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ON THE COUPLING OF FINITE AND BOUNDARY ELEMENT METHODS FOR THE HELMHOLTZ EQUATION*

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Abstract – Finite and boundary element methods have been used by many authors to solve mathematical physics problems. However, the coupling of these two methods happens to be more efficient as it combines their merits. In this paper, the mathematical analysis of the coupling of finite and boundary element methods for the Helmholtz equation is presented.

Keywords – Boundary element, boundary integral equation, finite element, Galerkin approximation, Helmholtz equation, symmetric method

1. INTRODUCTION

In recent years, the coupling of finite element and boundary element methods have become a famous method for solving many mathematical physics problems [1-5].

The creation of this method is due to the drawbacks of both finite element and boundary element methods. This can be seen, for example, in the solution of exterior problems by the finite element method and that of the nonlinear problems by the boundary element method [6, 7].

In early eighties, Johnson and Nedelec [8] presented a mathematical analysis of this new method which is nowadays commonly called, the traditional or standard coupled method.

Unfortunately, their study was very limited and did not concern many important cases in modeling a concrete physical situation, as for example, problems when the boundary is not smooth, systems of integral equations and higher order equations.

By the end of the eighties, Costabel succeeded in the establishment of the bases for another coupled method, more complex than the first one, but it deals with more complicated situations as well as those mentioned above [9].

Their approach relied on the reduction of interface problems to the study of functional, which regroup integral forms defined on a portion of the considered domain and integral operators defined on the boundary of the coupling, which is the common boundary between the sub-domains.

The plan of the paper is as follows. In Section 2, we deal with the boundary value problem of the Helmholtz equation, its variational formulation in a portion of the whole domain and, its reduction to a system of integral equations in the rest of the domain. Section 3 contains the weak formulation of the coupling problem. In Section 4, we present error analysis and discuss the convergence of the Galerkin approximation.

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2. THE MODEL PROBLEM AND ITS VARIATIONAL FORMULATION

We consider a boundary value problem on a domain $\Omega \subset \mathrm{IR}^2$ which is decomposed as:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_c \,, \ \ with \quad \Gamma_c = \partial\Omega_1 \cap \partial\Omega_2 \quad , \quad \Gamma_1 = \partial\Omega_1 / \Gamma_c \quad and \quad \Gamma_2 = \partial\Omega_2 / \Gamma_c \,.$$

Let us consider the following Helmholtz equation

$$\left(\Delta + k^2 I\right) u = -f_1 \qquad in \quad \Omega \tag{1}$$

where f_1 is a function in $IL^2(\Omega_j)$ with a bounded support in Ω_1.
Then for function g_2 on the boundary Γ_2 we can formulate the following problem:

$$\begin{cases} \left(\Delta + k^2 I\right) u_1 = -f_1 & in \quad \Omega_1 \\ \left(\Delta + k^2 I\right) u_2 = 0 & in \quad \Omega_2 \\ u_1 = 0 & on \quad \Gamma_1 \\ u_2 = g_2 & on \quad \Gamma_2 \\ u_1 = u_2 \quad and \quad \dfrac{\partial u_1}{\partial \eta_1} = \dfrac{\partial u_2}{\partial \eta_2} & on \quad \Gamma_c \end{cases} \tag{2}$$

where $\dfrac{\partial v}{\partial \eta}$ denotes the derivative of v with respect to the normal vector pointing from Ω_1 into Ω_2.
We will use the finite element method for Ω_1 and the boundary element method for Ω_2. If Ω_2 is unbounded, then a certain condition at infinity for u_2 will be required.
Problem (1) is the Dirichlet problem for the Helmholtz operator. It models, among others, a phenomenon of waves propagation in a fluid surrounding.
The finite element method in Ω_1 is based on the first Green formula [10]:
For $u_1 \in H^1(\Omega_1)$ and $f_1 \in IL^2(\Omega_1)$, we have

$$-\int_{\Omega_1} \left(\Delta + k^2 I\right) u_1 w \, dx = \Phi_{\Omega_1}\left(u, w\right) - \int_{\partial\Omega_1} \frac{\partial u_1}{\partial \eta_1} w \, ds \ , \ \ \forall w \in H^1(\Omega_1) \tag{3}$$

where

$$\Phi_{\Omega_1}\left(u, w\right) = \int \left(\nabla u \nabla w - k^2 u \, w\right) dx \tag{4}$$

with

$$w_{|\partial\Omega_1} \in H^{1/2}(\partial\Omega_1) \quad and \quad \frac{\partial u_1}{\partial \eta_1} \in H^{-1/2}(\partial\Omega_1).$$

Here Φ_{Ω_1} is a continuous bilinear form on the Sobolev space $H^1(\Omega_1)$.
The homogeneous Dirichlet condition on Γ_1 is incorporated into the function space:

$$H_0^1(\Omega_1) = \left\{ \ u \in H^1(\Omega_1) : \ u_{|\Gamma_1} = 0 \right\}.$$

By substituting equations (2-3) into equation (4), we have the following weak formulation on Ω_1:

$$\Phi_{\Omega_1}\left(u, w\right) - \int_{\Gamma_c} \frac{\partial u_2}{\partial \eta_2} w \, ds = \int_{\Omega_1} f_1 \, w \, dx \ , \ \ \forall w \in H_0^1(\Omega_1)$$

or

$$\Phi_{\Omega_1}\left(u,w\right)-\langle\varphi,z\rangle=\int_{\Omega_1} f_1 w\,dx\ ,\ \ \forall w\in H_0^1(\Omega_1)\ ,\quad with\quad z=w|_{\Gamma_0}\,. \tag{5}$$

Relation (5) is equivalent to the following problem:

$$\begin{cases} Find\ \left(u_1,\varphi\right)\ in\ H_0^1(\Omega_1)\times H^{-1/2}(\Gamma_c)\ such\ that \\ \left(\Delta+k^2 I\right)u_1=-f_1\ , \quad in\ \Omega_1 \\ \dfrac{\partial u_1}{\partial\eta_1}=\varphi\ , \quad on\ \Gamma_c\,. \end{cases} \tag{6}$$

The unknowns are, in this case, $u=u_1\in H_0^1(\Omega_1)\ \ and\ \ \varphi\in H^{-1/2}(\Gamma_c)$.
For the integral equation method in Ω_2, we need the knowledge of a fundamental solution of the partial differential operator. In our case it is given by

$$E\left(x,y\right)=-\frac{i}{4}H_1^{(0)}\left(k|x-y|\right)\ , \tag{7}$$

where $H_1^{(0)}$ is the Hankel function of the first kind [11].
Now, from the direct method of integral equations, which is based on the second Green formula, we obtain the representation formula on Ω_2 [12]:

$$u_2(x)=\int_{\Omega_2}\left\{\frac{\partial E(x,y)}{\partial\eta_2}v(y)-E(x,y)\psi(y)\right\}ds_y\ ,\ x\in\Omega_2 \tag{8}$$

where the Cauchy data $\left(v,\psi\right)$ are defined as:

$$\left(v,\psi\right)=\left(u_2|_{\partial\Omega_2},\frac{\partial u_2}{\partial\eta_2}\bigg|_{\partial\Omega_2}\right).$$

Using the normal derivative in (8), we find a second representation formula:

$$\frac{\partial u_2(x)}{\partial\eta_2}=\frac{\partial}{\partial\eta_2}\left[\int_{\partial\Omega_2}\left\{\frac{\partial E(x,y)}{\partial\eta_2}v(y)-E(x,y)\psi(y)\right\}ds_y\right],\ x\in\Omega_2 \tag{9}$$

Now, based on Potential theory [13], by a limiting process to the boundary Γ_2, we obtain a system of integral equations as follows:

$$\begin{cases} v=\dfrac{1}{2}[(K+I)v-V\psi]\ , & x\in\partial\Omega_2 \\ \psi=\dfrac{1}{2}[-Dv-(K'-I)\psi]\ , & x\in\partial\Omega_2 \end{cases} \tag{10}$$

where

$$V\psi(x)=-2\int_{\partial\Omega_2}\psi(y)E(x,y)ds_y\ \ ;\ \ Kv(x)=-2\int_{\partial\Omega_2}v(y)\frac{\partial E(x,y)}{\partial\eta_y}ds_y\ \ ,\ \ x\in\partial\Omega_2$$

$$K'\psi(x)=-2\int_{\partial\Omega_2}\psi(y)\frac{\partial E(x,y)}{\partial\eta_x}ds_y\ \ ;\ \ Dv(x)=-\frac{\partial}{\partial\eta_x}Kv(x)\ \ ,\ \ x\in\partial\Omega_2\,.$$

The operators V, K, K′, and D are pseudodifferential operators on Γ_2 of order -1, 0, 0 and +1, respectively [14, 15].

Theorem 2. 1. The operators V and D are continuous and coercive in the following sense:
There exists a constant c > 0 and compact operators T_V , T_D *on* $H^{-1/2}(\partial\Omega_2)$ into $H^{1/2}(\partial\Omega_2)$ and on $H^{1/2}(\partial\Omega_2)$ into $H^{-1/2}(\partial\Omega_2)$, respectively, such that:

$$\begin{aligned} \langle V\varphi,\varphi\rangle &\geq c\|\varphi\|_{-1/2} - \langle T_V\varphi,\varphi\rangle \quad , \quad \forall\varphi \in H^{-1/2}(\partial\Omega_2) \\ \langle Dv,v\rangle &\geq c\|v\|_{1/2} - \langle T_D v,v\rangle \quad , \quad \forall v \in H^{1/2}(\partial\Omega_2) \end{aligned} \tag{11}$$

Proof: See [15].

3. WEAK FORMULATION OF THE COUPLING PROBLEM

We distinguish between two different methods of coupling. One is a standard procedure and has been applied successfully in [8] for the Laplacian. The second method, being more efficient, is called the symmetric method for the coupling, and has been introduced by Costabel in [9]. In our case we have

Proposition 3. 1. The weak formulation of the coupled problem is given by

$$\begin{cases} Find \;\; (u,\varphi) \in H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2) \quad such \ that \\ a(u,\varphi;w,\psi) = l(w,\psi) \quad , \quad \forall (w,\psi) \in H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2) \end{cases} \tag{12}$$

where

$$a(u,\varphi\ ;w,\psi) = \Phi_{\Omega_1}(u,w) - \left\langle \varphi, z\right\rangle + \left\langle (I-P)\begin{pmatrix} v \\ \varphi \end{pmatrix}; \begin{pmatrix} z \\ -\psi \end{pmatrix}\right\rangle \tag{13}$$

and

$$l(u,w) = \int_{\Omega_1} f_1 w\, dx - \left\langle (I-P)\begin{pmatrix} g_2 \\ 0 \end{pmatrix}; \begin{pmatrix} z \\ -\psi \end{pmatrix}\right\rangle \ . \tag{14}$$

Here $z = w|_{\Gamma_c}$, $v = u_1|_{\Gamma_c}$ and P is the Calderon projector in Ω_2.

Proof: Let the test function $\psi \in H^{-1/2}(\partial\Omega_2)$ be given. Multiply the first equation of (10) by ψ and integrate on $\partial\Omega_2$.Then multiply the second equation of the same system (10) by z and integrate over Γ_c. Finally, adding the two equations with (5), we obtain the desired result.

Theorem 3. 2. The bilinear form a(.,.) defined on $H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2)$ is
i) Symmetric.
ii) Continuous on $H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2)$.

Proof:
i) The symmetry property is evident from the symmetry of the bilinear form $\Phi_{\Omega_1}(.,.)$ and the self-adjointness of operators V and D.
ii) Continuity results from the continuity of operator $(I-P)$ and bilinear form $\Phi_{\Omega_1}(.,.)$.

Since the bilinear form (13) is symmetric, the problem (12) is equivalent to the minimization problem of functional J_1 given as

$$J_1(u,\varphi) = \frac{1}{2} a\Big(u,\varphi; u,\varphi\Big) - l(u,\varphi) \quad , \quad \forall (u,\varphi) \in H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2)\,. \tag{15}$$

Now, following the idea of [16], we define the following functional:

Definition 3. 3. For $(u,\varphi) \in H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2)$, we define

a)
$$\begin{aligned} &q : H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2) \to IR \\ &(u,\varphi) \to q(u,\varphi) = \frac{1}{2}\Big\langle (K - I)\, v, \varphi \Big\rangle \quad , \quad where \quad v = u_1|_{\Gamma_c} \,, \end{aligned} \tag{16}$$

b)
$$\begin{aligned} &b : H^{-1/2}(\partial\Omega_2) \to IR \\ &\varphi \to b(\varphi) = \frac{1}{4}\Big\langle V\varphi, \varphi \Big\rangle \,, \end{aligned} \tag{17}$$

c)
$$\begin{aligned} &J_0 : H_0^1(\Omega_1) \to IR \\ &u \to J_0(u) = \frac{1}{2}\Phi_{\Omega_1}(u,u) - \int_{\Omega_1} f_1 u\, ds - \left\langle (I - P)\begin{pmatrix} g_2 \\ 0 \end{pmatrix}; \begin{pmatrix} v \\ -\varphi \end{pmatrix} \right\rangle. \end{aligned} \tag{18}$$

So we now consider the following problem:

$$\begin{cases} Find \quad (u,\varphi) \in H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2) \quad such\ that: \\ J_1(u_1,\varphi) = \inf \Big\{ J_1(w,\psi) = J_0(w) + q(w,\psi) - b(\psi) \Big\}, \\ or \quad \Big\langle J_1'\Big(u_1,\varphi\Big); \Big(w,\psi\Big) \Big\rangle = 0 \quad , \quad \forall (w,\psi) \in H_0^1(\Omega_1) \times H^{-1/2}(\partial\Omega_2). \end{cases} \tag{19}$$

where J' is the Gâteaux derivative of functional J.

Now for the rest of this work we consider a part ω_1 of the domain Ω_1 sufficiently small with a diameter ρ.

We will first prove the following lemma:

Lemma 3. 4. The functional J_0 is twice continuously differentiable and there exist two constants $\lambda_{\rho,k^2}, \mu > 0$ such that for all $u, w \in H_0^1(\Omega_1)$,

$$\lambda_{\rho,k^2} \| w \|_1^2 \le \Big\langle J_0''(u_1) w \,; w \Big\rangle \le \mu \| w \|_1^2 \;. \tag{20}$$

Proof: By calculating the second derivative of functional J_0 we find the relation

$$\Big\langle J_0''(u) w \,; w \Big\rangle = \Phi_{\Omega_1}(w,w)\,.$$

Now, the result is immediate from the Poincaré inequality and the continuity of Φ_{Ω_1}.

Theorem 3. 5.

i) If J_0 is Gâteaux differentiable, so is J_1.

ii) For $\forall u \in H_0^1(\Omega), \quad J_0'(u) = 0 \Leftrightarrow \exists \varphi \in H^{-1/2}(\partial\Omega_2) \quad with \quad J_1'(u,\varphi) = 0$.

In this case, $\varphi = V^{-1}(K' - I)\,u$.

Proof:

i) Follows from the definition of Gâteaux differentiability. So,

$$J_1'(u,\varphi) = \left(J_0'(u), -\frac{1}{2}(K' - I)u + \frac{1}{2}V\varphi\right).$$

As $J_0'(u)$ exists, i) is verified.
ii) If $J_0'(u) = 0$, it is sufficient to take $\varphi = V^{-1}(K' - I)u$, to have $J_1'(u,\varphi) = (0,0)$.
Conversely, if there exists $\varphi \in H^{-1/2}(\partial\Omega_2)$ *such that* $J_1'(u,\varphi) = (0,0)$, then

$$\left(J_0'(u), -\frac{1}{4}(K' - I)u + \frac{1}{4}V\varphi\right) = (0,0)$$

where $J_0'(u) = 0$ *and* $\varphi = V^{-1}(K' - I)u$.

Theorem 3. 6. The problem of finding $u \in H_0^1(\omega_1)$ such that

$$J_0(u) = \inf\left\{\frac{1}{2}\Phi_{\Omega_1}(w,w) - \int_{\Omega_1} f_1 w\,dx - \left\langle (I-P)\begin{pmatrix} g_2 \\ 0 \end{pmatrix}; \begin{pmatrix} v \\ -\varphi \end{pmatrix}\right\rangle, \quad w \in H_0^1(\Omega_1)\right\} \tag{21}$$

has exactly one solution in $H_0^1(\omega_1)$.

Proof: From Lemma 3. 4, it is clear that J_0 has exactly one critical point u which is a minimum, since J_0 is coercive, lower semi continuous, and strictly convex. Now, we can establish the following result.

Theorem 3. 7. The problem (19) has a unique solution in $H_0^1(\omega_1) \times H^{-1/2}(\partial\Omega_2)$.

Proof: From Theorem 3. 6, the Euler equation $\left\langle J_0'(u), w\right\rangle = 0$ has a critical point. So by part (ii) of the Theorem 3. 5, $\left\langle J_1'\left(u,\varphi\right);\left(w,\psi\right)\right\rangle = 0$.

Corollary 3. 8. The point $\left(u,\varphi\right)$ is a saddle point of functional J_1, i.e

$$J_1\left(u,\varphi+\psi\right) \le J_1\left(u,\varphi\right) \le J_1\left(u+w,\varphi\right) \quad , \quad \forall (w,\psi) \in H_0^1(\omega_1) \times H^{-1/2}(\partial\Omega_2).$$

Proof:
1) Let $L_\varphi(t) = J_1\left(t,\varphi\right)$ *for* $t \in H_0^1(\omega_1)$. We can easily verify that $L_\varphi''(t) = J_0''(t)$.
However, relation (21) implies that the minimization problem of functional L_φ has a solution. Then

$$L_\varphi(u) \le L_\varphi(t) \quad , \quad \forall t \in H_0^1(\Omega_1) \quad , \quad where \quad J_1\left(u,\varphi\right) \le J_1\left(t,\varphi\right) \quad , \quad \forall t \in H_0^1(\omega_1).$$

Thus

$$J_1\left(u,\varphi\right) \le J_1\left(u+w,\varphi\right) \quad , \quad \forall t \in H_0^1(\omega_1).$$

2) To show the left hand side inequality, we consider

$$J_1\left(u,\varphi+\psi\right) = J_0(u) + q\left(u,\varphi+\psi\right) - b(\varphi+\psi).$$

So, we replace φ by $V^{-1}(K' - I)u$ and use the fact that V is a positive operator.

4. GALERKIN APPROXIMATION

The most important contribution of this section is the proof of the error estimate and the convergence of the Galerkin approximation.

Let X_n and Y_n be two closed subspaces of $H_0^1(\omega_1)$ and $H^{-1/2}(\partial\Omega_2)$ respectively. The spaces X_n and Y_n are usually finite dimensional with

$$\lim_{n\to 0}\inf\|w-v\|_1=0 \quad , \quad v\in X_n \quad ; \quad \lim_{n\to 0}\inf\|\psi-\chi\|_{-1/2}=0 \quad , \quad \chi\in Y_n .$$

The restriction of J_1 to $X_n\times Y_n$ inherits all relevant properties from J_1 to

$$H_0^1(\Omega_1)\times H^{-1/2}(\partial\Omega_2).$$

It has exactly one critical point, $\left(u_n,\varphi_n\right)\in X_n\times Y_n$, which is the approximate solution of the variational problem. For the points $\left(u,\varphi\right)$ and $\left(u_n,\varphi_n\right)$, the following Euler-Lagrange equations hold

$$\left\langle J_1'\left(u,\varphi\right);\left(w,\psi\right)\right\rangle=0 \quad , \quad \forall w\in H_0^1(\Omega_1) \quad , \quad \forall\psi\in H^{-1/2}(\partial\Omega_2). \tag{22}$$

$$\left\langle J_1'\left(u_n,\varphi_n\right);\left(w,\psi\right)\right\rangle=0 \quad , \quad \forall w\in X_n \quad , \quad \forall\psi\in Y_n. \tag{23}$$

Definition 4. 1. The equations in (23) are called Galerkin equations corresponding to the weak form (22). In the rest of the paper we will need some elementary consequences of the following two lemmas.

Lemma 4. 2. Let the functional J_0 be defined by (18). Then, there exist two constants $\alpha>0$, $\beta>0$ such that:

i)
$$\alpha\|v-w\|_1^2\le\left\langle J_0'(v),v-w\right\rangle-\left\langle J_0'(w),v-w\right\rangle \quad , \quad \forall v,w\in H_0^1(\omega_1) \tag{24}$$

ii)
$$\frac{\alpha}{2}\|v-w\|_1^2\le J_0(v)-J_0(w)-\left\langle J_0'(w),v-w\right\rangle\le\frac{\beta}{2}\|v-w\|_1^2 \quad , \quad \forall v,w\in H_0^1(\omega_1) \tag{25}$$

iii) The restriction of J_0 to X_n has a unique minimum $u_n^*\in X_n$, and the following inequality holds:

$$\frac{\alpha}{2}\left\|u-u_n^*\right\|_1^2\le J_0\left(u_n^*\right)-J_0(u_1)\le\frac{\beta}{2}\inf\|u-w\|_1^2 \quad , \quad \forall\, w\in X_n . \tag{26}$$

Proof:
i) We select $h(\theta)=\left\langle J_0'(u+\theta(v-u)),v-u\right\rangle$, and apply the mean value theorem over h for $\theta\in\left[0,1\right]$. Finally, we use relation (21).
ii) Concerning (25), we put $h(\theta)=\left\langle J_0(u+\theta(v-u)),v-u\right\rangle$. We then apply the same theorem twice.
iii) It is evident according to (21) that the restriction of J_1 to X_n possesses a minimum which we denote by u_n^*, so $J_0(w)\ge J_0\left(u_n^*\right) \quad , \quad \forall w\in X_n$.
From ii) we obtain

$$\frac{\alpha}{2}\left\|u-u_n^*\right\|_1^2\le J_0(u)-J_0\left(u_n^*\right)-\left\langle J_0'(u),u-u_n^*\right\rangle\le\frac{\beta}{2}\left\|u-u_n^*\right\|_1^2 . \tag{27}$$

As u is a solution of the Euler equation (22), so (27) becomes

$$\frac{\alpha}{2}\left\|u-u_n^*\right\|_1^2\le J_0(u)-J_0\left(u_n^*\right)\le\frac{\beta}{2}\left\|u-u_n^*\right\|_1^2 .$$

Finally, we have

$$\frac{\alpha}{2}\left\|u-u_n^*\right\|_1^2 \leq J_0(u)-J_0\left(u_n^*\right) \leq \frac{\beta}{2}\inf\|u-w\|_1^2 \quad , \quad \forall\, w \in X_n .$$

Now, we denote by J_{1n}, the restriction of J_1 to $X_n \times Y_n$. In this case we have

$$\begin{aligned} J_{1n}\left(w,\psi\right) &= J_0(w)+q\left(w,\psi\right)-b(\psi) \quad , \quad \forall\left(w,\psi\right) \in X_n \times Y_n \\ &= J_0(w)+\frac{1}{2}\left\langle Q_n\psi,w\right\rangle-\frac{1}{4}\left\langle A_n\psi,\psi\right\rangle \end{aligned} \tag{28}$$

Here the operators A_n and Q_n are defined as follows:

$$\begin{aligned} &Q_n : Y_n \rightarrow X_n' \quad , \quad X_n' \ \textit{is the dual of} \ X_n, \\ &\psi \rightarrow \left\langle (K-I)\psi,w\right\rangle \quad , \quad \forall\left(w,\psi\right) \in X_n \times Y_n \end{aligned}$$

$$\begin{aligned} &A_n : Y_n \rightarrow Y_n' \quad , \\ &\psi \rightarrow \left\langle A_n\psi,\chi\right\rangle \quad , \quad \forall \psi,\chi \in Y_n \end{aligned}$$

A_n is a linear, continuous, bijective and positive self-adjoint operator.

Lemma 4. 3. Let $z \in H^{-1/2}(\Gamma_c)$ be given, $\psi = V^{-1}z$ and $\psi_n = V_n^{-1}P_n z$ with $P_n : H^{-1/2} \rightarrow Y_n'$ being the natural projection. Then

$$\|\psi-\psi_n\|_{-1/2} \leq C\inf\|\psi-\chi\|_{-1/2} \quad , \quad \forall \psi,\chi \in Y_n . \tag{29}$$

Proof: This lemma is a special case of the Cea lemma [10].

Theorem 4. 4. There exists a unique solution $\left(u_n,\varphi_n\right) \in X_n \times Y_n$ of the Galerkin equation

$$\left\langle J_1'\left(u_n,\varphi_n\right);\left(w,\psi\right)\right\rangle = 0 \quad , \quad \forall w \in X_n \quad , \quad \forall \psi \in Y_n .$$

In addition, there exists a constant c > 0 such that

$$\|u-u_n\|_{H_0^1}+\|\varphi-\varphi_n\|_{H^{-1/2}} \leq c\inf\left\{\|u-w\|_{H_0^1}+\|\varphi-\psi\|_{H^{-1/2}} \ \middle|\ (w,\psi) \in X_n \times Y_n\right\} \tag{30}$$

Proof: By using inequality (24), the definition of operators P_n, Q_n and Theorem 3. 5, we obtain

$$\left\|u_n-u_n^*\right\|_1^2 \leq c\left\{\inf\|w-u\|_1^2+\inf\|\varphi-\psi\|_{-1/2}^2 \ , \ \forall (w,\psi) \in X_n \times Y_n\right\}. \tag{31}$$

Now according to (18) we have

$$\left\|u-u_n^*\right\|_1^2 \leq c\inf\|w-u\|_1^2 \ , \ \forall w \in X_n. \tag{32}$$

On the other hand,

$$\|u_n-u\|_1 = \left\|u_n-u_n^*+u_n^*-u\right\|_1 \leq \left\|u_n-u_n^*\right\|_1+\left\|u_n^*-u\right\|_1 .$$

So, from (31) and (32), we have

$$\|u_n-u\|_1^2 \leq c\left\{\inf\|w-u\|_1^2+\inf\|\varphi-\psi\|_{-1/2}^2 \ , \ \forall (w,\psi) \in X_n \times Y_n\right\}. \tag{33}$$

Let us now estimate $\|\varphi-\varphi_n\|_{H^{-1/2}}$. Let us take $\varphi_n^* = V_n^{-1}P_n(K'-I)u$. Then

$$\|\varphi-\varphi_n\|_{-1/2}\le\left\|\varphi-\varphi_n^*\right\|_{-1/2}+\left\|\varphi_n^*-\varphi_n\right\|_{-1/2}.$$

So, according to the definition of Q_n and as $j=V^{-1}(K'-I)u$, we have

$$\left\|\varphi_n^*-\varphi_n\right\|_{-1/2}\le c\|u-u_n\|_1.$$

Finally, Lemma 4. 3 implies that

$$\left\|\varphi_n^*-\varphi\right\|_{-1/2}\le c\inf\|\varphi-\psi\|_{-1/2}.$$

Therefore

$$\|\varphi-\varphi_n\|_{-1/2}\le c\left\{\inf\|w-u\|_1^2+\inf\|\varphi-\psi\|_{-1/2}^2\quad,\quad\forall(w,\psi)\in X_n\times Y_n\right\}\tag{34}$$

which together with (33) and (34) completes the proof of inequality (31).

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