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Structure of quasi ordered $*$-vector spaces

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Abstract

Let (X, X^{+}) be a quasi ordered $*$-vector space with order unit, that is, a $*$-vector space X with order unite together with a cone $X^{+} \subseteq X$. Our main goal is to find a condition weaker than properness of X, which suffices for fundamental results of ordered vector space theory to work. We show that having a non-empty state space or equivalently having a non-negative order unit is a suitable replacement for properness of X^{+}. At first, we examine real vector spaces and then the complex case. Then we apply the results to self adjoint unital subspaces of unital $*$-algebras to find direct and shorter proofs of some of the existing results in the literature.

***Keywords**:* Quasi ordered $*$-vector space; bounded algebra; quasi operator system; Archimedeanization

1. Introduction

The theory of ordered $*$-vector spaces is a well established subject which has its roots in functional analysis where convexity arguments play an essential role. See for instance, Alfsen (1971), Gupta (1985), Kelley (1963), and Persini (1967) for the fundamentals and history of this subject. Beginning with Kadison's characterization of commutative operator systems (Kadison, 1951) ordered $*$-vector spaces were found to be a strong tool in the study of structure of operator systems. Choi and Effros (1977) constructed a non-commutative generalization of Kadison's result by characterizing operator systems in terms of their order structure. Paulsen, Tomforde, and Todorov (2011) considered ordered $*$-vector spaces as a general framework for abstract study of operator systems and introduced various operator system structures of Archimedean ordered $*$-vector spaces. See also Xhabli (2012).

Existing literature on ordered $*$-vector spaces and, in particular, the above mentioned references focus around the case where the positive cone is proper; a condition which we believe is not necessary and so far this belief has led to construction of algebraic analogs of Choi-Effros, Ruan, and Arveson's extension theorems (Esslamzadeh and Taleghani, 2012; Esslamzadeh and Taleghani (2013); and Esslamzadeh and Turowska (2013)). Indeed, the general question of investigating the role of algebraic structure of operator systems, in the fundamental results in operator system theory, led to results of the aforementioned references together with the abstract approach of Choi and Effros (1977), Paulsen and Tomforde (2009), and Paulsen, Todorov, and Tomforde (2011) motivated us to study quasi ordered $*$-vector spaces, that is, $*$-vector spaces with a (not necessarily proper) cone.

Our objective in this paper, is the study of the order structure of quasi ordered $*$-vector spaces with an order unit. Since state spaces are among the main tools in the study of structure of ordered $*$-vector spaces, in absence of properness of positive cones, it is quite natural to ask whether the state space of a given quasi ordered $*$-vector space is non-empty. This question has led to a simple and extremely useful characterization of such spaces. Indeed, the state space of a quasi ordered $*$-vector space with order unit is non-empty if and only if its order unit is non-negative. By means of some examples, we show that having a non-negative order unit is strictly weaker than having a proper cone. Then we demonstrate that the aforementioned condition suffices for many of the classical results including the main results of Paulsen and Tomforde (2009) to work with adapted arguments. Our demonstration in the next section begins with the study of real case, that is, quasi ordered real vector spaces and then extends to complex case in section 3 where we consider quasi ordered $*$-vector spaces. In the last section, we apply the results of previous sections to self adjoint unital subspaces of unital $*$-algebras that we call quasi operator systems to extend some existing results on operator systems to quasi operator systems and simplify some existing arguments in the literature.

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2. Quasi ordered real vector spaces

A *quasi ordered real vector space* (X, X^+) is a pair consisting of a real vector space X and a cone $X^+ \subseteq X$. We call X^+ a *proper cone* if it satisfies $X^+ \cap -X^+ = \{0\}$. We may define a relation $\geq$ on X by letting $a \geq b$ if and only if $a - b \in X^+$. The relation $\geq$ which is reflexive and transitive is called a *quasi order* on X. Notice that X^+ is a proper cone if and only if $\geq$ is antisymmetric, in which case $\geq$ turns out to be a partial order on X. In this case, we say that (X, X^+) is an *ordered real vector space*. Also, we say that X^+ is *full* if $X = X^+ - X^+$. An element $e \in X$ is called an *order unit* for X if for each $a \in X$ there exists a real number $r > 0$ such that $re \geq a$.

It is easy to see that the positive cone of a quasi ordered real vector space with order unit e is a full cone and $e \in X^+$. Moreover, if $a \in X$ and a real number $r > 0$ is chosen so that $re \geq a$, then $se \geq a$ for all $s \geq r$.

The cone X^+ is called *Archimedean* provided that given any $a_0 \in X^+$ if $a \in X$ and $ra_0 + a \geq 0$ for all $r > 0$ then $a \geq 0$. If X has an order unit e, one needs only to consider the element $a_0 = e$ to prove that X^+ is Archimedean. In this case, e is called Archimedean.

If (X, X^+) and (Y, Y^+) are two quasi ordered real vector spaces with order units e and e' respectively, then a linear map $\phi : X \to Y$ is *positive* if $a \in X^+$ implies $\phi(a) \in Y^+$, and *unital* if $\phi(e) = e'$. A linear map $\phi : X \to Y$ is an *order isomorphism* if ϕ is bijective and both ϕ and ϕ^{-1} are positive. An $\mathbb{R}$-linear functional $f : X \to \mathbb{R}$ is called *positive* if $f(X^+) \subseteq [0, \infty)$. Also, a positive $\mathbb{R}$-linear functional $f : X \to \mathbb{R}$ is called a *state* if $f(e) = 1$. We call the set of all states on X the *state space* of X, and denote it by $S(X)$.

If $E \subseteq X$, we say that E *majorizes* X^+ if for each $a \in X^+$ there exists $b \in E$ such that $b \geq a$. Paulsen and Tomforde (2009) proved an analogue of the Hahn-Banach Theorem for ordered real vector spaces, which is a generalization of Corollary 2.1 of Kadison (1951). The following is a generalization of Theorem 2.14 of Paulsen and Tomforde (2009) which can be verified by a similar argument.

Theorem 2.1. Suppose that (X, X^+) is a quasi ordered real vector space and X^+ is a full cone for X. If E is a subspace of X that majorizes X^+, then any positive $\mathbb{R}$-linear functional $f : E \to \mathbb{R}$ may be extended to a positive $\mathbb{R}$-linear functional $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f}\,|_E = f$.

Observe that in a quasi ordered real vector space with order unit e, any subspace E containing e satisfies the hypothesis of the preceding theorem. Next, we characterize quasi ordered real vector spaces with non-empty state space.

Theorem 2.2. If (X, X^+) is a quasi ordered real vector space with order unit e, then the state space of X is not empty if and only if $e \nleq 0$.

Proof: Suppose that $e \leq 0$. Given $a \in X$, we may choose $r > 0$ with $re \geq -a$. So $0 \geq -a$ and hence $a \geq 0$. Thus $X = X^+$. Now if f is a positive $\mathbb{R}$-linear functional on $E = span\{e\}$, it follows that $0 \leq f(e) \leq 0$ since $0 \leq e \leq 0$. So $f(e) = 0 \neq 1$, hence, the state space of X is empty.

Conversely, suppose that $e \nleq 0$. Since $e \geq 0$, the functional $f(re) = r$ defined on $E = span\{e\}$ is a state. So by the previous theorem, it can be extended to a state on X.

The above argument shows that trivialities $X = X^+$ and $S(X) = \emptyset$ can arise when $e \leq 0$. For this reason, henceforth we assume that e is non-negative.

Example 2.3. A quasi ordered real vector space with a non-negative order unit and non-proper cone: Let $X = \mathbb{R}^3$ and

$$X^+ = \{(x, y, z) : x > 0, y \in \mathbb{R}, and\ z \in \mathbb{R}\} \cup \{(0, y, z) : y \geq 0\ and\ z \in \mathbb{R}\}.$$

Since $X^+ \cap -X^+ = \{(0, 0, z) : z \in \mathbb{R}\}$, then X^+ is not proper. Moreover, $e = (1, 1, 1)$ is a non-negative order unit for (X, X^+), which is not Archimedean.

Indeed one has $re + (0, -1, 0) \in X^+$ for all $r > 0$, but $(0, -1, 0) \notin X^+$. For some other examples of this type see Esslamzadeh and Taleghani (2013).

Remark 2.4. Let (X, X^+) be a quasi ordered real vector space with non-negative order unit e. If $a \in X$, then by an argument similar to Theorem 2.17 of Paulsen (2009) we have

$$\begin{aligned} \alpha &= \sup\{r \in \mathbb{R} : re \leq a\} \\ &\leq inf\{s \in \mathbb{R} : a \leq se\} = \beta, \end{aligned}$$

and for every real number $\gamma \in [\alpha, \beta]$ there exists a state $f : X \to \mathbb{R}$ with $f(a) = \gamma$. Furthermore, $\{f(a) : f \in S(X)\} = [\alpha, \beta]$. When e is Archimedean, similar to Proposition 2.20 of Paulsen (2009) one can prove that if $a \in X$ and $f(a) \geq 0$ for every state $f : X \to \mathbb{R}$, then $a \in X^+$.

Definition 2.5. Let (X, X^+) be a quasi ordered real vector space with non-negative order unit e. For $a \in X$ let

$$\|a\| = inf\{r \in \mathbb{R} : -re \leq a \leq +re\}.$$

Note that the infimum exists and is a non-negative real number since e is non-negative. We call $\|\cdot\|$ the *order seminorm on X determined by e*.

Similar to Proposition 2.23 of Paulsen (2009) we can see that

$$\|a\| = max\{|\alpha|, |\beta|\} = max\{|f(a)| : f \in S(X)\}.$$

So, the order seminorm is actually a seminorm, which cannot be a norm when the positive cone is not proper, even if the unit is Archimedean. Indeed if X^+ is not proper, then there exists a non-zero element $a \in X^+ \cap -X^+$ so that $-re \leq 0 \leq a \leq 0 \leq re$ for all $r > 0$. Thus, $\|a\| = 0$, hence, $\|\cdot\|$ is not a norm.

Remark 2.6. (i) The previous argument shows, at the same time, that the state space of X does not separate elements of X whenever X^+ is not proper.
(ii) From Proposition 2.23 of Paulsen (2009), we know that having a proper Archimedean positive cone is sufficient for turning the order seminorm into a norm. But while properness of X^+ is a necessary condition, being Archimedean is not (Paulsen and Tomforde, 2009, Remark 2.25).

The following proposition generalizes Propositions 2.27, 2.28 of Paulsen and Tomforde (2009) with a similar proof.

Proposition 2.7. Let (X, X^+) be a quasi ordered real vector space with non-negative order unit e, and let $\|\cdot\|$ be the order seminorm on X determined by e. Then an $\mathbb{R}$-linear functional f on X is positive if and only if it is continuous with respect to the topology induced by $\|\cdot\|$ and $\|f\| = f(e)$.

The first half of the following result which characterizes the order seminorm is an extension of Theorem 2.29 of Paulsen and Tomforde (2009). The second half gives equivalent conditions for the Archimedean property of positive cone as in Theorem 2.30 of Paulsen and Tomforde (2009).

Theorem 2.8. Let (X, X^+) be a quasi ordered real vector space with non-negative order unit e. Then the order seminorm $\|\cdot\|$ is the unique seminorm on X satisfying the next three conditions
(i) $\|e\| = 1$,
(ii) if $-b \leq a \leq b$ then $\|a\| \leq \|b\|$,
(iii) for any state $f : X \to \mathbb{R}$, $|f(a)| \leq \|a\|$.
Moreover, the following are equivalent:
(iv) e is Archimedean,
(v) X^+ is a closed subset of X in the order topology induced by $\|\cdot\|$,
(vi) $-\|a\|e \leq a \leq \|a\|e$ for all $a \in X$.

Now, we consider a process similar to §2.3 of Paulsen and Tomforde (2009) for turning a quasi ordered vector space to an Archimedean *ordered* vector space which is called *Archimedeanization*.
Let (X, X^+) be a quasi ordered real vector space with non-negative order unit e. Define

$$D = \{a \in X : re + a \in X^+ \text{ for all } r > 0\},$$

$N = D \cap -D$ and $(X/N)^+ = D + N = \{a + N : a \in D\}$. Notice that D is a cone with $X^+ \subseteq D$ and N is a real subspace of X. An argument similar to Proposition 2.34 and Theorem 2.35 of Paulsen and Tomforde (2009), shows that D is equal to the closure of X^+ in the order topology,

$$N = \{a \in X : \|a\| = 0\} = \bigcap \{\ker f : f : X \to \mathbb{R} \text{ is a state}\},$$

and $(X/N, (X/N)^+)$ is an ordered vector space with Archimedean order unit $e + N$ which is denoted by X_{Arch}, we call it the *Archimedeanization* of X. Moreover, if the order seminorm is a norm on X, then (X, D) itself is an ordered vector space with Archimedean order unit e.

The following result whose proof is similar to Theorem 2.38 of Paulsen and Tomforde (2009) describes a universal property that characterizes Archimedeanization.

Theorem 2.9. Let (X, X^+) be a quasi ordered real vector space with non-negative order unit e, and let X_{Arch} be the Archimedeanization of X. Then there exists a unital surjective positive linear map $q : X \to X_{\mathrm{Arch}}$ with the property that whenever (Y, Y^+) is an ordered vector space with Archimedean order unit e', and $\phi : X \to Y$ is a unital positive linear map, then there exists a unique positive linear map $\widetilde{\phi} : X_{\mathrm{Arch}} \to Y$ with $\phi = \widetilde{\phi} \circ q$.

In addition, this property characterizes X_{Arch}: if X' is any ordered vector space with an Archimedean order unit and $q' : X \to X'$ is a unital surjective positive linear map with the above property then X' is isomorphic to X_{Arch} via a unital order isomorphism $\psi : X_{\mathrm{Arch}} \to X'$ with $\psi \circ q = q'$.

In order to study quotients of quasi ordered real vector spaces, we need the notion of order ideal which is a generalization of a concept introduced in Definition 2.2 of Kadison (1951) and Definition 2.40. of Paulsen and Tomforde (2009). A subspace $J \subseteq X$ of a quasi ordered vector space (X, X^+) is called an *order ideal* provided that $p \in J$ and $0 \leq q \leq p$ implies that $q \in J$. If X has an order unit e, then it is easy to see that $(X/J, X^+ + J)$ is an ordered real vector space with order unit $e + J$ which is called the *quotient of X by J*, and its Archimedeanization is called the *Archimedean quotient of X by J*. Note that in the construction of

Archimedean quotient, as it is described in the next proposition, the order unit of X need not be Archimedean, hence, its assumption is unnecessary in the definition of Archimedean quotient.

Also note that if $\phi : X \to Y$ is a positive map between quasi ordered vector spaces then $\ker\phi$ need not be an order ideal unless the positive cone of Y is proper. By definition, the Archimedean quotient of X by an order ideal J is a quotient of X/J therefore, a quotient of a quotient of X. The following proposition whose proof is similar to Proposition 2.44 of Paulsen and Tomforde (2009) shows how this may be viewed as a quotient of X.

Proposition 2.10. Let (X, X^+) be a quasi ordered real vector space with non-negative order unit e, and let J be an order ideal of X. If we define
$N_J = \{a \in X : \forall r > 0 \ \exists j, k \in J$ such that

$$j + re + a \in X^+, \ k + re - a \in X^+\},$$

then N_J is an order ideal of X. Furthermore, if we let

$$(X/N_J)^+ = \{a + N_J : \forall r > 0 \ \exists j \in J \text{ such that } j + re + a \in X^+\}$$

then $(X/N_J, (X/N_J)^+)$ is an ordered vector space with Archimedean order unit $e + N_J$. Moreover, this space is unitally ordered isomorphic to the Archimedean quotient of X by J.

The following theorem whose proof is similar to Theorem 2.45 of Paulsen and Tomforde (2009) shows that neither properness of positive cone nor Archimedean property of order unit of the domain of ϕ are necessary assumptions in Theorem 2.45 of Paulsen and Tomforde (2009). Indeed, only non-negativity of order unit is needed to avoid trivialities. Moreover $\widetilde{\phi}$ need not be an order isomorphism without the assumption $\phi(X^+) = Y^+ \cap \phi(X)$.

Theorem 2.11. Let (X, X^+) be a quasi ordered real vector space with non-negative order unit e, and let (Y, Y^+) be an ordered real vector space with Archimedean order unit e'. If $\phi : X \to Y$ is a unital positive linear map, then $\ker\phi$ is an order ideal and the Archimedean quotient of X by $\ker\phi$ is unitally ordered isomorphic to $X/\ker\phi$ with positive cone

$$(X/\ker\phi)^+ = \{a + \ker\phi : \forall\, r > 0 \,\exists\, j \in \ker\phi \text{ such that } j + re + a \in X^+\}$$

and Archimedean order unit $e + \ker\phi$. In addition, the map $\widetilde{\phi} : X/\ker\phi \to Y$ given by $\widetilde{\phi}(a + \ker\phi) = \phi(a)$ is a unital positive linear map. Moreover, if $\phi(X^+) = Y^+ \cap \phi(X)$, then $\widetilde{\phi}$ is an order isomorphism from $X/\ker\phi$ onto $\phi(X)$.

3. Quasi ordered ∗-vector spaces

A *∗-vector space* consists of a complex vector space X together with a conjugate linear map $* : X \to X$. We let $X_{sa} := \{x \in X : x^* = x\}$, and we call the elements of X_{sa} the *self-adjoint* elements of X. It is easy to see that X_{sa} is a real subspace of the complex vector space X. Also, note that any $a \in X$ may be written uniquely as $a = x + iy$ with $x, y \in X_{sa}$, in fact, $x = (a + a^*)/2$ and $y = (a - a^*)/2i$ which we call the *real* and *imaginary parts* of a respectively, and we write $\mathrm{Re}(a) = x$, $\mathrm{Im}(a) = y$. So we have $X \cong X_{sa} \oplus iX_{sa}$ as real vector spaces.

Definition 3.1. If X is a ∗-vector space, then we say that (X, X^+) is a *quasi ordered ∗-vector space* if X^+ is a cone of X_{sa}. In this case for $a, b \in X_{sa}$ by $a \geq b$ we mean $a - b \in X^+$.

Notice that (X, X^+) is a quasi ordered ∗-vector space if and only if (X_{sa}, X^+) is a quasi ordered real vector space. Similar statements hold for e to be an order unit for (X, X^+) and e to be Archimedean. The notions of positive map, unital map, order isomorphism, and state are defined as in the real case. It is easy to see that a positive linear map ϕ from a quasi ordered ∗-vector space (X, X^+) with an order unit into a quasi ordered ∗-vector space (Y, Y^+) is *self-adjoint*, that is, $\phi(a^*) = \phi(a)^*$ for all $a \in X$; Remark 3.5 of Esslamzadeh and Taleghani (2013) shows that this is not necessarily true when X does not have any order unit.

Let (X, X^+) be a quasi ordered ∗-vector space. If $f : X_{sa} \to \mathbb{R}$ is $\mathbb{R}$-linear, then we define the $\mathbb{C}$-linear map $\widetilde{f} : X \to \mathbb{C}$ by $\widetilde{f}(a) = f(\mathrm{Re}(a)) + if(\mathrm{Im}(a))$. Observe that f is positive if and only if f is positive, and f is a state if and only if f is a state. In addition, any $\mathbb{C}$-linear functional f on a quasi ordered ∗-vector space (X, X^+) with non-enegative order unit e is positive if and only if $f = \widetilde{g}$ for some positive $\mathbb{R}$-linear functional $g : X_{sa} \to \mathbb{R}$.

Remark 2.6 clarifies how Propositions 3.12 and 3.13 of Paulsen and Tomforde (2009) fail whenever X^+ is not proper. Now, we extend the process of Archimedeanization for real vector spaces described in the previous section to complex ∗-vector spaces. Let (X, X^+) be a quasi ordered ∗-vector space with non-negative order unit e. Define

$$D = \{a \in X_{sa} : re + a \in X^+, \text{ for all } r > 0\},$$

and $N_{\mathbb{R}} = D \cap -D$ which is a real subspace of X_{sa}. It is easy to see that

$$N_{\mathbb{R}} = \{\ker f : f : X_{sa} \to \mathbb{R} \text{ is a state}\}.$$

In analogy, we define

$$N = \{\ker f : f : X \to \mathbb{C} \text{ is a state}\}.$$

One can verify that $N = N_{\mathbb{R}} \oplus iN_{\mathbb{R}}$ and N is a complex subspace of X that is closed under the $*$-operation. Thus, we form the quotient X/N with the well-defined $*$-operation $(a+N)^* = a^* + N$. So $(X/N)_{sa} = \{a + N : a \in X_{sa}\}$. We define the positive elements of $(X/N)_{sa}$ to be the set $(X/N)^+ = \{a + N : a \in D\}$. Then $((X/N)_{sa}, (X/N)^+)$ is ordered isomorphic to $(X_{sa}/N_{\mathbb{R}}, D + N_{\mathbb{R}})$ via the map $a + N \to a + N_{\mathbb{R}}$. Thus, $(X/N, (X/N)^+)$ is an *ordered* $*$-vector space with Archimedean order unit $e + N$ which is called *Archimedeanization of* X and is denoted by X_{Arch}.

Remark 3.2. If we replace the term "real vector space" in Theorem 2.9 with "$*$-vector space", then we obtain the complex version of Theorem 2.9 which can be proved with an argument similar to the proof of theorem 3.16 of Paulsen and Tomforde (2009).

Let (X, X^+) be a quasi ordered $*$-vector space. A complex subspace $J \subseteq X$ is called *self adjoint* if $J = J^*$ where $J^* = \{a^* : a \in J\}$. A self adjoint subspace J is called an *order ideal* if $p \in J$ and $0 \le q \le p$ implies that $q \in J$. Notice that if J is a self adjoint subspace of a quasi ordered $*$-vector space X then we may define a $*$-operation on X/J by $(a+J)^* = a^* + J$. In addition, if we define $J_{\mathbb{R}} = J \cap X_{sa}$, then $J_{\mathbb{R}}$ is a real subspace of X and $J = J_{\mathbb{R}} \oplus iJ_{\mathbb{R}}$.

It is easy to see that if (X, X^+) is a quasi ordered $*$-vector space with non-negative order unit e and if $J \subseteq X$ is an order ideal then $(X/J, X^+ + J)$ is an ordered $*$-vector space with order unit $e + J$ which is called the *quotient of* X *by* J and its Archimedeanization is called the *Archimedean quotient of* X *by* J. If we define

$$N_{J_{\mathbb{R}}} = \{a \in X_{sa} : \forall r > 0 \ \exists j, k \in J_{\mathbb{R}} \text{ such that } j + re + a \in X^+, \quad k + re - a \in X^+\}$$

and we set $N_J = N_{J_{\mathbb{R}}} \oplus iN_{J_{\mathbb{R}}}$, then N_J is an order ideal of X . Furthermore, if we let

$$(X/N_J)^+ = \{a + N_J : \forall r > 0 \ \exists j \in J \text{ such that } j + re + a \in X^+\}$$

then by an argument similar to that of Proposition 3.21 of Paulsen and Tomforde (2009), we see that $(X/N_J, (X/N_J)^+)$ is an ordered vector space with Archimedean order unit $e + N_J$, and the Archimedean quotient of X by J is unitally ordered isomorphic to $(X/N_J, (X/N_J)^+)$. Also, we can prove the complex version of Theorem 2.11 with the same argument.

Convention: Henceforth, (X, X^+) is a quasi ordered $*$-vector space with non-negative order unit e and $\|\cdot\|$ denotes the order seminorm on X_{sa} unless otherwise specified.

Definition 3.3. A seminorm $|||\cdot|||$ on X is called a $*$-*seminorm* if $|||a^*||| = |||a|||$ for all $a \in X$. An *order seminorm* on X is a $*$-seminorm $|||\cdot|||$ on X with the property that $|||a^*||| = \|a\|$ for all $a \in X_{sa}$.

Remark 3.4. Let X be a quasi ordered $*$-vector space with order unit e. Applying the argument of Theorem 2.2, one can show that the following are equivalent:
(i) The state space $S(X)$ is empty,
(ii) $e \le 0$,
(iii) $X_{sa} = X^+ = -X^+$,
(iv) $X_{sa} = X^+ \cap -X^+$,
(v) The zero seminorm is an order seminorm on X and hence is the only order seminorm on X, by Theorem 3.6 below,
(vi) The only positive linear functional on X is the zero functional.

Definition 3.5. The *minimal order seminorm* $\|\cdot\|_m$, the *maximal order seminorm* $\|\cdot\|_M$, and the *decomposition seminorm* $\|\cdot\|_{\text{dec}}$ on X are defined by the identities

$$\|a\|_m = sup\{|f(a)| : f : X \to \mathbb{C} \text{ is a state}\},$$
$$\|\cdot\|_M = inf\left\{\sum_{i=1}^{n} |\lambda_i| \|a_i\| : a = \sum_{i=1}^{n} \lambda_i v_i, a \in X_{sa}, \lambda_i \in \mathbb{C}\right\}$$
$$\|a\|_{\text{dec}} = inf\left\{\left\|\sum_{i=1}^{n} |\lambda_i| p_i\right\| : a = \sum_{i=1}^{n} \lambda_i p_i, p_i \in X^+, \lambda_i \in \mathbb{C}\right\}$$

In the following theorem, we have collected some fundamental facts regarding these order seminorms which are extensions of Theorems 4.5, 4.7 and Propositions 4.9 and 4.11of Paulsen and Tomforde (2009) with similar proofs.

Theorem 3.6. Let (X, X^+) be a quasi ordered $*$-vector space with non-negative order unit e. Then the following statements hold.
(i) $\|\cdot\|_m$, $\|\cdot\|_M$, and $\|\cdot\|_{\text{dec}}$ are all order seminorms on X. Moreover, for any order seminorm $|||\cdot|||$ on X we have

$$\|a\|_m \leq |||a||| \leq \|a\|_M \quad (a \in X).$$

(ii) Any two order seminorms on X are equivalent, and if $||| \cdot |||$ is an arbitrary order seminorm on X then

$$\{a \in X : |||a||| = 0\} = \{\ker f : f: X \to \mathbb{C} \text{ is a state}\}.$$

In particular, the order seminorm on X_{sa} is a norm if and only if one and hence all of the order seminorms on X are norms.

The last statement of the preceding theorem shows that the topology generated by an order seminorm on X is independent of which order seminorm is used. This allows us to use the term *order topology* for this topology without any ambiguity. One can easily see that any positive linear functional f on X is continuous with respect to the order topology, and $\|f\| = f(e)$. In addition, Remark 4.4 shows that if X^+ is not proper then X_{sa} is not necessarily closed in the order topology.

Given a linear map $\phi : X \to Y$ between quasi ordered $*$-vector spaces, we let $\|\phi\|_m$ (resp. $\|\phi\|_M$/ $\|\phi\|_{\text{dec}}$) denote the seminorm of the map ϕ when both X and Y are given the minimal order (resp. maximal order/ decomposition) seminorm.

The following theorem which is used in the next section can be proved similar to Theorem 4.22 of Paulsen and Tomforde (2009), with an argument based on the extension of Lemma 4.20 of Paulsen and Tomforde (2009) to quasi ordered $*$-vector spaces.

Theorem 3.7. Let (X, X^+) and (Y, Y^+) be two quasi ordered $*$-vector spaces with non-negative order units. If $\phi : X \to Y$ is a unital positive linear map, then $\|\phi\|_m = \|\phi\|_{\text{dec}} = 1$.

4. Properties of $\|\cdot\|_{[X]}$ on a Quasi Operator System X

In this section, A is a complex unital $*$-algebra whose unit is denoted with e. We call any subspace (resp. unital self adjoint subspace) X of A a *quasi operator space* (resp. *quasi operator system*) in A or briefly a quasi operator space (resp. quasi operator system) when the algebra A is clear from the context. An element a of A is called *positive* if $a = \sum_{k=1}^{n} a_k{}^* a_k$ for some $n \in \mathbb{N}$ and some $a_1, \ldots, a_n \in A$. An element x of A is said to be *bounded* if there exists a positive real number k such that $x^* x \leq ke$. The set of all such elements is a $*$-subalgebra of A which is denoted by A_0 and is called the *bounded subalgebra* of A. If X is a subspace of A, then by X^+ we mean $X \cap [X]^+$ where $[X]$ is the unital self adjoint subalgebra of A generated by X; While by X_0 we mean $X \cap [X]_0$. It was shown in Theorem 2.8 of Esslamzadeh and Taleghani (2013) that A_0 has a C^*-seminorm defined by

$$\|a\|_A = inf\{k \in \mathbb{R}^+ : a^* a \leq k^2 e\}, \qquad (a \in A_0).$$

We warn the reader that the unit of a quasi operator system X is not an order unit, unless $X = X_0$.

Theorem 4.1. Let X be a bounded quasi operator system. If E is a subspace of X containing e where $E^+ = X^+ \cap E$, then any positive linear functional $f: E \to \mathbb{C}$ may be extended to a positive linear functional $F: X \to \mathbb{C}$.

Proof: Using notations of section 3, we know $f = \tilde{g}$ for some positive $\mathbb{R}$-linear map $g: E_{sa} \to \mathbb{R}$. But E_{sa} is a real subspace of X_{sa} and by Theorem 2.1 g may be extended to a positive $\mathbb{R}$-linear map $G: X_{sa} \to \mathbb{R}$ such that $G|_{E_{sa}} = g$. Now, it is easy to see that the map $\tilde{G} : X \to \mathbb{C}$ defined by $\tilde{G}(a + ib) = G(a) + iG(b)$ for all $a, b \in X_{sa}$ is a positive linear functional and $\tilde{G}|_E = \tilde{g} = f$.

Notice that there is a quasi operator system with unit e such that $e \leq 0$. For instance, see Example 5.4 of Esslamzadeh (2013). Moreover there are quasi operator systems with Archimedean non-negative order unit whose positive cone is not proper, namely, Example 4.5 below. See also Examples 5.5, 5.6 of Esslamzadeh (2013). This fact, together with Remark 3.4 show that assuming $e \nleq 0$ would avoid trivialities. Indeed, examples such as the one introduced in Example 5.4 of Esslamzadeh (2013) lack interest since they do not have any non-trivial induced C^*-seminorm or a non-zero positive linear functional. As such, from now on we assume that the order units of quasi operator systems are always non-negative.

A seminorm $||| \cdot |||$ on a quasi ordered real vector space X is called *absolutely monotone* if $-x \leq y \leq x$ implies that $|||y||| \leq |||x|||$. Moreover, $||| \cdot |||$ is called *regular* if it is absolutely monotone and $|||y||| < 1$ implies the existence of $x \in X$ such that $|||x||| < 1$ and $-x \leq y \leq x$. A seminorm on a quasi ordered $*$-vector space X is called regular if its restriction to X_{sa} is regular. In the next proposition we show that $\|\cdot\|_{[X]}$ satisfies this property.

Proposition 4.2. Let X be a quasi operator system. The C^*-seminorm $\|\cdot\|_{[X]}$ is an order seminorm on $(X_0)_{sa}$ which is regular and for each $x \in (X_0)_{sa}$ we have

$$\|x\|_{[X]} = \max\{|f(x)| : f \in S(X_0)\}.$$

Proof: Being an order seminrom follows from the definition of $\|\cdot\|_{[X]}$. By Theorem 2.8 the restriction of $\|\cdot\|_{sa}$ to $\|\cdot\|_{[X]}$ is an absolutely monotone seminorm on $(X_0)_{sa}$. If $y \in (X_0)_{sa}$ with $\|y\|_{[X]} < 1$, then there is an $r \in \mathbb{R}$ such that $\|y\|_{[X]} < r < 1$ and $-re \leq y \leq re$ by definition of $\|\cdot\|_{[X]}$. So if $x = re$, we have $x \in (X_0)_{sa}$, $-x \leq y \leq x$, and $\|x\|_{[X]} < 1$. Therefore $\|\cdot\|_{[X]}$ is a regular seminorm. Now suppose $x \in (X_0)_{sa}$. The last identity of the proposition follows from the observation of preceding Remark 2.6.

Theorem 4.3. Let X be a bounded quasi operator system in A with unit e. Then the following statements are equivalent:
(i) e is Archimedean,
(ii) X^+ is a closed subset of X_{sa} in the order topology induced by $\|\cdot\|_{[X]}$,
(iii) $-\|x\|_{[X]}e \leq x \leq \|x\|_{[X]}e$ for all $x \in X_{sa}$.
If in addition to these conditions $[X]^+$ is proper, then $\|\cdot\|_{[X]}$ is an algebra norm and X_{sa} and X^+ are $\|\cdot\|_{[X]}$-closed subsets of X.

Proof: Equivalence of these conditions follows from Theorem 2.8. Suppose that X^+ is proper and Archimedean. Then by Theorem 2.8 of Esslamzadeh and Taleghani (2013), $\|\cdot\|_{[X]}$ is a C^*-norm. In particular it is an algebra norm and X_{sa} is a $\|\cdot\|_{[X]}$-closed subset of X. By (ii), X^+ is $\|\cdot\|_{[X]}$-closed in X_{sa}, hence, X^+ is closed in X.

Remark 4.4. The Archimedean (not necessarily proper) positive cone of a quasi operator system X, even with the assumption $e \nleq 0$, is not necessarily a $\|\cdot\|_{[X]}$-closed subset of X_0, and so is $(X_0)_{sa}$. In addition, $\|\cdot\|_{[X]}$ is not necessarily a norm in such spaces. See the next example.

Example 4.5. A bounded quasi operator system with Archimedean positive cone which is neither proper nor closed: Let $A = \mathbb{C} \oplus S(2)$ as in Example 5.5 of Esslamzadeh and Taleghani (2013). Then A is a $*$-algebra with order unit $e = (1,1,0)$, positive cone $A^+ = \mathbb{R}^+ \oplus \mathbb{R} \oplus \mathbb{R}$, and $A_{sa} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$. So A^+ is not proper and $\|(\alpha,\beta,\gamma)\|_A = |\alpha|$ for all $(\alpha,\beta,\gamma) \in A$. Moreover, e is Archimedean, since if $(\alpha,\beta,\gamma) + r(1,1,0) \geq 0$, then $\alpha + r \geq 0, \beta + r, \gamma \in \mathbb{R}$ and so $(\alpha,\beta,\gamma) \in \mathbb{R}^+ \oplus \mathbb{R} \oplus \mathbb{R} = A^+$.

But $\|\cdot\|_A$ is not a norm since $\|(0,1,i)\|_A = |0| = 0$. Moreover, A_{sa} and A^+ are not $\|\cdot\|_A$-closed in A since $(1,i,-i)$ is a limit point of A_{sa} although it is not self adjoint. Indeed, for all $r > 0$, $(1,0,0) \in A_{sa} \cap \mathrm{B}_r(1,i,-i)$ since $\|(1,i,-i) - (1,0,0)\|_A = 0$. Similarly A^+ is not closed.

The argument for proving the equivalence of order seminorms in section 3 yields the next theorem.

Theorem 4.6. The C^*-seminorm $\|\cdot\|_{[X]}$ is an order seminorm on X_0 that satisfies the inequalities

$$\tfrac{1}{2}\|x\|_{[X]} \leq \|x\|_m \leq \|x\|_{[X]} \leq \|x\|_M \leq 2\|x\|_{[X]}, \quad (x \in X_0).$$

Moreover, we have

$$\{x \in X : \|x\|_{[X]} = 0\} = \{x \in X : \|x\|_m = 0\} = \bigcap_{f \in S(X_0)} \ker f.$$

We drop the assumption that A is unital just in the next theorem and we denote its unitization, that is, $A \oplus \mathbb{C}$ with usual product and involution by A^1. In the next theorem, we give a direct elementary proof for Theorem 10.2.4 of Palmer (2001) that any T^*-algebra A is a BG^*-algebra. But first we need to recall some notations. If for every $a \in A$ $sup\{\|\pi(a)\| : \pi$ is a $*$-representation of A on some Hilbert space$\} < \infty$, then we say that A is a *G^*-algebra* and the above supremum defines a C^*-seminorm on A which is called the *Gelfand-Naimark seminorm* and is denoted with $\|\cdot\|_\gamma$. Let H be a pre-Hilbert space and $L(H)$ be the set of all linear maps from H into H. A *pre-$*$-representation* of a $*$-algebra A on H is a $*$-homomorphism of A into $L_*(H)$ where $L_*(H)$ is the set of all elements of $L(H)$ that have an adjoint in $L(H)$. We say that A is a *BG^*-algebra* if every pre-$*$-representation π of A on a pre-Hilbert space H is normed, that is, π maps into $B(H)$. We say that A is a *T^*-algebra* if for every self adjoint element $a \in A$, there is a $t \in \mathbb{R}$ satisfying $t1_A + a \in A^+$.

Theorem 4.7. Any T^*-algebra A is a BG^*-algebra satisfying

$$inf\{t \in \mathbb{R}^+ : t^2 - a^*a \in (A^1)^+\} = \|a\|_{A^1} \quad \forall a \in A.$$

Proof: Let A be a T^*-algebra and $(a,s) \in A^1$ be arbitrary. Then $b = -(a^*a + sa^* + \bar{s}a)$ is in A_{sa}, so, by assumption there is a $t \in \mathbb{R}$ such that

$$(-(a^*a + sa^* + \bar{s}a), t) = t(0,1) + (b,0) \in (A^1)^+.$$

Thus for $r = |s|^2 + t$,

$$\begin{aligned} r(0,1) - (a,s)^*(a,s) &= r(0,1) - (a^*,\bar{s}) \\ &= (-(a^*a + sa + \bar{s}a), t) \in (A^1)^+. \end{aligned}$$

So $A^1 = (A^1)_0$. Now, let π be a $*$-representation of A on a Hilbert space H. Then π can be extended to a $*$-representation $\tilde{\pi}$ of A^1 on H with $\tilde{\pi}(a,s) = \pi(a) + sI$. Since $\tilde{\pi}(0,1) = I$ is positive,

$\|\pi(a)\| = inf\{r \in \mathbb{R}^+ : \pi(a)^*\pi(a) \leq r^2 I\}$
$= inf\{r \in \mathbb{R}^+ : \widetilde{\pi}(a,0)^* \widetilde{\pi}(a,0) \leq$
$r^2 \widetilde{\pi}(0,1)\} \leq inf\{r \in \mathbb{R}^+ : a^*a \leq r^2(0,1)\} = \|a\|_{A^1}$.

So $\|a\|_\gamma \leq \|a\|_{A^1}$ and hence A is a G^*-algebra. With a similar argument we see that any pre-$*$-representation of A on a pre-Hilbert space is normed and A is a BG^*-algebra. By Theorem 9.5.4 of Palmer (2001), there is a $*$-representation π of A satisfying $\pi(a) = \|a\|_{A^1}$, so $\|a\|_\gamma \geq \|a\|_{A^1}$. Therefore $\|a\|_\gamma = \|a\|_{A^1}$, $(a \in A^1)$.

The next lemma which is an extension of Lemma 3.4 of Esslamzadeh and Taleghani (2013) is proved by using the arguments of previous sections. Note that Remark 3.5 of Esslamzadeh and Taleghani (2013) shows that next Lemma is not necessarily true for unbounded elements of X. Hence when a quasi ordered $*$-vector space (X, X^+) does not have any order unit, X^+ is not necessarily full and a positive linear map on X is not necessarily self adjoint.

Lemma 4.8. Let X be a quasi operator system and Y be a quasi ordered $*$-vector space with an order unit. Then each element of X_0 can be written as a linear combination of four positive elements of X_0. In addition, restriction of any positive linear map $\phi : X \to Y$ to X_0 is self-adjoint.

Proof: Since $(X_0)_{sa}$ is a quasi ordered real vector space with an order unit e, $(X_0)^+$ is a full cone for $(X_0)_{sa}$. But $X_0 = (X_0)_{sa} \oplus i(X_0)_{sa}$. Therefore, we can write any element of X_0 as a linear combination of four positive elements of X_0. Now suppose that $\phi : X \to Y$ is a positive linear map. Then its restriction to X_0 which is a quasi ordered $*$-vector space with order unit is positive. So if $a = \sum_{j=1}^4 \lambda_j a_j \in X_0$ where $a_j \in (X_0)^+$, then $\phi(a^*) = \sum_{j=1}^4 \overline{\lambda}_j \phi(a_j) = \phi(a)^*$. Therefore, ϕ is self-adjoint.

In the next theorem we provide an alternate shorter proof for the main statement of Theorem 3.6 of Esslamzadeh and Taleghani (2013). Note that as in Theorem 3.6 of Esslamzadeh and Taleghani (2013) we may replace the assumption of ϕ being unital with boundedness of $\phi(e)$. Besides, Example 5.12 of Esslamzadeh and Taleghani (2013) shows that 2 is the best bound for $\|\phi\|$ in this theorem.

Theorem 4.9. Let X and Y be quasi operator systems with non-negative units e and e', respectively. If $\phi : X \to Y$ is a unital positive linear map, then ϕ is bounded and $\|\phi\| \leq 2$.

Proof: First we show that $\phi(X_0) \subseteq Y_0$. To see this, suppose that $a \in (X)_{sa}$ is bounded. Then there is an $r \in \mathbb{R}^+$ such that $-re \leq a \leq re$. Since ϕ is positive, we have

$$-re' = -r\phi(e) \leq \phi(a) \leq r\phi(e) = re',$$

so, $\phi(a)$ is bounded in Y. Now let a be a bounded element of X. Then $\phi(\mathrm{Re}(a))$, $\phi(\mathrm{Im}(a))$ are self-adjoint and bounded in Y. Since Y_0 is a subspace of Y, it yields $\phi(a) = \phi(\mathrm{Re}(a)) + i\phi(\mathrm{Im}(a)) \in Y_0$.

Now the first statement of proof shows that restriction of ϕ is a unital positive linear map from the quasi ordered $*$-vector space X_0 with order unit e to the quasi ordered $*$-vector space Y_0 with order unit e'. So by Theorem 3.7, $\|\phi\|_m = 1$. Let $a \in X_0$ and $\|a\|_{[X]} \leq 1$. By Theorem 4.6, $\|a\|_m \leq \|a\|_{[X]} \leq 1$, hence $\|\phi(a)\|_m \leq 1$. Therefore by reusing Theorem 4.6 we have $\|\phi(a)\|_{[Y]} \leq 2\|\phi(a)\|_m \leq 2$.

The next Theorem yields an alternate proof for Theorem 3.8 and Proposition 3.10 of Esslamzadeh and Taleghani (2013).

Theorem 4.10. A linear functional f on a quasi operator system X with unit e is positive on X_0 if and only if it is bounded and $\|f\| = f(e)$. Moreover, if Ω is a compact subset of $\mathbb{C}$, X is a quasi operator system in A with non-negative unit and if $\phi : X \to C(\Omega)$ is a unital linear map which is positive on X_0, then $\|\phi\| = 1$.

Proof: First assume that e is non-negative. If f is positive on X_0 then by Theorem 4.9 it is bounded. Let $a \in (X_0)_{sa}$. Then there is a $k > 0$ such that $-ke \leq a \leq ke$, hence $|f(a)| \leq kf(e)$. This implies that if $f(e) = 0$ then $f|_{X_0} = 0$. So we may assume that $f(e) \neq 0$. Consequently, $g = \frac{1}{f(e)} f$ is a state on X_0 and by the definition of $\|\cdot\|_m$ for every $a \in X_0$ we have

$$|f(a)| = f(e)|g(a)| \leq f(e)\|a\|_m \leq f(e)\|a\|_{[X]}.$$

Therefore $\|f\| = f(e)$. Conversely, let f be a bounded linear functional on X with $\|f\| = f(e)$. Then the restriction g of f to $(X_0)_{sa}$ is bounded too and its bound is equal to $f(e)$. So by Proposition 2.7, it is positive on $(X_0)_{sa}$. Therefore using notations of section 3, we see that $f|_{X_0} = \widetilde{g}$ is positive on X_0.

Next, assume that $e \leq 0$. Then by Remark 3.5, the only positive linear functional on X_0 is the zero functional and $\|f\| = f(e) = 0$. Conversely, suppose that f is bounded and $\|f\| = f(e)$. Since for all $a \in X_0$, $\|e-a\|_{[X]} = 0$, then $|f(e-a)| \leq f(e)$ and $f(a) \geq 0$. Thus, f is identically zero and positive on X_0.

Now let $\phi : X \to C(\Omega)$ be a unital linear map which is positive on X_0. Theorem 3.7 applied to X_0 implies that $\|\phi\|_m = 1$. Let $a \in X_0$ and $\|a\|_{[X]} \leq 1$. By Theorem 4.6, $\|a\|_m \leq \|a\|_{[X]} \leq 1$, so $\|\phi(a)\|_m \leq 1$. From Corollary 5.5 of Paulsen and Tomforde (2009) we conclude that $\|\phi(a)\| = \|\phi(a)\|_m \leq 1$. Therefore, $\|\phi\| = 1$.

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