

N*-ARY HYPERGROUPS

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Abstract – In this paper the class of n -ary hypergroups is introduced and several properties are found and examples are presented. n -ary hypergroups are a generalization of hypergroups in the sense of Marty. On the other hand, we can consider n -ary hypergroups as a good generalization of n -ary groups. We define the fundamental relation β^* on an n -ary hypergroup H as the smallest equivalence relation such that H / β^* is the n -ary group, and then some related properties are investigated.

Keywords – hypergroup, n -ary hypergroup, n -ary group, fundamental equivalence relation

1. INTRODUCTION

Hypergroup, which is based on the notion of hyperoperation, has been introduced by Marty in [1] and studied extensively by many mathematicians. For example, the connection between hypergraphs and hypergroups is studied by Corsini [2]. In [3], Corsini and Leoreanu described hypergroups associated with trees and in [4] some applications of hyperstructures in rough sets are given. The hypergroup theory both extends some well-known group results and introduces new topics, thus leading to a wide variety of applications, as well as to a broadening of the investigation fields. A comprehensive review of the theory of hyperstructures appears in [5-8].

The notion of an n -ary group was introduced by Dörnte [9], which is a natural generalization of the notion of a group. n -ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. Since then many papers concerning various n -ary algebra have appeared in the literature, (for example see [10-15]).

In this paper, n -ary hypergroups are defined and considered. Examples of n -ary hypergroups are given and some of their properties described. n -ary hypergroups are a generalization of hypergroups in the sense of Marty. Also, we can consider n -ary hypergroups as a good generalization of n -ary groups. We define the fundamental relation β^* on an n -ary hypergroup H as the smallest equivalence relation such that H / β^* is the n -ary group, and then some related properties are investigated.

2. BASIC DEFINITIONS AND RESULTS

Let H be a non-empty set and f be a mapping $f : H \times H \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all non-empty subsets of H . Then f is called a *binary hyperoperation* on H . We denote by H^n the cartesian product $H \times \cdots \times H$, where H appears n times. An element of H^n will be denoted by (x_1, \dots, x_n) , where $x_i \in H$ for any i with $1 \leq i \leq n$. In general, a mapping $f : H^n \longrightarrow P^*(H)$ is called an *n -ary*

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hyperoperation and n is called the *arity of hyperoperation*.

Let f be an n -ary hyperoperation on H and A_1, \dots, A_n subsets of H . We define

$$f(A_1, \dots, A_n) = \bigcup \{ f(x_1, \dots, x_n) \mid x_i \in A_i, i = 1, \dots, n \}.$$

We shall use the following abbreviated notation: the sequence x_i, x_{i+1}, \dots, x_j will be denoted by x_i^j . For $j < i$, x_i^j is the empty set. In this convention

$$f(x_1, \dots, x_i, y_{i+1}, \dots, y_j, z_{j+1}, \dots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$.

Definition 2. 1. A non-empty set H with an n -ary hyperoperation $f : H^n \longrightarrow P^*(H)$ will be called an n -ary hypergroupoid and will be denoted by (H, f) . An n -ary hypergroupoid (H, f) will be called an n -ary semihypergroup if and only if the following associative axiom holds:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{x+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for every $i, j \in \{1, 2, \dots, n\}$ and $x_1, x_2, \dots, x_{2n-1} \in H$.

If for all $(a_1, a_2, \dots, a_n) \in H^n$, the set $f(a_1, a_2, \dots, a_n)$ is singleton, then f is called an n -ary operation and (H, f) is called an n -ary groupoid (resp. n -ary semigroup).

If $m = k(n-1) + 1$, then the m -ary hyperoperation g given by

$$g(x_1^{k(n-1)+1}) = f(\underbrace{f(\dots, f(x_1^n), x_{n+1}^{2n-1}), \dots}_{k}, x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role, or when it will differ depending on additional assumptions, we write f_{\circ} , to mean $f_{(k)}$ for some $k = 1, 2, \dots$.

Definition 2. 2. An n -ary semihypergroup (H, f) , in which the equation

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n) \tag{*}$$

has the solution $x_i \in H$ for every $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b \in H$ and $1 \leq i \leq n$, is called an n -ary hypergroup.

In Definition 2.2, if f is n -ary operation then the equation (*) is as follows:

$$b = f(a_1^{i-1}, x_i, a_{i+1}^n). \tag{**}$$

In this case (H, f) is an n -ary group.

The important question is the solvability of the equation (*). The classical n -ary semigroup is an n -ary group if and only if the equation (**) is solvable at the place $i = 1$ and $i = n$, or at least one place $1 < i < n$, (see [12] or [13]). The following theorem shows that it is true for hypergroups.

Theorem 2. 3. Let (H, f) be an n -ary semihypergroup. Then (H, f) is an n -ary hypergroup if and only if the equation (*) is solvable at the place $i = 1$ and $i = n$ or at least one place $1 < i < n$.

Proof: If (*) is solvable at the place $i = 1$ and $i = n$, then for every $a_1, \dots, a_n, b \in H$ there exist $x_0, z_0 \in H$ such that

$$b \in f(x_0, a_2^n) \text{ and } x_0 \in f(a_1^{n-1}, z_0).$$

Assume that $1 < j < n$ be arbitrary. Then

$$b \in f(f(a_1^{n-1}, z_0), a_2^n) = f(a_1^{j-1}, f(a_j^{n-1}, z_0, a_2^j), a_{j+1}^n).$$

Therefore there exists $x \in f(a_j^{n-1}, z_0, a_2^j)$ such that $b \in f(a_1^{j-1}, x, a_{j+1}^n)$.

Now, let (*) be solvable at place $1 < i < n$. Assume that $j < i$, then for every $a_1, \dots, a_n, b \in H$ there exists $y_1 \in H$ such that

$$b \in f(a_1^{i-1}, y_1, \underbrace{f(a_1, \dots, a_1, a_{j+1}^{i+1})}_{n-(i-j+1)}, a_{i+2}^n)$$

and so

$$b \in f(a_1^{j-1}, f(a_j^{i-1}, y_1, \underbrace{a_1, \dots, a_1}_{n-(i-j+1)}), a_{j+1}^n).$$

Therefore there exists $x \in f(a_j^{i-1}, y_1, a_1, \dots, a_1)$ such that $b \in f(a_1^{j-1}, x, a_{j+1}^n)$. If we choose $i < j$, then similarly we can prove that (*) is solvable.

Definition 2.2 is a generalization of Marty's formulation of axiom of a hypergroup. Let \circ be a binary algebraic hyperoperation on H , then (H, \circ) is called a *hypergroupoid*. A *hypergroup* is a hypergroupoid (H, \circ) that satisfies:

- 1) $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$,
- 2) $x \circ H = H \circ x = H$ for all $x \in H$.

The second condition is frequently used in the form: Given $a, b \in H$, there exist $u, v \in H$ such that $b \in a \circ u$ and $b \in v \circ a$.

Condition 2 can be formulated for n -ary hypergroups as follows:

$$f(H^{i-1}, x, H^{n-i}) = H$$

for all $x \in H$ and $i = 1, \dots, n$.

Let (H, f) be an n -ary hypergroup, $a_2^{n-1} \in H$ be fixed and let $x \odot y = f(x, a_2^{n-1}, y)$. Then the hypergroupoid (H, \odot) is a hypergroup and it is called a *retract of the n -ary hypergroup* (H, f) .

Example 2. 4. Let $H = \{x, y, z\}$ be a set with a 3-ary hyperoperation f as follows:

$$\begin{array}{lll} f(x, x, x) = x & f(y, y, x) = \{x, z\} & f(z, x, x) = z \\ f(x, x, y) = y & f(y, y, y) = \{y, z\} & f(z, x, y) = \{y, z\} \\ f(x, x, z) = z & f(y, y, z) = H & f(z, x, z) = \{x, y\} \\ f(x, y, x) = y & f(y, x, x) = y & f(z, y, x) = \{y, z\} \\ f(x, y, y) = \{x, z\} & f(y, x, y) = \{x, z\} & f(z, y, y) = H \\ f(x, y, z) = \{y, z\} & f(y, x, z) = \{y, z\} & f(z, y, x) = H \\ f(x, z, x) = z & f(y, z, x) = \{y, z\} & f(z, z, x) = \{x, y\} \\ f(x, z, y) = \{y, z\} & f(y, z, y) = H & f(z, z, y) = H \\ f(x, z, z) = \{x, y\} & f(y, z, z) = H & f(z, z, z) = \{y, z\}. \end{array}$$

For every $x_i \in H$ ($i = 1, \dots, 5$), we have

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$$

i.e., f is associative, and it is easy to see that f is a 3-ary hypergroup.

Let (H, f) be an n -ary hypergroup. If the value of $f(x_1, x_2, \dots, x_n)$ is independent on the permutation of elements x_1, x_2, \dots, x_n , then (H, f) is called a *commutative n -ary hypergroup*.

The element $a \in H$ is called a *scalar* if

$$\left| f(x_1^i, a, x_{i+2}^n) \right| = 1$$

for all $x_1, \dots, x_i, x_{i+2}, \dots, x_n \in H$.

Element e of an n -ary hypergroup (H, f) is called a *neutral (identity) element* if

$$f(\underbrace{e, \dots, e}_{i-1}, x, \underbrace{e, \dots, e}_{n-i})$$

includes x , for all $x \in H$ and all $1 \leq i \leq n$.

Lemma 2. 5. *Let (H, f) be a commutative n -ary hypergroup and $a \in H$ a scalar element such that $f(a, e, \dots, e) = a$ for some $e \in H$. Then e is a neutral element.*

Proof: We have

$$f(f(x, a, \underbrace{e, \dots, e}_{n-2}, \underbrace{e, \dots, e}_{n-1}), \underbrace{e, \dots, e}_{n-1}, \underbrace{e, \dots, e}_{n-2}) = f(x, a, \underbrace{e, \dots, e}_{n-2}).$$

Since every element of H is representable in the form $f(x, a, e, \dots, e)$ and f is commutative, this means that e is a neutral element.

It is to be noted that in Lemma 2.4, the condition $f(a, x, \dots, x) = a$ can be replaced by the condition $f(x, \dots, x, a, x, \dots, x) = a$, where a appears at one fixed place $i = 1, \dots, n$.

Proposition 2. 6. *If the set of all scalar neutral elements of a given commutative n -ary hypergroup is non-empty, then it is an n -ary group.*

Proof: To prove that the set N_H of all scalar neutral elements is closed under the hyperoperation f , let $a = f(e_1^n)$, where $e_1, \dots, e_n \in N_H$. Then

$$\begin{aligned} f(\underbrace{a, \dots, a}_{i-1}, x, \underbrace{a, \dots, a}_{n-i}) &= f(f(\underbrace{e_1^n}_{i-1}), \dots, f(\underbrace{e_1^n}_{i-1}), x, f(\underbrace{e_1^n}_{i-1}), \dots, f(\underbrace{e_1^n}_{i-1})) \\ &= f(\underbrace{e_1, \dots, e_1}_{n-1}, f(\underbrace{e_2, \dots, e_2}_{n-1}), f(\dots, f(\underbrace{e_{n-1}, \dots, e_{n-1}}_{n-1}), f(\underbrace{e_n, \dots, e_n}_{n-1}, x)), \dots)) \\ &= f(\underbrace{e_1, \dots, e_1}_{n-1}, f(\underbrace{e_2, \dots, e_2}_{n-1}), f(\dots, f(\underbrace{e_{n-1}, \dots, e_{n-1}}_{n-1}, x), \dots)) \\ &= \dots = f(\underbrace{e_1, \dots, e_1}_{n-1}, x) = x, \end{aligned}$$

which proves that an element $a = f(e_1^n)$ is neutral. Therefore, N_H is closed under f . Also, for all $e_2, \dots, e_n, e \in N_H$, the equation $e = f(x, e_2^n)$ has the solution

$$x = f(\dots, \underbrace{e, e_n, \dots, e_n}_{n-2}, \underbrace{e_{n-1}, \dots, e_{n-1}}_{n-2}, \dots, \underbrace{e_3, \dots, e_3}_{n-2}, \underbrace{e_2, \dots, e_2}_{n-2})$$

which is contained in N_H .

Definition 2. 7. Let (H, f) be an n -ary hypergroup and B be a non-empty subset of H . Then B is an *n -ary Iranian Journal of Science & Technology, Trans. A, Volume 30, Number A2*

subhypergroup of H if the following conditions hold:

- 1) B is closed under the n -ary hyperoperation f , i.e., for every $(x_1, \dots, x_n) \in B^n$ implies that $f(x_1, \dots, x_n) \subseteq B$.
- 2) Equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has the solution $x_i \in B$ for every $b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n, b \in B$ and $1 \leq i \leq n$.

Definition 2. 8. Let (A, f) and (B, g) be two n -ary hypergroups. A *homomorphism* from A to B is a mapping $\varphi : A \longrightarrow B$ such that

$$\varphi(f(a_1, \dots, a_n)) = g(\varphi(a_1), \dots, \varphi(a_n))$$

holds for all $a_1, \dots, a_n \in A$.

If φ is injective, then it is called an *embedding*. The map φ is an *isomorphism* if φ is injective and onto. We say that A is *isomorphic* to B , denoted by $A \cong B$, if there is an isomorphism from A to B .

Theorem 2. 9. Let (A, f) and (B, g) be two n -ary hypergroups and $\varphi : A \longrightarrow B$ a homomorphism. Then

- 1) If S is an n -ary subhypergroup of A , then $\varphi(S)$ is an n -ary subhypergroup of B ,
- 2) If K is an n -ary subhypergroup of B such that $\varphi^{-1}(K) \neq \phi$, then $\varphi^{-1}(K)$ is an n -ary subhypergroup of A .

Proof: 1) Suppose that $y_1, \dots, y_n \in \varphi(S)$. Then there exist $x_1, \dots, x_n \in S$ such that $\varphi(x_i) = y_i$ for all $1 \leq i \leq n$. We have $\varphi(f(x_1, \dots, x_n)) \subseteq \varphi(S)$ and so $g(\varphi(x_1), \dots, \varphi(x_n)) \subseteq \varphi(S)$ or $g(y_1, \dots, y_n) \subseteq \varphi(S)$. Therefore the first condition of Definition 2.7 is satisfied. For the second condition of Definition 2.7, we consider the equation $b \in g(b_1^{i-1}, x_i, b_{i+1}^n)$ for all $b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n, b \in \varphi(S)$. Then there exist $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, a \in S$ such that $\varphi(a) = b$ and $\varphi(a_i) = b_i$. Since S is an n -ary subhypergroup of A , the equation

$$a \in f(a_1^{i-1}, y_i, a_{i+1}^n)$$

has a solution $y_i \in S$. From the equation $a \in f(a_1^{i-1}, y_i, a_{i+1}^n)$ we obtain the equation $\varphi(a) \in \varphi(f(a_1^{i-1}, y_i, a_{i+1}^n))$ or $b \in g(b_1^{i-1}, \varphi(y_i), b_{i+1}^n)$. Therefore the equation $b \in g(b_1^{i-1}, x_i, b_{i+1}^n)$ has the solution $\varphi(y_i)$.

2) The proof of this part is similar to (1).

3. QUOTIENT N-ARY HYPERGROUPS

Let (H, f) be an n -ary hypergroup. An equivalence relation θ on H is called *compatible* if $a_1\theta b_1, \dots, a_n\theta b_n$, then for all $a \in f(a_1, \dots, a_n)$ there exists $b \in f(b_1, \dots, b_n)$ such that $a\theta b$. An equivalence relation θ is called *strongly compatible* if $a_1\theta b_1, \dots, a_n\theta b_n$ implies that $a\theta b$ for all $a \in f(a_1, \dots, a_n)$ and $b \in f(b_1, \dots, b_n)$.

Theorem 3. 1. Let (H, f) be an n -ary hypergroup and θ a compatible relation on H . Then $(H/\theta, f|_\theta)$ is an n -ary hypergroup where

$$f|_\theta(\theta(a_1), \dots, \theta(a_n)) = \{\theta(a) \mid a \in f(a_1, \dots, a_n)\}.$$

Proof: We shall use the following abbreviated notation: the sequence $\theta(a_i), \theta(a_{i+1}), \dots, \theta(a_j)$ will be denoted by $\theta_{a_i}^{a_j}$. Since θ is a compatible relation, then we conclude that $f|_\theta$ is well-defined. We show that $f|_\theta$ is associative. We have

$$\begin{aligned}
f|_{\theta}(\theta_{a_1}^{a_i-1}, f|_{\theta}(\theta_{a_i}^{a_{n+i-1}}, \theta_{a_{n+i}}^{a_{2n-1}})) &= \cup \{ f|_{\theta}(\theta_{a_1}^{a_i-1}, \theta(y), \theta_{a_{n+i}}^{a_{2n-1}}) \mid y \in f(a_i^{n+i-1}) \} \\
&= \{ \theta(a) \mid a \in f(a_1^{i-1}, y, a_{n+i}^{2n-1}), y \in f(a_i^{n+i-1}) \} \\
&= \{ \theta(a) \mid a \in f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1}) \} \\
&= \{ \theta(a) \mid a \in f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1}) \} \\
&= f|_{\theta}(\theta_{a_1}^{a_j-1}, f|_{\theta}(a_j^{n+j-1}), \theta_{a_{n+j}}^{a_{2n-1}}).
\end{aligned}$$

Therefore $f|_{\theta}$ is associative. Now, we consider the equation

$$\theta(b) \in f|_{\theta}(\theta_{a_1}^{a_i-1}, \theta(x_i), \theta_{a_{i+1}}^{a_n}) \quad (*)$$

for every $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b \in H$. Since H is an n -ary hypergroup, the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has the solution $x_i \in H$, and so $\theta(x_i)$ is a solution for (*).

The natural map $\pi : H \longrightarrow H/\theta$, where $\pi(x) = \theta(x)$ is an onto homomorphism.

Definition 3. 2. Let (A, f) and (B, g) be two n -ary hypergroups and let $\varphi : A \longrightarrow B$ be a homomorphism. Then the *kernel* φ , written $\ker\varphi$, is defined by

$$\ker\varphi = \{ (a, b) \in A^2 \mid \varphi(a) = \varphi(b) \}.$$

It is easy to see that $\ker\varphi$ is a compatible relation.

Theorem 3. 3. Let (A, f) and (B, g) be two n -ary hypergroups and let $\varphi : A \longrightarrow B$ be a homomorphism. Then there exists a compatible relation θ on A and a monomorphism $\psi : A/\theta \longrightarrow B$ such that $\psi \circ \pi = \varphi$.

Proof: We consider $\theta = \ker\varphi$. Now, let $\theta(a) \in A/\theta$ and define $\psi(\theta(a)) = \varphi(a)$.

Theorem 3. 4. Let ρ and θ be compatible relations on an n -ary hypergroup (H, f) such that $\rho \subseteq \theta$. Then there exists a compatible relation μ on $(H/\rho, f|_{\rho})$ such that $(H/\rho)/\mu$ is isomorphic to H/θ .

Proof: We consider the map $\varphi : H/\rho \longrightarrow H/\theta$ by $\varphi(\rho(x)) = \theta(x)$. Since $\rho \subseteq \theta$, φ is well-defined. Clearly φ is a homomorphism. Now, by Theorem 3.3, there exists a compatible relation μ and a monomorphism $\psi : (H/\rho)/\mu \longrightarrow H/\theta$ such that $\psi \circ \pi = \varphi$, and so ψ is an isomorphism.

The diagonal relation Δ on H is the set $\{ (a, a) \mid a \in H \}$ and the full relation H^2 is denoted by ∇ . The set of all equivalence relations on a set H , with \subseteq as the partial ordering, is a complete lattice. Let θ_1 and θ_2 be two equivalence relations on H . It is clear that $\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$. Also, we have

$$\theta_1 \vee \theta_2 = \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \cup \dots$$

We suppose that analogous results on other products of hyperstructures can be obtained [7], [16].

Definition 3. 5. Let (A_1, f_1) and (A_2, f_2) be two n -ary hypergroups. Define the direct hyperproduct $(A_1 \times A_2, f_1 \times f_2)$ to be the n -ary hypergroup whose universe is the set $A_1 \times A_2$ and such that for $a_i \in A_1$, $a'_i \in A_2$, $1 \leq i \leq n$,

$$(f_1 \times f_2)((a_1, a'_1), \dots, (a_n, a'_n)) = \{ (a, a') \mid a \in f_1(a_1, \dots, a_n), a' \in f_2(a'_1, \dots, a'_n) \}.$$

The mapping $\pi_i : A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, defined by $\pi_i((a_1, a_2)) = a_i$, is called the *projection map* on the i th coordinate of $A_1 \times A_2$. For $i = 1, 2$, the mapping $\pi_i : A_1 \times A_2 \longrightarrow A_i$ is an onto homomorphism. Furthermore, we have

- i) $\ker \pi_1 \cap \ker \pi_2 = \Delta$,
 - ii) $\ker \pi_1$ and $\ker \pi_2$ permute,
 - iii) $\ker \pi_1 \wedge \ker \pi_2 = \nabla$,
- where

$$\ker \pi_i = \{ ((a_1, a_2), (b_1, b_2)) \mid \pi_i(a_1, a_2) = \pi_i(b_1, b_2) \}, \quad i = 1, 2.$$

Note that

$$((a_1, a_2), (b_1, b_2)) \in \ker \pi_i \Leftrightarrow \pi_i((a_1, a_2)) = \pi_i((b_1, b_2)) \Leftrightarrow a_i = b_i.$$

Thus $\ker \pi_1 \cap \ker \pi_2 = \Delta$. Also, if $(a_1, a_2), (b_1, b_2)$ are any two elements of $A_1 \times A_2$, then

$$\begin{aligned} &(a_1, a_2) \ker \pi_1 (a_1, b_2), \\ &(a_1, b_2) \ker \pi_2 (b_1, b_2), \end{aligned}$$

so $\nabla = \ker \pi_1 \circ \ker \pi_2$. But, then $\ker \pi_1$ and $\ker \pi_2$ permute, and their joining is ∇ .

Definition 3. 6. Let (H, f) be an n -ary hypergroup. A compatible relation θ on H is a *factor compatible relation* if there is a compatible relation θ^* on H such that $\theta \cap \theta^* = \Delta$, $\theta \wedge \theta^* = \nabla$ and θ permutes with θ^* .

The pair θ, θ^* is called a *pair of factor compatible relations* on H .

Theorem 3. 7. If θ, θ^* is a pair of factor compatible relations on H , then

$$H \cong H / \theta \times H / \theta^*$$

under the map $\psi \langle a \rangle = (\theta \langle a \rangle, \theta^* \langle a \rangle)$.

Proof: If $a, b \in H$ and $\psi \langle a \rangle = \psi \langle b \rangle$, then $\theta \langle a \rangle = \theta \langle b \rangle$ and $\theta^* \langle a \rangle = \theta^* \langle b \rangle$, so $(a, b) \in \theta \cap \theta^*$; hence $a = b$. This means that ψ is injective. Next, given $a, b \in H$, there is $c \in H$ such that $a\theta c$ and $c\theta^* b$, hence $\psi \langle c \rangle = (\theta \langle c \rangle, \theta^* \langle c \rangle) = (\theta \langle a \rangle, \theta^* \langle b \rangle)$, so ψ is onto. Finally, for $a_1, \dots, a_n \in H$, we show that

$$\psi(f(a_1, \dots, a_n)) = (f \mid_{\theta} \times f \mid_{\theta^*})(\psi(a_1), \dots, \psi(a_n)).$$

We have

$$\begin{aligned}
\psi(f(a_1, \dots, a_n)) &= \{ \psi \langle a \rangle \mid a \in f(a_1, \dots, a_n) \} \\
&= \{ (\theta \langle a \rangle, \theta^* \langle a \rangle) \mid a \in f(a_1, \dots, a_n) \} \\
&\subseteq \{ (\theta \langle a \rangle, \theta^* \langle b \rangle) \mid a \in f(a_1, \dots, a_n), b \in f(a_1, \dots, a_n) \} \\
&\subseteq f|_{\theta}(\theta(a_1), \dots, \theta(a_n)) \times f|_{\theta^*}(\theta^*(a_1), \dots, \theta^*(a_n)) \\
&= (f|_{\theta} \times f|_{\theta^*})(\theta(a_1), \theta^*(a_1), \dots, \theta(a_n), \theta^*(a_n)) \\
&= (f|_{\theta} \times f|_{\theta^*})(\psi(a_1), \dots, \psi(a_n))
\end{aligned}$$

and so $\psi(f(a_1, \dots, a_n)) \subseteq (f|_{\theta} \times f|_{\theta^*})(\psi(a_1), \dots, \psi(a_n))$.

Conversely, suppose that $(\theta \langle x \rangle, \theta^* \langle y \rangle) \in (f|_{\theta} \times f|_{\theta^*})(\psi(a_1), \dots, \psi(a_n))$, then $(\theta \langle x \rangle, \theta^* \langle y \rangle) \in \{ (\theta \langle a \rangle, \theta^* \langle b \rangle) \mid \theta \langle a \rangle \in f|_{\theta}(\theta(a_1), \dots, \theta(a_n)), \theta^* \langle b \rangle \in f|_{\theta^*}(\theta^*(a_1), \dots, \theta^*(a_n)) \}$. Now there exists $c \in H$ such that $x\theta c$ and $c\theta^* y$, and so $(\theta \langle x \rangle, \theta^* \langle y \rangle) = (\theta \langle c \rangle, \theta^* \langle c \rangle)$ where $c \in f(a_1, \dots, a_n)$. Therefore $(\theta \langle x \rangle, \theta^* \langle y \rangle) \in \psi(f(a_1, \dots, a_n))$.

4. FUNDAMENTAL N-ARY GROUPS

If (H, f) is an n -ary hypergroup, then $\hat{\beta}$ denotes the transitive closure of the relation $\beta = \bigcup_{k \geq 1} \beta_k$, where β_1 is the diagonal relation, i.e., $\beta_1 = \{ (x, x) \mid x \in H \}$ and for every integer $k > 1$, β_k is the relation defined as follows:

$$x\beta_k y \text{ if and only if } \{x, y\} \subseteq f_{\cdot},$$

where f_{\cdot} means that $f_{(k)}$ for some $k = 1, 2, \dots$. When $x\beta_1 y$ (i.e., $x = y$) then we write $\{x, y\} \subseteq f_{(0)}$, we define β^* as the smallest equivalence relation such that the quotient $(H/\beta^*, f/\beta^*)$ is an n -ary group, where H/β^* is the set of all equivalence classes. The β^* is called *fundamental equivalence relation*. The equivalence relation β^* was first introduced on hypergroups by Koskas [17] and studied mainly by Corsini [6] concerning hypergroups, Vougiouklis [16] and Davvaz [7] concerning H_V -structures.

Theorem 4. 1. *The fundamental relation β^* is the transitive closure of the relation β , i.e., $(\beta^* = \hat{\beta})$.*

Proof: First we show that the quotient set $H/\hat{\beta}$ is an n -ary semigroup. The n -ary operation $f/\hat{\beta}$ in $H/\hat{\beta}$ is defined in the usual manner:

$$f/\hat{\beta}(\hat{\beta}(x_1), \dots, \hat{\beta}(x_n)) = \{ \hat{\beta}(y) \mid y \in (\hat{\beta}(x_1), \dots, \hat{\beta}(x_n)) \}$$

for all $x_1, \dots, x_n \in H$. Suppose $a_1 \in \hat{\beta}(x_1), \dots, a_n \in \hat{\beta}(x_n)$. Then we have $a_1 \hat{\beta} x_1$ if there exist $x_{11}, \dots, x_{1m_1+1}$ with $x_{11} = a_1, x_{1m_1+1} = x_1$ such that

$$\{x_{1i_1}, x_{1i_1+1}\} \subseteq f_{(k_1)} \quad (0 < i_1 \leq m_1)$$

$a_n \hat{\beta} x_n$ if there exist $x_{n1}, \dots, x_{nm_n+1}$ with $x_{n1} = a_n, x_{nm_n+1} = x_n$ such that

$$\{x_{ni_n}, x_{ni_n+1}\} \subseteq f_{(k_n)} \quad (0 < i_n \leq m_n).$$

Therefore, we obtain

$$\begin{aligned}
 f(\{x_{1i_1}, x_{1i_1+1}\}, x_{21}, \dots, x_{n1}) &\subseteq f_{(k_1)} & 1 \leq i_1 \leq m_1, \\
 f(x_{1m_1+1}, \{x_{2i_2}, x_{2i_2+1}\}, \dots, x_{n1}) &\subseteq f_{(k_2)} & 1 \leq i_2 \leq m_2, \\
 &\vdots & \vdots \\
 f(x_{1m_1}, x_{21m_2+1}, \dots, \{x_{ni_n}, x_{ni_n+1}\}) &\subseteq f_{(k_n)} & 1 \leq i_n \leq m_n.
 \end{aligned}$$

So, every element $z \in f(x_{11}, x_{21}, \dots, x_{n1}) = f(a_1, a_2, \dots, a_n)$ is equivalent to every element $t \in f(x_{1m_1+1}, x_{2m_2+1}, \dots, x_{nm_n+1}) = f(x_1, x_2, \dots, x_n)$. Therefore

$$f / \hat{\beta}(\hat{\beta}(x_1), \dots, \hat{\beta}(x_n))$$

is singleton. So we can write $f / \hat{\beta}(\hat{\beta}(x_1), \dots, \hat{\beta}(x_n)) = \hat{\beta}(y)$ for all $y \in f(\hat{\beta}(x_1), \dots, \hat{\beta}(x_n))$.

Moreover, since f is associative, it is obvious that $f / \hat{\beta}$ is associative, and consequently, $H / \hat{\beta}$ is an n -ary semigroup.

Now, let θ be an equivalence relation on H such that H / θ is an n -ary semigroup. Denote $\theta \langle a \rangle$ the class of a . Then for all $x_1, \dots, x_n \in H$, $f|_{\theta}(\theta(x_1), \dots, \theta(x_n)) = \theta(y)$ for all $y \in f(\theta(x_1), \dots, \theta(x_n))$. But also, for every $x_1, \dots, x_n \in H$ and $A_i \subseteq \theta(x_i)$ ($i = 1, \dots, n$) we have

$$f|_{\theta}(\theta(x_1), \dots, \theta(x_n)) = \theta(f(x_1, \dots, x_n)) = \theta(f(A_1, \dots, A_n)).$$

Therefore $\theta \langle x \rangle = \theta(f_{(k)})$ for all $k \geq 0$ and for all $x \in f_{(k)}$. So for every $a \in H$, $x \in \beta \langle a \rangle$ implies $x \in \theta \langle a \rangle$. But θ is transitively closed, so we obtain $x \in \hat{\beta} \langle a \rangle$ implies $x \in \theta \langle a \rangle$. Hence, the relation $\hat{\beta}$ is the smallest equivalence relation on H such that $H / \hat{\beta}$ is an n -ary semigroup, i.e., $\hat{\beta} = \beta^*$.

Theorem 4.2. β^* is a strongly compatible relation.

Proof: If $a_1\beta^*b_1, \dots, a_n\beta^*b_n$, then $\beta^*(a_1) = \beta^*(b_1), \dots, \beta^*(a_n) = \beta^*(b_n)$. For every $a \in f(a_1, \dots, a_n)$ and $b \in f(b_1, \dots, b_n)$ we have

$$\begin{aligned}
 \beta^* \langle a \rangle &= \beta^*(f(a_1, \dots, a_n)) \\
 &= f / \beta^*(\beta^*(a_1), \dots, \beta^*(a_n)) \\
 &= f / \beta^*(\beta^*(b_1), \dots, \beta^*(b_n)) \\
 &= \beta^*(f(b_1, \dots, b_n)) \\
 &= \beta^* \langle b \rangle.
 \end{aligned}$$

Theorem 4.3. Let (A, f) and (B, g) be two n -ary hypergroups and let β_A^* , β_B^* and $\beta_{A \times B}^*$ be fundamental equivalence relations on A , B and $A \times B$ respectively. Then

$$A \times B / \beta_{A \times B}^* \cong A / \beta_A^* \times B / \beta_B^*.$$

Proof: First we define the relation $\tilde{\beta}$ on $A \times B$ as follows:

$$(a_1, b_1)\tilde{\beta}(a_2, b_2) \Leftrightarrow a_1\beta_A^*a_2 \text{ and } b_1\beta_B^*b_2.$$

$\tilde{\beta}$ is an equivalence relation. We define h on $A \times B / \tilde{\beta}$ as follows:

$$h(\tilde{\beta}(a_1, b_1), \dots, \tilde{\beta}(a_n, b_n)) = \tilde{\beta}(a, b)$$

for all $a \in f(\beta_A^*(a_1), \dots, \beta_A^*(a_n))$, $b \in g(\beta_B^*(b_1), \dots, \beta_B^*(b_n))$. Since f, g are associative, we see that h is

associative, and consequently, $A \times B / \tilde{\beta}$ is an n -ary semigroup. Now let θ be an equivalence relation on $A \times B$ such that $A \times B / \theta$ is an n -ary group. Similar to the proof of Theorem 4.1, we get

$$(a_1, b_1) \tilde{\beta} (a_2, b_2) \Rightarrow (a_1, b_1) \theta (a_2, b_2).$$

Therefore the relation $\tilde{\beta}$ is the smallest equivalence relation on $A \times B$ such that $A \times B / \tilde{\beta}$ is an n -ary group, i.e., $\tilde{\beta} = \beta_{A \times B}^*$. Now we consider the map $\varphi : A / \beta_A^* \times B / \beta_B^* \longrightarrow A \times B / \beta_{A \times B}^*$ by

$$\varphi(\beta_A^*(a), \beta_B^*(b)) = \beta_{A \times B}^*(a, b).$$

It is easy to see that φ is an isomorphism.

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