an integer N>1 such that $B \subset NU$. X is called locally bounded if there is a bounded neighbourhood of zero.

Theorem 2. Let $0 < p_k \leq 1$. Then $[B_{\Delta^n}, p]_0$ and $[B_{\Delta^n}, p]_\infty$ are locally bounded if $\inf p_k > 0$. If (3) holds, then $[B_{\Delta^n}, p]$ has the same property.

Proof: We consider only $[B_{\Delta^n}, p]_\infty$. Let $\inf p_k = \Theta > 0$. If $x \in [B_{\Delta^n}, p]_\infty$ then there exists a constant T>0 such that

$$\sum_k a(k,m) |\Delta^n x_k|^{p_k} \leq T \quad (\forall m).$$

For this T and given $\delta > 0$ choose an integer N>1 such that $N^\Theta \geq \frac{T}{\delta}$. Since $\frac{1}{N} < 1$ and $p_k \geq \Theta$, we have $\frac{1}{N^{p_k}} \leq \frac{1}{N^\Theta}$ $(\forall k)$. Then for all m, we get

$$\sum_k a(k,m) \left| \frac{\Delta^n x_k}{N} \right|^{p_k} \leq \frac{1}{N^\Theta} \sum_k a(k,m) |\Delta^n x_k|^{p_k} \leq \frac{T}{N^\Theta} \leq \delta .$$

Therefore by taking supremum over m, we have

$$\{x : G(x) \leq T\} \subset N \{x : G(x) \leq \delta\} \tag{6}$$

For every $\delta > 0$ there exists an integer N>1 for which (6) holds and so

$$\{x : G(x) \le T\}$$

is bounded. This completes the proof.

Theorem 3. Let $0 < p_k \le 1$. Then $[B_{\Delta^n}, p]_0$ and $[B_{\Delta^n}, p]_\infty$ are r-convex for all r, $0 < r < \liminf p_k$. Morever, if $p_k = p \le 1$ for all k, then they are p-convex. If (3) holds, $[B_{\Delta^n}, p]$ has the same property.

Proof: We shall prove only for $[B_{\Delta^n}, p]_\infty$. Let $x \in [B_{\Delta^n}, p]_\infty$ and $0 < r < \liminf p_k$. Then there exists k_0 such that $r < p_k$ for all $k > k_0$. Now define

$$f(x) = \sup_m \left[\sum_{k=1}^{k_0} a(k,m) |\Delta^n x_k|^\gamma + \sum_{k=k_0+1}^{\infty} a(k,m) |\Delta^n x_k|^{p_k} \right].$$

Since $r < p_k \le 1$ for all $k > k_0$, f is subadditive. Further, for $0 < |\lambda| \le 1$ and for all $k > k_0$, $|\lambda|^{p_k} \le |\lambda|^\gamma$. Therefore for such λ, we have

$$f(\lambda x) \le |\lambda|^\gamma f(x).$$

Now for $0 < \delta < 1$,

$$\Psi = \left\{ x : f(x) \le \delta \right\}$$

is an absolutely r-convex set, for if $|\lambda|^\gamma + |\mu|^\gamma \le 1$ and $x, y \in \Psi$ then

$$f(\lambda x + \mu y) \le f(\lambda x) + f(\mu y)$$

$$\le |\lambda|^\gamma f(x) + |\mu|^\gamma f(y)$$

$$\le (|\lambda|^\gamma + |\mu|^\gamma)\delta \le \delta.$$

If $p_k = p$ for all k, then for $0 < \delta < 1$, $\{x : f(x) \le \delta\}$ is an absolutely p-convex set. This can be obtained by a similar analysis. This completes the proof.

Theorem 4. (i) Let $0 < \inf_k p_k = \Theta \le p_k \le 1$. Then $[B_{\Delta^n}, p] \subset [B_{\Delta^n}]$.
(ii) Let $1 \le p_k \le \sup_k p_k < \infty$. Then $[B_{\Delta^n}] \subset [B_\Delta, p]$.
Where

$$[B_{\Delta^n}] = \left\{ x : \sum_k a(k,m) |\Delta^n x_k - L| \to 0, \ \textit{as} \ \ m \to \infty, \ \textit{for some } L \right\}.$$

(iii) Suppose that (3) holds. Let $0 < p_k \le q_k$ and $\left(\frac{q_k}{p_k}\right)$ be bounded. Then $[B_{\Delta^n}, q] \subset [B_{\Delta^n}, p]$.

Proof: (i) Let $x \in [B_{\Delta^n}, p]$, since $0 < \inf_k p_k = \Theta \le p_k \le 1$, we get

$$\sum_k a(k,m) |\Delta^n x_k - L| \le \sum_k a(k,m) |\Delta^n x_k - L|^{p_k}$$

for each m, and hence $x \in [B_{\Delta^n}]$.

(ii) Let $1 \le p_k \le \sup_k p_k < \infty$ and $x \in [B_{\Delta^n}]$. Then for each $0 < \varepsilon < 1$, there exists a positive integer N such that

$$\sum_k a(k,m) |\Delta^n x_k - L| \le \varepsilon < 1$$

for all m ≥ N. This implies that

$$\sum_k a(k,m)|\Delta^n x_k - L|^{p_k} \le \sum_k a(k,m)|\Delta^n x_k - L|.$$

Thus we get $x \in [B_{\Delta^n}, p]$.

(iii) Define

$$u_{k.p} = \begin{cases} y_{k,p}, & y_{k,p} \ge 1 \\ 0 \quad , & y_{k,p} < 1 \end{cases}$$

and

$$v_{k.p} = \begin{cases} y_{k,p}, & y_{k,p} \ge 1 \\ 0 \quad , & y_{k,p} < 1 \end{cases}$$

where

$$y_{k,p} = |\Delta^n x_k - L|^{q_k}$$

Therefore $y_{k,p} = u_{k,p} + v_{k,p}$ and $y_{k,p}^{\lambda_k} = u_{k,p}^{\lambda_k} + v_{k,p}^{\lambda_k}$, where $\lambda_k = \frac{p_k}{q_k}$. Now it follows that $u_{k,p}^{\lambda_k} \le u_{k,p} \le y_{k,p}$ and $v_{k,p}^{\lambda_k} \le v_{k,p}^{\lambda}$ for $0 < \lambda < \lambda_k \le 1$. We have the inequality Maddox ([1], p.351)

$$\sum_k a(k,m) y_{k,p}^{\lambda_k} \le \sum_k a(k,m) y_{k,p} + \left(\sum_k a(k,m) v_{k,p}\right)^{\lambda} \|A\|^{(1-\lambda)}.$$

Hence $x \in [B_{\Delta^n}, q]$ if (3) holds and $x \in [B_{\Delta^n}, p]$.

Acknowlegements-The authors thanks the referees for the suggestions that improved the presentation of the article**.**

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