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AMENABILITY OF WEIGHTED MEASURE ALGEBRAS*

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Abstract – Let G be a locally compact group, and let ω be a weight on G. We show that the weighted measure algebra $M(G,\omega)$ is amenable if and only if G is a discrete, amenable group and $\sup\left\{\omega(g)\ \omega(g^{-1}) : g \in G\right\} < \infty$, where $\omega(g) \geq 1 \quad (g \in G)$.

Keywords – Amenability, measure algebra, weight

1. INTRODUCTION

Let A be a Banach algebra, and let X be a Banach A-bimodule. A derivation from A into X is a continuous linear map $D : A \to X$ satisfying

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

For example, let $x \in X$, and define $\delta_x(a) = a \cdot x - x \cdot a$, $a \in A$. Then it is a derivation; maps of this form are called inner derivations. The cohomology group $H^1(A, X)$ is the quotient of the space of continuous derivations from A into X by the space of inner derivations.

Let A be a Banach algebra and let X be a Banach A-bimodule. Then the dual space X' is also a Banach A-bimodule for the products $a \cdot \lambda$ and $\lambda \cdot a$ specified by

$$a \cdot \lambda(x) = \lambda(x \cdot a),\ \ \lambda \cdot a(x) = \lambda(a \cdot x) \qquad (a \in A, x \in X, \lambda \in X')$$

The Banach algebra A is amenable if $H^1(A, X') = 0$ for every Banach A-bimodule X; this definition was introduced by Johnson in [1]. In his paper Johnson showed that $L^1(G)$ is amenable if and only if G is amenable.

Grønbæk in [2] showed that $L^1(G,\omega)$ is amenable if and only if G is amenable and

$$\sup\{\omega(g)\ \omega(g^{-1}) : g \in G\} < \infty.$$

Dales et al. in [3] showed that $M(G)$ is amenable if and only if G is amenable and discrete. In this paper we will generalize this result. We will show that the weighted measure algebra $M(G,\omega)$ is amenable if and only if G is amenable, discrete and $\sup\left\{\omega(g)\omega(g^{-1}) : g \in G\right\} < \infty$, where $\omega(g) \geq 1 \quad (g \in G)$.

Throughout G is a locally compact group. A weight on G is a continuous function $\omega : G \to R^+$ satisfying

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$$\omega(e) = 1, \quad \omega(gh) \le \omega(g)\omega(h) \qquad (g, h \in G).$$

In the case where $\omega(g) \ge 1 \quad (g \in G)$, we denote by $M(G,\omega)$ the Banach algebra of all complex-valued regular Borel measures μ on G such that

$$\|\mu\|_\omega = \int_G \omega(t)d|\mu|(t) < \infty.$$

Note that $M(G,\omega)$ is identified with the dual of $C_0(G,\omega^{-1})$.

The subspace of the continuous measures is denoted by $M_c(G,\omega)$. The subspace of the discrete measures is denoted by $M_d(G,\omega)$, and is identified with $\ell^1(G,\omega)$. Note that $M_d(G,\omega)$ is a closed subalgebra of $M(G,\omega)$, and $M_c(G,\omega)$ is a closed ideal of $M(G,\omega)$. We have $M(G,\omega) = M_d(G,\omega) \oplus M_c(G,\omega)$ as a Banach space. In the case where G is discrete we have $M(G,\omega) = \ell^1(G,\omega)$ and $M_c(G,\omega) = \{0\}$; if G is not discrete, then $M_c(G,\omega) \ne \{0\}$.

2. THE MAIN RESULTS

In this section we investigate the amenability of the weighted measure algebra $M(G,\omega)$ for a locally compact group G, where $\omega(g) \ge 1$ for every $g \in G$. The method that we use is the same as [3], though in order to deal with the difficulty of the weighted case our presentation differs from [3] in some cases.

The subset $\{1,\ldots,n\}$ of N is denoted by N_n, and N_4^m is the set of elements $(i_1,\ldots,i_m)$ such that $i_k \in N_4$ for $k \in N_m$. Let (K_n) be a sequence of compact subsets of group H as specified in [3]. The family of sets $K_n(i_1,\ldots,i_n)$ for $n \in N$ and $(i_1,\ldots,i_n) \in N_4^n$ is denoted by Ω. We set $K = \bigcap\{K_n : n \in N\}$.

Lemma 2. 1. Let H be a non-discrete, metrizable, locally compact group, and let ω be a weight on H with $\omega(t) \ge 1$ for every $t \in H$. Then for every $L \in \Omega$, there exists $\mu_L \in M_c(H,\omega)^+$ such that

$$\frac{1}{M_L} \le \int_{L\cap K} \omega(t)d\mu_L(t) \le 1 \quad \text{and} \quad \int_{H\setminus(L\cap K)} \omega(t)d\mu_L(t) = 0,$$

where $M_L = \sup\left\{\omega(t) : t \in L\right\}$.

Proof: Let $L = K_m(i_1,\ldots,i_m) \in \Omega$ be fixed, where $(i_1,\ldots,i_m) \in N_4^n$. Let $n \in N$ and $i \in N_4^n$. Then $\int_{K_n(i)} \omega(t)d(t) > 0$, because $\mathrm{int}K_n(i) \ne \varnothing$ and $\omega(t) \ge 1$.

For every $n \ge m$ and $i \in N_4^n$ such that $K_n(i) \subset L$, we define

$$\mu_{n,i}(E) = \frac{\int_{E\cap K_{n(i)}} \omega(t)d(t)}{4^{n-m}\int_{K_n(i)} \omega(t)dt},$$

clearly $\mu_{n,i}(H) = \mu_{n,i}(K_n(i)) = 4^{m-n}$ and $\mu_{n,i}(H \setminus K_n(i)) = 0$. Thus every $\mu_{n,i}$ is a positive measure with compact support. Now for every $n \in N$ we define

$$\mu_n = \frac{1}{M_L}\sum\{\mu_{n,i} : i \in N_4^n, K_n(i) \subset L\}.$$

Since for $n \ge m$ the set $\{i \in N_4^n, K_n(i) \subset L\}$ has 4^{n-m} elements, we have $\mu_n(H) = \frac{1}{M_L}$ and

$$\mu_n(H) \le \int_H \omega(t)\, d\mu_n(t) \le \sum_{\substack{i\in N_4^n \\ K_n(i)\subset L}} \int_{K_n(i)} \frac{\omega(t)}{M_L} d\mu_{n,i}(t)$$
$$\le \sum_{\substack{i\in N_4^n \\ K_n(i)\subset L}} \mu_{n,i}(K_n(i)) = 1.$$

Thus every μ_n is a positive measure with $\frac{1}{M_L} \le \|\mu_n\|_\omega \le 1$. We claim that the sequence $\{\mu_n\}$ has a weak accumulation point μ_L in $M_c(H,\omega)^+$ such that $\|\mu_L\|_w \le 1$. The proof of the claim is similar to [3].

Now fix $n \in N$. Let $f \in C_0(H)$ be such that $f(K_n \cap L) = \{1\}$ and $f(H) \subset [0,1]$. Since $f\omega \in C_0(H, \omega^{-1})$ and for every $r \ge n$, $\text{supp}\mu_r \subseteq K_n \cap L$, then we have

$$\langle f\omega, \mu_r\rangle = \int_H f(t)\omega(t)d\mu_r(t) = \int_{K_n\cap L} \omega(t)d\mu_r(t) = \|\mu_r\|_\omega \qquad (r \ge n).$$

Hence $\frac{1}{M_L} \le \langle f\omega, \mu_L\rangle \le 1.$ If we consider f to be the characteristic function of an arbitrary neighborhood U of $K_n \cap L$, then we have $\frac{1}{M_L} \le \int_U \omega(t)d\mu_L(t) \le 1,$ and since μ_L is regular we have $\frac{1}{M_L} \le \int_{K_n\cap L} \omega(t)d\mu_L(t) \le 1$. But $(K_n \cap L : n \in N)$ is a decreasing sequence with $\bigcap_{n=1}^{\infty} K_n \cap L = K \cap L$. This implies

$$\frac{1}{M_L} \le \int_{K\cap L} \omega(t)d\mu_L(t) \le 1 \qquad \text{and} \qquad \int_{H\setminus(K\cap L)} \omega(t)d\mu_L(t) = 0.$$

With the same method as in [3] one can 'lift' the above result in the case where the underlying group is not metrizable.

Theorem 2. 2. Let G be a non-discrete, locally compact group. Then
1. $\overline{M_c(G,\omega)^2}$ has infinite co-dimension in $M_c(G,\omega)$.
2. There is a continuous, positive linear functional Ψ on $M(G,\omega)$ and $\mu_0 \in M_c(G,\omega)^+$ with $\frac{1}{M_L} \le \|\mu_0\|_\omega \le 1$ such that $\langle \mu_0, \Psi\rangle = \|\mu_0\|_\omega$, $\Psi|_{M_d(G,\omega)} = 0$ and $\Psi|_{M_c(G,\omega)^2} = 0.$

Proof: Let $\mu \in M_c(G,\omega)$, and let V be a set which is defined in [3]. Define

$$E_k(\mu) = \left\{x \in G : \int_{xV} \omega(t)d|\mu|(t) > \frac{1}{k}\right\}$$

for every $k \in N$. With the same argument as in [1], one can show that $E_k(\mu)$ is a finite set. Now we define $E(\mu) = \bigcup_{k\in N} E_k(\mu) = \left\{x : \int_{xV} \omega(t)d|\mu|(t) > 0\right\}$. Then $E(\mu)$ is a countable set and $\int_{xV} \omega(t)d|\mu|(t) = 0$ whenever $x \in G \setminus E(\mu)$. For every $L \in \Omega$ and every μ in $M_c(G,\omega)$, we define

$$\langle \mu, \Psi_L\rangle = \int_G \omega(t)\chi_{V_L}(t)d\mu(t).$$

It is clear that $\Psi_L \in M_c(G,\omega)'^+$ and

$$\|\Psi_L\| = \sup\left\{|\langle \mu, \Psi_L\rangle| : \mu \in M_c(G,\omega), \|\mu\|_\omega \le 1\right\} \le 1.$$

Let μ_L be a measure on H as in the Lemma 2.1. Then with the same method as in [3] μ_L can be transferred to a measure $\nu_L \in M_c(G,\omega)^+$ and $\frac{1}{M_L} \le \|\nu_L\|_\omega \le 1$. Also

$$\langle \nu_L, \Psi_L\rangle = \int_G \chi_{V_L}(t)\omega(t)d\nu_L(t) = \|\nu_L\|_\omega.$$

Let $\mu,\nu \in M_c(G,\omega)$. Since for every $x \in G$, where $x^{-1} \notin E(\nu)$, we have $\int_{x^{-1}V_L} \omega(t)d\nu(t) = 0$. Thus

$$\begin{aligned}\langle \mu * \nu, \Psi_L \rangle &= \int_G \chi_{V_L}(t)\omega(t)d\mu * \nu(t) \\ &\le \int_G \int_G \chi_{V_L}(ty)\omega(t)\omega(y)d\mu(t)d\nu(y) \\ &= \int_G \left\{ \int_{t^{-1}V_L} \omega(y)d\nu(y) \right\} \omega(t)d\mu(t). \\ &\le \|\nu\|_\omega \int_{E(\nu)^{-1}} \omega(t)d\mu(t).\end{aligned}$$

However $E(\nu)$, hence $E(\nu)^{-1}$ is a countable set. Thus $\int_{E(\nu)^{-1}} \omega(t)d\mu(t) = 0$, so $\langle \mu * \nu, \Psi_L \rangle = 0$ and this implies that $\Psi_L |_{M_c(G,\omega)^2} = 0$, so $\Psi_L |_{\overline{M_c(G,\omega)^2}} = 0$.

Now let $\{L_n : n \in N\}$ be an infinite, pairwise disjoint subfamily of Ω, and let $a_n = \langle \nu_{L_n}, \Psi_{L_n} \rangle$. Then $\frac{1}{M_L} \le a_n \le 1$ and

$$\langle \mu_{L_n}, \Psi_{L_m} \rangle = \int_G \omega(t)\chi_{V_{L_m}}(t)d\mu_{L_n}(t) = \begin{cases} 0 & m \ne n, \\ a_n & m = n. \end{cases}$$

Therefore the set $\left\{ \mu_{L_n} + \overline{M_c(G,\omega)^2} : n \in N \right\}$ is a linearly independent subset of $M_c(G,\omega)/\overline{M_c(G,\omega)^2}$. This shows that $\overline{M_c(G,\omega)^2}$ has infinite codimension in $M_c(G,\omega)$.

The proof of (ii) is similar to the proof of [3].

Theorem 2. 3. Let G be a non-discrete, locally compact group. Then $M(G,\omega)$ is not amenable, and for every weight ω on G such that $\omega(g) \ge 1$, $g \in G$.

Proof: We recall that $M_c(G,\omega)$ is a complemented closed ideal in $M(G,\omega)$. By Theorem 1 we have $M_c(G,\omega) \ne M_c(G,\omega)^{[2]}$ and also, by [4, Theorem 2.9.58], $M(G,\omega)$ is not amenable. Our final result is as follows.

Theorem 2. 4. Let G be a locally compact group and let ω be a weight of G such that $\omega(g) \ge 1$ for all $g \in G$. Then $M(G,\omega)$ is amenable if and only if G is discrete, amenable and $\sup\left\{ \omega(g)\omega(g^{-1}) : g \in G \right\} < \infty$.

Proof: If G is discrete, then $M(G,\omega) = \ell^1(G,\omega)$. So by [2] $M(G,\omega)$ is amenable if and only if G is amenable and $\sup\left\{ \omega(g)\omega(g^{-1}) : g \in G \right\} < \infty$. If G is not discrete, then by Theorem 2.3 $M(G,\omega)$ is not amenable.

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