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Existence solutions for nonlocal fractional differential equation with nonlinear boundary conditions

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Abstract

In this paper, by employing the Guo-Krasnoselskii fixed point theorem in a cone, we study the existence of positive solutions to the following nonlocal fractional boundary value problems

$$\begin{cases} {}^{c}D_{0^+}^{\alpha}u(t) = f(t,u(t)), \qquad t \in (0,1), \\ u(t) + u'(0) = \dfrac{1}{2}\left[H_1(\varphi(u)) + \displaystyle\int_E H_2(s,u(s))ds\right], \\ u(1) + u'(1) = 0, \end{cases}$$

where ${}^{c}D_{0^+}^{\alpha}$ is the standard Caputo derivative of order $\alpha, 1 < \alpha < 2, E \subseteq (0,1)$ is some measurable set. We provide conditions on f, H_1, H_2 and φ such that the problem exhibits at least one positive solution.

Keywords: Cone; fixed point theorem; standard Caputo; derivative

1. Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order, the fractional calculus may be considered an old and yet novel topic.

Recently, fractional differential equations have been of great interest. This is because of both the intensive development of the theory of fractional calculus itself and its applications in various sciences, such as physics, mechanics, chemistry, engineering, etc. For example, for fractional initial value problems, the existence and multiplicity of solutions were discussed in (Babakhani and Gejji, 2003; Baleanu et al., 2012; Delbosco and Rodino, 1996; Kilbas and Trujillo, 2001; Kilbas and Trujillo, 2002), moreover, fractional derivative arises from many physical processes, such as a charge transport in amorphous semiconductors (Scher and Montroll, 1975), electrochemistry and material science are also described by differential equations of fractional order (Diethelm and Freed, 1999; Gaul et al., 1991; Glockle and Nonnenmacher, 1995; Mainar, 1997; Metzler et al., 1995).

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The existence of solutions of initial value problems for fractional order differential equations have been studied in the literature (Agarwal et al., 2010; Lakshmikantham and Vatsala, 2008; Nyamoradi, 2012; Nyamoradi, 2013; Nyamoradi, 2012; Nyamoradi and Bashiri, 2012; Nyamoradi and Bashiri, 2013; Podlubny, 1999; Razminia et al., 2013; Samko et al., 1993) and the references therein.

On the other hand, certain authors have investigated nonlocal, nonlinear boundary conditions including, for example, (Ehme et al., 2002; Graef and Webb, 2009; Kong and Kong, 2005; Liu et al., 2010; Webb and Infante, 2006; Webb and Infante, 2009; Yang, 2005; Yang, 2006). In addition, there are examples of nonlocal boundary conditions in the context of fractional differential and difference equations (see, for example, (Goodrich, 2011; Goodrich, 2011; Goodrich, 2011; Goodrich, 2012; Goodrich, 2011) and the references therein).

Our purpose in this paper is to show the existence and multiplicity of positive solutions for the boundary value problem of fractional differential equation:

$$\begin{cases} {}^{c}D_{0^+}^{\alpha}u(t) = f(t,u(t)), \quad t \in (0,1), \\ u(t) + u'(0) = \frac{1}{2}\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right], \\ u(1) + u'(1) = 0, \end{cases} \tag{1}$$

where ${}^{c}D_{0^+}^{\alpha}$ is the standard Caputo derivative of order $\alpha, 1 < \alpha < 2, E \Subset (0,1)$ is some measurable set and φ is a linear functional having the form

$$\varphi(u) := \int_0^1 u(t)d\alpha(t), \tag{2}$$

where the integral appearing in (2) is taken in the Lebesgue-Stieltjes sense and $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function with suitable conditions.

Prior to describing the novel contributions of this work, let us place problem (1) in an appropriate context and review briefly some recent results on such problems. In particular, rich literature on nonlocal boundary value problems exists along, with the important work by Debbouche, Baleanu and Agarwal (Debbouche et al., 2012), in which a unified theory of a nonlocal onlinear fractional problem which the authors proved the existence of mild and strong solutions of a nonlocal nonlinear fractional problem with the nonlocal condition $u(0) + \sum_{k=1}^{p} u(t_k) = u_0$.

The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. In Section 3, we give the existence of one positive solution for the problem (1) by using the Guo-Krasnoselskii fixed point theorem. Finally, in Section 4, an example is given to demonstrate the application of our main result.

2. Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in Sections 3 and 4.

Definition 1. Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:
(1) $x \in P, \mu \geq 0$ *implies* $\mu x \in P$,
(2) $x \in P, -x \in P$ *implies* $x = 0$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 3. The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 1. ((Kilbas, et al., 2006)). The equality $D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t), \alpha > 0$ holds for $f \in L^1(0,1)$.

Definition 4. ((Kilbas, et al., 2006; Podlubny, 1999)) The fractional derivative of f in the Caputo sense is defined as

$${}^{c}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \ n-1 < \alpha < n,$$

where $n = [\alpha] + 1$.

Lemma 2. ((Kilbas, et al., 2006)). Let $\alpha > 0$
Then the differential equation

$${}^{c}D_{0^+}^{\alpha} u(t) = 0$$

has a unique solution $u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where $n - 1 < \alpha \leq n$.

Lemma 3. ((Kilbas, et al., 2006)). Assume that $h \in C(0,1) \cap L^1(0,1)$ with a derivative of order $\alpha > 0$ that belongs to $(0,1) \cap L^1(0,1)$. Then

$$I_{0^+}^{\alpha}\, {}^{c}D_{0^+}^{\alpha} h(t) = h(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 1, \ldots, n-1$, where $n - 1 < \alpha \leq n$.

Lemma 4. Suppose that $h \in C[0,1]$, then the boundary value problem (1) has a unique solution

$$u(t) = (1-t)\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right] + \int_0^1 G(t,s)h(s)ds, \tag{3}$$

where

$$G(t,s)=\begin{cases}\frac{(1-s)^{\alpha-1}(1-t)+(t-s)^{\alpha-1}}{\Gamma(\beta)}+\frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & o\le s\le t\le 1,\\ \frac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\beta)}+\frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & o\le t\le s\le 1,\end{cases} \tag{4}$$

Proof: The proof is similar to that of Lemma 3.1 in (Zhang, 2006), so we omit it here.

Lemma5. Let $\theta\in\left(0,\frac{1}{2}\right)$, then the function $G(t,s)$ defined by (4) satisfies the following conditions:
(i) $G(t,s)\in C([0,1]\times[0,1))$ and $(t,s)>0$, for any $(t,s)\in(0,1)\times(0,1)$;
(ii) There exists a positive function $\gamma\in C(0,1)$ such that

$$\min_{0\le t\le 1-\theta} G(t,s)\ge\gamma(s)M(s),$$
$$\max_{0\le t\le 1-\theta} G(t,s)\le M(s),\quad s\in(0,1),$$

where

$$M(s)=\frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)},\quad s\in[0,1),$$

$$\gamma(s)=\theta\frac{(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}}{2(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}},\quad s\in(0,1).$$

Proof: The proof is similar to that of Lemma 2.5 in (Yang et al., 2012), so we omit it here.

Now, we consider the system (1). By applying lemma 4, $u\in C(0,1)$ is a solution of the system (1) if and only if $u\in C[0,1]$ is a solution of the following nonlinear integral system:

$$u(t)=(1-t)\left[H_1(\varphi(u))+\int_E H_2(s,u(s))ds\right]+\int_0^1 G(t,s)f(s,u(s))ds. \tag{5}$$

The basic space used in this paper is a real Banach space $\beta=C[0,1]$ with the norm $\|u\|=max_{0\le t\le 1}|u(t)|$.

3. Main result for the single-valued case

Now we are able to present the existence results for problem (1). In this section, to establish the existence one positive solution of system (1), we will employ the following Guo-Krasnoselskii fixed point theorem.

Our approach is based on the following Guo-Krasnoselskii fixed point theorem of cone expansion-compression type (Krasnoselskii, 1964).

Theorem 1. Let E be Banach space and $K\subseteq E$ a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0\in\Omega_1$ and $\overline{\Omega}_1\subseteq\Omega_2$. Let $T:K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ be a completely continuous operator. In addition suppose either
(A) $\|Tu\|\le\|u\|,\forall u\in K\cap\partial\Omega_1$ and $\|Tu\|\ge\|u\|,\forall u\in K\cap\partial\Omega_2$ or
(B) $\|Tu\|\ge\|u\|,\forall u\in K\cap\partial\Omega_1$ and $\|Tu\|\le\|u\|,\forall u\in K\cap\partial\Omega_2$
holds. Then T has a fixed point in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Also, we introduce the following notations. Define

$$\sigma=min\left\{\frac{min_{\theta\le t\le 1-\theta}(1-t)}{max_{0\le t\le 1}(1-t)},\frac{min_{\theta\le t\le 1-\theta}\gamma(t)}{3}\right\}.$$

Then, choose a cone $K\subset\beta$, by

$$K=\left\{\begin{matrix}u\in\beta: u\ge 0 \min\limits_{0\le t\le 1-\theta}(\mathrm{u(t)})\\ \ge\sigma\|\mathrm{u}\|,\varphi_1(u),\varphi_2(u)\ge 0\end{matrix}\right\}.$$

and define an operator $T:E\to E$ by

$$T(u)(t)=(1-t)\left[H_1(\varphi(u))+\int_E H_2(s,u(s))ds\right]+\int_0^1 G(t,s)f(s,u(s))ds. \tag{6}$$

Throughout the forthcoming analysis, the following conditions are assumed:
(H1) Let $H_1:[0,+\infty)\to[0,+\infty)$ and $H_2:[0,1]\times[0,+\infty)\to[0,+\infty)$ be two real-value, continuous function;
(H2) The functional $\varphi(u):=\int_0^1 u(t)d\alpha_1(t)+\int_0^1 u(t)d\alpha_2(t)$ can be written in the form

$$\varphi(u)=\varphi_1(u)+\varphi_2(u):=\int_0^1 u(t)d\alpha_1(t)+\int_0^1 u(t)d\alpha_2(t), \tag{7}$$

where $\alpha,\alpha_1,\alpha_2:[0,1]\to\mathbb{R}$ satisfy $\alpha\in BV([0,1])$ and φ_1,φ_2 are linear functionals;
(H3) For each i=1,2 both

$$\int_0^1(1-t)d\alpha_i(t)\ge 0 \tag{8}$$

and

$$\int_0^1 G(t,s)d\alpha_i(t)\ge 0 \tag{9}$$

hold, where (9) holds for every $s\in[0,1]$.
(H4) There is a constant $C_1\in[0,1)$ such that the functional φ in (7) satisfies the inequality

$$|\varphi(u)| \le C_1\|u\|, \ \ \forall u \in C([0,1]). \tag{10}$$

Moreover, there is a constant $C_2 > 0$ such that the functional φ in (7) satisfies $\varphi_2(u) \ge C_2\|u\|$ whenever $u \in K$;

(H5) $\limsup_{u\to+\infty} \frac{H_1(u)}{u} = 0$;

(H6) There exists a function $p: [0,+\infty) \to [0,+\infty)$ satisfying the growth condition

$$p(u) \le C_4 u, \quad \text{for some } C_4 \ge 0, \tag{11}$$

having the property

$$\limsup_{u\to+\infty} \frac{H_2(t,u)}{u} = 0, \tag{12}$$

uniformly with respect to $(t,u) \in [0,1] \times \mathbb{R}^+$;

(H7) $\limsup_{u\to 0^+} \frac{f(t,u)}{u} = +\infty$ uniformaly with respect to $(t,u) \in [0,1] \times \mathbb{R}^+$.

(H8) $\limsup_{u\to+\infty} \frac{f(t,u)}{u} = 0$ uniformaly with respect to $(t,u) \in [0,1] \times \mathbb{R}^+$.

Lemma 6. If (H1), (H2), and (H3) hold, then the operator $T: K \to K$ is well-defined, i.e. $T(K) \subseteq K$ and completely continuous.

Proof: For any $u \in K$, by Lemma 5, $T(u)(t) \ge 0, t \in [0,1]$, and it follows from (6) that

$$\begin{aligned}
&\|T(u)\| \\
&\le \max_{0\le t\le 1}(1-t)\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right] + \int_0^1 M(s)f(s,u(s))ds \\
&= \max_{0\le t\le 1}(1-t)\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right] + \left(\int_0^\theta + \int_0^{1-\theta} + \int_{1-\theta}^1\right)(M(s)f(s,u(s))ds) \\
&\le \max_{0\le t\le 1}(1-t)\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right] + 3\int_\theta^{1-\theta} M(s)f(s,u(s))ds.
\end{aligned} \tag{13}$$

Thus, for any $u \in K$, it follows from Lemma 5 and (6) that

$$\begin{aligned}
&\min_{0\le t\le 1-\theta} T(u)(t) \\
&= \min_{0\le t\le 1-\theta}\left\{(1-t)\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right] + \int_0^1 G(t,s)f(s,u(s))ds\right\} \\
&\ge \min_{0\le t\le 1-\theta}(1-t)\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right] + \int_\theta^{1-\theta} \gamma(s)M(s)f(s,u(s))ds \\
&\ge \frac{\min_{0\le t\le 1-\theta}(1-t)}{\max_{0\le t\le 1}(1-t)}\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right] + \min_{0\le t\le 1-\theta}\gamma(t)\int_\theta^{1-\theta} M(s)f(s,u(s))ds \\
&\ge \sigma\|T(u)\|.
\end{aligned}$$

Finally, for $u \in K, i = 1,2$ and (H3), one can obtain

$$\begin{aligned}
&\varphi_i(T(u)) \\
&\le \int_0^1 (1-t)\left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right]d\alpha_i(t) + \int_0^1\left(\int_0^1 G(t,s)f(s,u(s))ds\right)d\alpha_i(t) \\
&= \left[H_1(\varphi(u)) + \int_E H_2(s,u(s))ds\right]\int_0^1 (1-t)d\alpha_i(t) + \int_0^1\left(\int_0^1 G(t,s)d\alpha_i(t)\right)f(s,u(s))ds \\
&\ge 0.
\end{aligned}$$

Therefore, from the above, we conclude that $T(u)(t) \in K$, that is, $T(K) \subset K$. Thus, The operator T by an application of the Ascoli-Arzela theorem, is completely continuous. This completes the proof.

It is clear that the existence of a positive solution for the problem (1) is equivalent to the existence of a nontrivial fixed point of T in K.

Theorem 2. If (H1)-(H8) hold, $E \Subset (0,1)$ and

$$C_1 + C_4 m(E) < 1, \tag{14}$$

then (1) has at least one positive solution.

Proof: By (H7), there exists $r_1 > 0$ such that $f(t,u) \ge \eta_1 u, 0 < u \le r_1$. Set

$$\Omega_1 := \{u \in \beta: \|u\| < r_1\}, \tag{15}$$

and let $\eta_1$satisfy

$$\eta_1\sigma\int_\theta^{1-\theta} M(s)ds \ge 1. \tag{16}$$

Then, for any $u \in K \cap \partial\Omega_1$, one can get

$$T(u)\left(\frac{1}{2}\right) \geq \int_0^1 G\left(\frac{1}{2}, s\right) f(s, u(s)) ds$$
$$\geq \|u\| \eta_1 \sigma \int_\theta^{1-\theta} M(s) ds \geq \|u\|.$$

which implies that

$$T(u) \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1. \tag{17}$$

On the other hand, the condition (14) implies the existence of $\epsilon > 0$ such that

$$C_1 + C_4 m(E) + \epsilon < 1.$$

But then by choosing ϵ even closer to zero if necessary, we may also assume that

$$\epsilon C_1 + m(E) C_4 \epsilon + \epsilon < 1. \tag{18}$$

Henceforth, we let ϵ be fixed such that (18) holds. Now, since (H5) holds, for $\epsilon > 0$, there exists $M_\epsilon > 0$ such that

$$H_1(\varphi(u)) \leq \epsilon \varphi(u), \quad \text{for each } \varphi(u) > M_\epsilon. \tag{19}$$

Since $\varphi_1(u) \geq 0$, (H4), it follows that $\varphi(u) \geq \varphi_2(u) \geq C_2\|u\|$, so that if

$$\|u\| \geq \frac{M_\epsilon}{C_2}, \tag{20}$$

then (19) holds. Similarly, condition (H6) implies the existence of a number $M_\epsilon > 0$, which we do not relabel, such that

$$H_2(t\varphi(u(s))) \leq \epsilon F(u(s)), \tag{21}$$

for each $t \in [0,1]$, $u(s) > M_\epsilon$. In order to ensure that this occurs, note that it is sufficient to assume that

$$\|u\| \geq \frac{M_\epsilon}{\sigma}. \tag{22}$$

Indeed, it then holds that

$$\min_{t\in E} T(u)(t) \geq \min_{0\leq t\leq 1-\theta} u(u) \geq \sigma\|u\| \geq \sigma . \frac{M_\epsilon}{\sigma} = M_\epsilon. \tag{23}$$

Note that (23) is allowable since $E \Subset (\theta, 1-\theta)$. In any case, then, both (20) and (24) hold provided that

$$\|u\| \geq max\left\{\frac{M_\epsilon}{C_2}, \frac{M_\epsilon}{\sigma}\right\}. \tag{24}$$

Since (H8) holds, for $\eta_2 > 0$, there exists $r_2 > r_1 > 0$ such that $f(t,u) \leq \eta_2 u$ for $u > r_2$, where η_2 satisfy

$$\eta_2 \int_0^1 M(s) ds \leq \epsilon. \tag{25}$$

We consider two cases:
Case (i): Suppose that f is unbounded, then define a function $f^*: [0, +\infty) \to [0, +\infty)$ by

$$f^*(r) := max\{f(t,u) : \ t \in [0,1], 0 \leq u \leq r\}.$$

It is easy to see that f^* is nondecreasing and $\limsup_{r\to+\infty} \frac{f^*(r)}{r} = 0$, and

$$f^*(r) \leq \epsilon r, \quad for\ r > r_2. \tag{26}$$

Taking $r_2^* > max\left\{r_2, \frac{2r_1}{\sigma}, \frac{M_\epsilon}{C_2}, \frac{M_\epsilon}{\sigma}\right\}$, then from (26), one has

$$f(t,u) \leq f^*(r_2^*) \leq \epsilon r_2^*, \quad for\ t \in [0,1], 0 \leq u \leq r_2^*. \tag{27}$$

Set

$$\Omega_2 := \{u \in \beta: \|u\| < r_2^*\}. \tag{28}$$

Then, for any $u \in K \cap \partial\Omega_2$, one may obtain

$$\|T(u)\| \leq H_1(\varphi(u)) + \int_E H_2(s, u(s)) ds + \int_0^1 M(s) f(s, u(s)) ds$$
$$\leq \epsilon\varphi(u) + \int_E \epsilon F(u(s))\, ds + r_2^* \eta_2 \int_0^1 M(s) ds$$
$$\leq \epsilon C_1\|u\| + m(E) C_4 \in \|u\| + \in \|u\|$$
$$\leq [\epsilon C_1 + m(E) C_4 \in + \in]\|u\| \leq \|u\|.$$

which implies that

$$T(u) \leq \|u\|, \qquad for\ \ u \in K \cap \partial\Omega_2. \tag{29}$$

Case (ii): Suppose that f is bounded, say $\max_{t\in[0,1]} f(t,u) \leq r_2$ for some $r_2 \geq 0$ is sufficiently large. In fact, without loss of generality, we may assume that

$$f(t,u) \leq \frac{r_2}{\int_0^1 M(s) ds} \epsilon. \tag{30}$$

Taking $r_2^* > max\left\{r_2, \frac{2r_1}{\sigma}, \frac{M_\epsilon}{C_2}, \frac{M_\epsilon}{\sigma}\right\}$, for any $u \in K \cap \partial\Omega_2$, we have

$$\|T(u)\| \leq H_1(\varphi(u)) + \int_E H_2(s, u(s)) ds + \int_0^1 M(s) f(s, u(s)) ds$$
$$\leq \epsilon\varphi(u) + \int_E \epsilon F(u(s))\, ds + \frac{r_2}{\int_0^1 M(s) ds} \int_0^1 M(s) ds$$
$$\leq \epsilon C_1\|u\| + m(E) C_4 \in \|u\| + \in \|u\|$$
$$\leq [\epsilon C_1 + m(E) C_4 \in + \in]\|u\|$$
$$\leq \|u\|.$$

which implies that

$$T(u) \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2. \quad (31)$$

Thus, in either case, we may combine estimates (17) and (29) or (31) together with Theorem 1 to deduce the existence of a function $u_0 \in K \cap (\bar{\Omega}_2/\Omega_1)$ such that $(u_0) = u_0$. Therefore the problem (1) has at least on positive solution. So, the proof is complete.

4. Application

Example 3. Consider the following singular boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2}} u(t) = \left(\frac{t+1}{2}\right)^4 \left(ue^{\frac{1}{u}} - u\right), \quad t \in (0,1), \\ u(0) + u'(0) = \frac{1}{2}\left[H_1(\varphi(u)) + \int_E H_2(s, u(s))ds\right], \\ u(1) + u'(1) = 0, \end{cases} \quad (32)$$

here, $= \frac{3}{2}$, $f(t,u) = \left(\frac{t+1}{2}\right)^4 \left(ue^{\frac{1}{u}} - u\right)$. Now, we define

$$H_1(u) = u - ue^{\frac{1}{\sqrt{u}}}, \quad p(u) = 4u, \quad H_2(t,u) = \frac{t}{10}\left(ue^{\frac{1}{u}} - u\right),$$

and $\varphi(u) = \varphi_1(u) + \varphi_2(u)$ where

$$\varphi_1(u) = \frac{1}{6}u\left(\frac{2}{3}\right) - \frac{1}{8}u\left(\frac{1}{2}\right), \quad \varphi_2(u) = \int_{\frac{1}{3}}^{\frac{2}{3}} u(t)dt.$$

Also, we choose $\theta = \frac{1}{4}, E = \left[\frac{9}{30}, \frac{1}{3}\right]$, which implies that $m(E) = \frac{1}{30}$. In addition, we have

$$|\varphi(u)| \leq \frac{1}{6}\|u\| + \frac{1}{8}\|u\| + \left(\frac{2}{5} - \frac{1}{3}\right)\|u\| = \frac{129}{360},$$

and

$$\varphi_2(u) \geq \frac{1}{15}\sigma\|u\|,$$

for each $u \in K$. Thus, upon putting $C_1 := \frac{129}{360} \in [0,1)$ and $C_2; = \frac{1}{15}\sigma > o$. Clearly, f, φ, H_1 and H_2 satisfy the conditions (H1)-(H8). Then, all conditions of Theorem 2 hold. Hence, the system (32) has at least one solution.

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