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The axisymmetric bifurcation analysis of an elastic cylindrical shell subjected to external pressure and axial loading

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Abstract

In this paper, the deformation of a thick-walled circular cylindrical shell of incompressible isotropic elastic material is considered. The shell, which is made of Three-Term strain energy function is subjected to the combined external and axial loading pressure. In order to obtain the relevant eigenvalues, which is the main objective of the work, the incremental equilibrium equations are solved with two numerical, i.e. Adams-Moulton and Compound matrix methods. Finally the bifurcation behavior is investigated by plotting the radius changes with respect to the changes of the length of the cylinder.

***Keywords**:* Compound matrix method; Adams-Moulton method; eigenvalues; axisymmetric bifurcation

1. Introduction

The elastic structures are affected by the heavy pressure. While the stresses of an elastic object are normal, the deformation can be analyzed as an eigenvalue problem. Sierakowski et al. (1975), have investigated the axisymmetric modes of deformation of a tube subjected to internal pressure and axial compression for a Neohookean elastic material. The prismatic, axisymmetric, asymmetric bifurcations for axial tension and compression combined with internal and external pressure has been discussed by Haughton and Ogden (1979). A more general description of a bifurcation behavior of thick walled tubes subject to external pressure combined with axial loading, including axial compression and extension has been analyzed by Zhu et al. (2008). They analyzed the asymmetric bifurcation and solved the eigenvalue problem by Adams-Moulton method for a specific elastic material. Based on the analysis of Zhu et al. (2008) and Haughton and Ogden (1979), the axisymmetric bifurcation of a Three-Term elastic cylindrical shell under axial loading and external pressure was investigated in this work. In order to evaluate the efficiency and advantages of the above mentioned numerical methods, we have calculated and compared the results of the two methods together. Finally, by plotting the graph which shows the changes of the radius with respect to the changes of the length, the bifurcation points were highlighted. Fortran 90 programming (Metcalf et al., 2011) and Mathematica software (Wolfram, 2008) were employed for solving the relevant eigenvalue problem.

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2. The finitely-deformed circular cylindrical configuration

It was hypothesized that the thick–walled cylindrical shell is defined by Haughton and Orr (1979) as

$$A \le R \le B,\ 0 \le \Theta \le 2\pi,\ 0 \le Z \le L, \tag{1}$$

where R, Θ and Z are the cylindrical polar coordinates, A and B are the inner and outer radii of the shell respectively and L is the length, all in un-deformed configuration. The cylindrical shell is subjected to an axial loading and external pressure so that the circular cylindrical shape is maintained, and the shell is defined by

$$a \le r \le b,\ 0 \le \theta \le 2\pi,\ 0 \le z \le l, \tag{2}$$

where r, θ and z are the cylindrical polar coordinates, a and b are respectively the inner and outer radii and l is the length of the tube, all in the current configuration. The deformation is described by the following equations

$$r^2 = \lambda_z^{-1}(R^2 - a^2) + A^2, \theta = \Theta, z = \lambda_z Z, \tag{3}$$

where λ_z is the axial extension ratio. Let e_1, e_2 and e_3 denote the unit basis vectors corresponding to the coordinates (θ, z, r) respectively and let $\lambda_1, \lambda_2, \lambda_3$ be the corresponding principal stretches. From the incompressibility constraint we have

$$det(\boldsymbol{F})=\lambda_1\lambda_2\lambda_3=1, \quad (4)$$

where $\boldsymbol{F}$ is the deformation gradient. While from (3)

$$\lambda_2=\lambda_z,\ \lambda_1=\frac{r}{R}=\lambda,\ \lambda_3=(\lambda\lambda_z)^{-1}, \quad (5)$$

where we have used (4) and introduced the azimuthal stretch λ. We regard λ_z as a given constant and λ as a function of r (or R) from (3). In view of $(5)_2$ the deformed inner and outer radii are defined by $a=\lambda_a A$ and $b=\lambda_b B$, where λ_a and λ_b are constants that, for the present problem satisfy $\lambda_a>0$ and $\lambda_b<1$. For incompressible material, the equilibrium equations reduce to the following single equation

$$r\frac{d\sigma 33}{dr}+\sigma_{33}-\sigma_{11}=0, \quad (6)$$

and also the principal Cauchy stretches are given by $\sigma_{ii}=\sigma_i-p=\lambda_i\frac{\partial W}{\partial\lambda_i}-p, (i=I=1,2,3)$, where p is the hydrostatic pressure that arises due to the incompressible conditions and W=W(λ_1, λ_2, λ_3) is the strain-energy function which for Three-Term materials is defined by Haughton and Ogden (1979) as

$$W=\sum_{r=1}^{3}\frac{\mu_r(\lambda_1^{\alpha_r}+\lambda_2^{\alpha_r}+\lambda_3^{\alpha_r}-3)}{\alpha_r}. \quad (7)$$

In the above relation, α_r and $\mu_r (r=1,2,3)$ are pairs of material constants and summation over a finite number of terms is implied by the repetition of the subscript r and we have

$$\left.\begin{array}{lll}\alpha_1=1.3, & \alpha_2=5, & \alpha_3=-2,\\ \mu_1^*=1.491, & \mu_2^*=0.003, & \mu_3^*=-0.023.\end{array}\right\} \quad (8)$$

It should be noted that, $\mu_r^*=\frac{\mu_r}{\mu}$ and μ is the ground state shear modulus. For the mentioned deformation, the boundary conditions are

$$\sigma_{33}|_{r=a}=0,\ \ \sigma_{11}|_{r=b}=-P, \quad (9)$$

where P denotes the external hydrostatic pressure.

3. Bifurcation criterion

Here we write down a brief description of the equilibrium equation and the boundary condition in view of the derivation of Haughton and Ogden (1979). In the absence of the body forces the incremental equilibrium equations can be written

$$div\,\dot{\boldsymbol{s}}=0, \quad (10)$$

where div is the divergence operator and $\dot{\boldsymbol{s}}$ is the increment of the nominal stress, both in the current configuration. The incremental boundary conditions with respect to a hydrostatic pressure loading are

$$\dot{\boldsymbol{s}}^T\boldsymbol{n}=P\boldsymbol{\eta}^T\boldsymbol{n}-\dot{P}\boldsymbol{n}, \quad (11)$$

where T, $\boldsymbol{n}$ and $\boldsymbol{\eta}$ respectively denotes the transpose, the unit normal in the current configuration and the incremental deformation gradient. We should note that, P is not equal to zero for the problems with the external hydrostatic pressure and is zero for the everted problems. The incremental constitutive law is

$$\dot{\boldsymbol{s}}=\boldsymbol{B}\boldsymbol{\eta}+p\boldsymbol{\eta}-\dot{p}\boldsymbol{I}, \quad (12)$$

where $\boldsymbol{B}$ is the fourth order tensor of instantaneous moduli in the current configuration, and $\boldsymbol{I}$ is the identity tensor. The non-zero components of $\boldsymbol{B}$ for a general isotropic material are given by Haughton and Orr (1995)

$$\left.\begin{array}{l}B_{ijij}=\lambda_i^2\frac{\sigma_i-\sigma_j}{\lambda_i^2-\lambda_j^2},\ \ \lambda_i\neq\lambda_j\\ B_{iijj}=B_{jjii}=\lambda_i\lambda_j\frac{\partial^2 w}{\partial\lambda_i\partial\lambda_j},\\ B_{ijji}=B_{jiij}=B_{ijij}-\sigma_i,\ i\neq j\end{array}\right\} \quad (13)$$

where λ_i and $\lambda_j (i,j=1,2,3)$ are introduced in the previous section. By considering the incremental displacement $\dot{\mathbf{u}}=(u(\theta,z,r),v(\theta,z,r),w(\theta,z,r))$, the components of $\boldsymbol{\eta}=grad\ \dot{\mathbf{u}}$ are displayed by

$$\boldsymbol{\eta}=\begin{bmatrix}\frac{(u+v_\theta)}{r} & v_z & v_r\\ \frac{w_\theta}{r} & w_z & w_r\\ \frac{(u_\theta-v)}{r} & u_z & u_r\end{bmatrix}, \quad (14)$$

where subscripts here denote the partial derivatives. The incremental form of the incompressibility condition can then be written by

$$\mathrm{tr}\,\boldsymbol{\eta}\equiv u_r+\frac{(u+v_\theta)}{r}+w_z=0. \quad (15)$$

By using relations (10)-(15) and in view of the main governing equations of Haughton and Orr (1995), the governing equations for the Three-Term strain energy function are simplified as follows:

$$f'''(r)=[\frac{m^2}{r^3B_{3131}}\{(-rB'_{3131}+B_{3131}-B_{1111}+B_{2112})\}+\frac{1}{r^3B_{3232}}\{rB'_{3232}-B_{3232}+m^2B_{1212}\}-\frac{\alpha^2r^2}{r^3B_{3232}}\{(rB'_{3223}+rp'+B_{3223}-B_{2222})\}]f(r)+[\frac{m^2}{r^2B_{3131}}\{(-B_{1133}-B_{3113}+B_{2112})\}+\frac{1}{r^2B_{3232}}\{-rB'_{3232}+B_{3232}+m^2B_{1212}+\alpha^2r^2(-B_{3223})\}]f'(r)+\{-rB'_{3232}-2B_{3232}\}\frac{f''(r)}{rB_{3232}}+$$

$[\frac{m}{r^3B_{3232}}\{rB'_{3232} - B_{3232} + m^2B_{1212} + a^2r^2(B_{2222} - B_{2112})\} + \frac{m}{r^3B_{3131}}\{-rB'_{3131} - B_{3131} - m^2(B_{1111} - B_{2112}) - a^2r^2B_{2121}\}]\, g(r) + [\frac{m}{r^2B_{3232}}\{-rB'_{3232} + B_{3232}\} + \frac{m}{r^2B_{3131}}\{rB'_{3131} + B_{3131}\}]g'(r) + [\frac{\alpha^2}{B_{3232}} + \frac{m^2}{r^2B_{3131}}]\, k(r),$ (16)

$g''(r) = \{(rB'_{3131} + B_{3131} + m^2(B_{1111} - B_{2112}) + \alpha^2r^2B_{2121})\}\frac{g(r)}{r^2B_{3131}} + \{(-rB'_{3131} - B_{3131})\}\frac{g'(r)}{rB_{3131}} + \{(rB'_{3131} + B_{3131} + B_{1111} - B_{2112})m\}\frac{f(r)}{r^2B_{3131}} + \{m(B_{3113} - B_{2112})\}\frac{f'(r)}{rB_{3131}} - \frac{m}{rB_{3131}}k(r),$ (17)

$k'(r) = \{-B_{1111} + B_{3223} - m^2B_{1313} - \alpha^2r^2B_{2323}\}\frac{f(r)}{r^2} + \{rB'_{3333} + rp' + B_{3333} - B_{3223}\} + \frac{f'(r)}{r} + (B_{3333} - B_{3223})f''(r) + \{(-B_{1111} + B_{3223} - B_{1313})m\}\frac{g(r)}{r^2} + \{(B_{3113} - B_{3223})m\}\frac{g'(r)}{r}.$ (18)

The corresponding boundary conditions on the curved surface are then given by

$$\left.\begin{array}{c} rg'(r) - g(r) - mf(r) = 0, \\ r^2f''(r) + rf'(r) + (\alpha^2r^2 + m^2 - 1)f(r) = 0, \\ (B_{3333} + \sigma_3)rf'(r) - rk(r) = 0, \end{array}\right\} \quad r = a, b \qquad (19)$$

where $\sigma_3 = \frac{\partial W}{\partial \lambda_3}$ and m is the azimuthal mode number, $\alpha = \frac{n\pi}{(\lambda_2 L)}(n = 1,\ 2,\ 3, \ldots)$ and n is the axial mode number.

4. Application of Compound matrix method to the eigenvalue problem

In this section, the method of obtaining the eigenvalues of equations (16)-(19) by using Compound matrix method (Ng and Reid, 1985; Haughton, 2008) is shown. In order to be able to do so, the equations are written in the following form:

$$f''' = \alpha_1 f + \alpha_2 f' + \alpha_3 f'' + \alpha_4 g + \alpha_5 g' + \alpha_6 k, \quad (20)$$

$$g'' = \beta_1 f + \beta_2 f' + \beta_3 f'' + \beta_4 g + \beta_5 g' + \beta_6 k, \quad (21)$$

$$k' = \gamma_1 f + \gamma_2 f' + \gamma_3 f'' + \gamma_4 g + \gamma_5 g' + \gamma_6 k, \quad (22)$$

where the prime denotes differentiation with respect to r and the coefficients α_i, β_i and γ_i ($i = 1,\ 2,\ 3, \ldots$) will depend on the parameter λ (eigenvalue) and we have

$\alpha_1 = [\frac{m^2}{r^3B_{3131}}\{(-rB'_{3131} + B_{3131} - B_{1111}) + B_{2112})\} + \frac{1}{r^3B_{3232}}\{rB'_{3232} - B_{3232} + m^2B_{1212}\} - \frac{\alpha^2r^2}{r^3B_{3232}}\{(rB'_{3223} + rp' + B_{3223} - B_{2222})\}],$

$\alpha_2 = [\frac{m^2}{r^2B_{3131}}\{(-B_{1133} - B_{3113} + B_{2112})\} + \frac{1}{r^2B_{3232}}\{-rB'_{3232} + B_{3232} + m^2B_{1212} + a^2r^2(B_{2222} - B_{3223})\}],$

$\alpha_3 = \{\{-rB'_{3232} - 2B_{3232}\}\frac{1}{rB_{3232}}\},$

$\alpha_4 = [\frac{m}{r^3B_{3232}}\{rB'_{3232} - B_{3232} + m^2B_{1212} + a^2r^2(B_{2222} - B_{2112})\} + \frac{m}{r^3B_{3131}}\{-rB'_{3131} - B_{3131} - m^2(B_{1111} - B_{2121}) - a^2r^2B_{2121}\}],$

$\alpha_5 = [\frac{m}{r^2B_{3232}}\{-rB'_{3232} + B_{3232}\} + \frac{m}{r^2B_{3131}}\{rB'_{3131} + B_{3131}\}],$

$\alpha_6 = [\frac{a^2}{B_{3232}} + \frac{m^2}{r^2B'_{3131}}],$

$B_1 = \{rB'_{3131} + B_{3131} + B_{1111} - B_{2112})m\}\frac{1}{r^2B_{3131}},$

$B_2 = \{m(B_{3113} - B_{2112})\}\frac{1}{rB_{3131}},\ B_3 = 0,$

$B_4 = \{(rB'_{3131} + B_{3131} + m^2(B_{1111} - B_{2112}) + a^2r^2B_{2121})\}\frac{1}{r^2B_{3131}},$

$B_5 = \{(-rB'_{3131} - B_{3131})\}\frac{1}{rB_{3131}},$

$B_6 = -\frac{m}{rB_{3131}},\ \gamma_1 = \{-B_{1111} + B_{3232} - m^2B_{1313} - a^2r^2B_{2323}\}\frac{1}{r^2},$

$\gamma_2 = \{rB'_{3333} + rp' + B_{3333} - B_{3223}\}\frac{1}{r},\ \gamma_3 = (B_{3333} - B_{3223}),$

$\gamma_4 = \{(-B_{1111} + B_{3232} - B_{1313})m\}\frac{1}{r^2},$

$\gamma_5 = \{(B_{3113} - B_{3223})m\}\frac{g'}{r},\ \gamma_6 = 0.$

Also, the obtained boundary conditions (19), may be written as follows

$$a_1 f + a_2 f' + a_3 g + a_4 g' = 0, r = a, \quad (23)$$

$$b_1 f + b_2 f' + b_3 f'' = 0, \quad r = a, \quad (24)$$

$$c_1 f + c_2 f' + c_3 g + c_4 k = 0, \ r = a, \quad (25)$$

and

$$d_1 f + d_2 f' + d_3 g + d_4 g' = 0, r = b, \quad (26)$$

$$e_1 f + e_2 f' + e_3 f'' = 0, \quad r = b, \quad (27)$$

$$f_1 f + f_2 f' + f_3 \mathrm{g} + f_4 \mathrm{k} = 0, \ r = b. \quad (28)$$

The coefficients of the boundary conditions will also depend on the parameter λ and we have

$$\left.\begin{array}{l} a_1 = d_1 = -\mathrm{m}, a_2 = d_2 = 0, a_3 = d_3 = -1, \\ a_4 = d_4 = \mathrm{r}, b_1 = e_1 = (\alpha^2 r^2 + m^2 - 1), \\ b_2 = e_2 = \mathrm{r}, b_3 = e_3 = r^2, c_1 = f_1 = 0, \\ c_2 = f_2 = (B_{3333} + \sigma_3) r, c_3 = f_3 = 0, \\ \quad c_4 = f_4 = -r. \end{array}\right\} \quad (29)$$

The general solution can then be written as

$$\boldsymbol{y} = k_1 \boldsymbol{u} + k_2 \boldsymbol{v} + k_3 \boldsymbol{w}, \quad (30)$$

where $\boldsymbol{y} = [f, f', f'', g, g', k]^T$ and k_1, k_2, k_3 are arbitrary constants, $\boldsymbol{u} = [u_1, u_2, u_3, u_4, u_5, u_6]$, $\boldsymbol{v} = [v_1, v_2, v_3, v_4, v_5, v_6]$, $\boldsymbol{w} = [w_1, w_2, w_3, w_4, w_5, w_6]^T$, are linearly independent solutions which satisfy the boundary conditions (23)–(25). Now we define the solution matrix as

$$\boldsymbol{Y} = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \\ u_5 & v_5 & w_5 \\ v_6 & v_6 & w_6 \end{pmatrix}. \quad (31)$$

The twenty 3×3 minors of the solution matrix (Compound matrix variables) are

$$y_{ijk} = \begin{vmatrix} u_i & v_i & w_i \\ u_j & v_j & w_j \\ u_k & v_k & w_k \end{vmatrix}, \ (\mathrm{i} = 1, 2, 3, 4),$$
$$(\mathrm{j} = \mathrm{i} + 1, \dots, 5), \ (\mathrm{k} = \mathrm{j} + 1, \dots, 6). \quad (32)$$

By considering the Laplace expansion (Haughton and Orr, 1997; Ng and Reid, 1985) with the complementary minors of this determinant, we obtain

$$y_{123} = u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1,$$
$$y_{124} = u_1 v_2 w_4 - u_1 v_4 w_2 - u_2 v_1 w_4 + u_2 v_4 w_1 + u_4 v_1 w_2 - u_4 v_2 w_1,$$
$$\vdots \quad \vdots \quad \vdots$$
$$y_{456} = u_4 v_5 w_6 - u_4 v_6 w_5 - u_5 v_4 w_6 + u_5 v_6 w_4 + u_6 v_4 w_5 - u_6 v_5 w_4 . \quad (33)$$

Now by differentiating the above Compound matrix variables with respect to r and in view of relations (20)–(22) and that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are three linearly independent solutions, we have

$$y'_{123} = \alpha_3 y_{123} + \alpha_4 y_{124} + \alpha_5 y_{125} \alpha_6 y_{126},$$
$$y'_{124} = y_{125} + y_{134},$$
$$\vdots \quad \vdots \quad \vdots$$
$$y'_{456} = -\beta_1 y_{146} - \beta_2 y_{246} + \beta_5 y_{456} + \gamma_1 y_{145} + \gamma_2 y_{245} + \gamma_3 y_{345}. \quad (34)$$

In view of the boundary conditions (19), we obtain the following relations at $r = a$

$$g' = \frac{m}{a} f + \frac{1}{a} g, \ f' = \frac{1}{B_{3333} + \sigma_3} k,$$

$$f'' = \left(\frac{1 - m^2 - \sigma^2 \sigma^2}{\sigma^2}\right) f - \frac{1}{a(B_{3333} + \sigma_3)} k.$$

Since, $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are three linearly independent solutions at $r = a$, which satisfies the boundary conditions (23)-(25) and in view of the above and due to the following relations

$$u_1 = 1, \ u_4 = 0, \ u_6 = 0, \ v_1 = 0, v_4 = 1,$$
$$v_6 = 0, w_1 = 0, w_4 = 0, w_6 = 1,$$

we get

$$u_2 = 0, \ u_3 = \frac{1 - m^2 - \alpha_2 a^2}{\sigma^2}, \ u_5 = \frac{m}{a},$$
$$v_2 = 0, \ v_3 = 0, \ v_5 = \frac{1}{a}, w_2 = \frac{1}{B_{3333} + \sigma_3},$$
$$w_3 = -\frac{1}{a\left(B_{3333 + \sigma_3}\right)}, \ w_5 = 0.$$

Hence, three linearly independent vectors which satisfy the boundary conditions at $r = a$ are shown by

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \\ u_5 & v_5 & w_5 \\ u_6 & v_6 & w_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{B_{3333} + \sigma_3} \\ \frac{1 - m^2 - \alpha_2 a^2}{\sigma^2} & 0 & -\frac{1}{a(B_{3333+\sigma_3})} \\ 0 & 1 & 0 \\ \frac{m}{a} & \frac{1}{a} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (35)$$

In view of (35), and by using the boundary conditions (23)–(25) in Compound matrix variables (33), we obtain the initial conditions at $r = a$ as follows:

$$y_{123}(a) = 0, \ y_{125}(a) = -\frac{a_3}{a_4} y_{124}(a),$$
$$y_{126}(a) = -\frac{c_3}{c_4} y_{124}(a), \ y_{134}(a) = -\frac{b_2}{b_3} y_{124}(a),$$
$$y_{135}(a) = \frac{b_2 a_3}{b_3 a_4} y_{124}(a), \ y_{136}(a) = \frac{c_3 b_2}{c_4 b_3} y_{124}(a),$$
$$y_{145}(a) = -\frac{a_2}{a_4} y_{124}(a), \quad y_{146}(a) = \frac{c_2}{c_4} y_{124}(a),$$
$$y_{156}(a) = -\frac{a_3 c_2}{a_4 c_4} y_{124}(a), \ y_{234}(a) = \frac{b_1}{b_3} y_{124}(a),$$
$$y_{235}(a) = -\frac{b_1 a_3}{b_3 a_4} y_{124}(a), y_{236}(a) = -\frac{b_1 c_3}{b_3 c_4} y_{124}(a),$$
$$y_{245}(a) = -\frac{a_1}{a_4} y_{124}(a), \ y_{246}(a) = -\frac{c_1}{c_4} y_{124}(a),$$
$$y_{256}(a) = \frac{(a_3 c_1 - a_1 c_3)}{a_4 c_4} y_{124}(a),$$
$$y_{345}(a) = \frac{b_2 a_1}{b_3 a_4} y_{124}(a),$$
$$y_{346}(a) = \frac{(b_2 c_1 - b_1 c_2)}{b_3 c_4} y_{124}(a),$$
$$y_{356}(a) = \frac{(-b_2 a_3 c_1 + b_1 c_2 a_3 + a_1 c_3 b_2)}{b_3 a_4 c_4} y_{124}(a),$$
$$y_{456}(a) = \frac{a_1 c_2}{a_4 c_4} y_{124}(a). \quad (36)$$

Now it is necessary to show that the boundary conditions are satisfied at $r = b$.

By using the general solution (30) and substituting into the boundary conditions (26)–(28), we have

$$k_1(d_4u_5 + d_3u_4 + d_1u_1 + d_2u_2)+ k_2(d_4v_5 + d_3v_4 + d_1v_1 + d_2v_2) + k_3(d_4w_5 + d_3w_4 + d_1w_1 + d_2w_2) = 0,$$
$$k_1(e_3u_3 + e_2e_2.+e_1u_1) + k_2(e_3v_3 + e_2v_2 +e_1v_1 + k_3(e_3w_3 + e_2w_2 + e_1w_1) = 0,$$
$$k_1(f_2u_2 + f_1u_1 + f_3u_4 + f_4u_6) + k_2(f_2v_2 +f_1v_1 + f_3v_4 + f_4v_6) + k_3(f_2w_2 + f_1w_1 +f_3w_4 + f_4w_6) = 0.$$

For the existence of the non-trivial solution, the determinant of the coefficients matrix of the above system of equations for the variables k_1, k_2, k_3, should be zero. The above 3×3 determinant of the coefficient matrix may be written in terms of y_{ijk}'s and finally the target condition is obtained as follows:

$$(d_2e_3f_1 - d_1e_3f_2)y_{123} + (d_1e_2f_3 + d_3e_1f_2 -d_3e_2f_1 - d_2e_1f_3)y_{124} + d_4e_1f_2y_{125} + d_1e_2f_4y_{126} + (d_1e_3f_3 - d_3e_3f_1)y_{131} - d_4e_2f_1y_{135} + d_1e_3f_4y_{136} - d_4e_1f_3y_{145} - d_3e_1f_4y_{146} - d_4e_1f_4y_{156} + d_2e_3f_4y_{236} +(d_2e_3f_3 - d_3e_3f_2)y_{234} - d_4e_3f_2y_{235} + d_4e_2f_3y_{245} - d_3e_2f_4y_{246} - d_4e_1f_4y_{156} + d_4e_1f_4y_{156} + (d_2e_3f_3 - d_3e_3f_2)y_{234} - d_4e_3f_2y_{235} + d_2e_3f_4y_{236} + d_4e_2f_3y_{245} - d_3e_2f_4y_{246} - d_4e_2f_4y_{256} - d_4e_3f_3y_{345} - d_3e_3f_4y_{346} - d_4e_3f_4y_{356} = 0,\ r = b. \quad (37)$$

By substituting the coefficients (29) in equation (37), we obtain the final simplified relation of the target condition as

$$\mathrm{m}b^2(B_{3333} + \sigma_3)y_{123} - b(-1 + m^2 + a^2b^2)(B_{3333} + \sigma_3)y_{124} + b^2(-1 + m^2 + a^2b^2)(B_{3333} + \sigma_3)y_{125} + mb^2y_{126} + mb^3y_{136} - b(-1 + m^2 + a^2b^2)y_{146} + b^2(-1 + m^2 + a^2b^2)y_{156} + b^3(B_{3333} + \sigma_3)y_{234} + b^4(B_{3333} + \sigma_3)y_{235} + b^2y_{246} + b^3y_{256} - b^3y_{346} + b^4y_{356} = 0, \quad r = b, \quad (38)$$

where $\sigma_3 = \frac{\partial w}{\partial \lambda_3}$ and $B_{3333} = \lambda_3^2 \frac{\partial^2 w}{\partial \lambda_3 \partial \lambda_3}$. In order to obtain the eigenvalues λ, we first integrate equations (34) in view of the initial conditions (36) with the use of the so called Runge-Kutta-Fehlberg method. Then by using the shooting method, we adjust λ such that the target condition (38) is satisfied. The numerical results which were obtained by the use of Fortran 90 programming (Metcalf, 2011) are shown in Fig. 1.

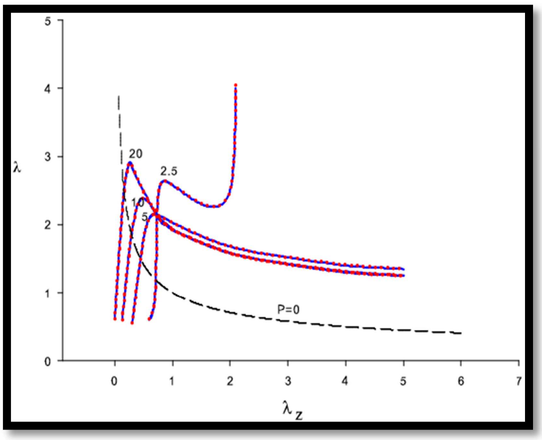

Fig. 1. Comparing the radius changes with respect to the changes of the length for A=0.85 and $\frac{L}{B}$ =2.5, 5, 10, 20

5. Discussion

One of the numerical methods mostly used on solving the eigenvalue problems and/or the higher order ODE's with variable coefficients or even the system of equations is Compound matrix method.

It seems that the reason of applying this method on solving the finite elasticity problems compared to the other numerical methods goes back to the power of the so called Laplace expansion.

By increasing the order of the system and due to the properties of the Laplace expansion, the accuracy and validity of this method do not change, which shows the performance of this method, while Adams-Moulton method has a vice versa ratio with the order of the system.

The obtained results are illustrated in Fig. 1. As we noted before, the aim of this article is to calculate the eigenvalues and also analyze the axisymmetric bifurcation of the cylindrical elastic shell under axial loading and external pressure. The dotted line curve in Fig. 1 is related to deformation of shell in the case of axial loading where we eliminated the external pressure, and the other four curves are the axisymmetric curves for A= 0.85 and the aspect ratio $\frac{L}{B}$ = 2.5, 5,10, 20. Since the azimuthal mode number considered is zero (m=0), these curves are called axisymmetric.

Our discussion is divided in two parts i.e. $0 < \lambda_z < 1$ and $\lambda_z > 1$. In the region $0 < \lambda_z < 1$ and due to the action of the curves, the cylinder is under an axial compression type of loading. It means that, the ends of the cylindrical shell are, compressed along the axis. Also, the external pressure causes the decreasing of the radius ratio of the cylinder. The second region corresponds to the case, where cylinder is under an axial extension type of loading. In this region the external pressure is also effective on the decreasing of the radius ratio. The bifurcation points of this problem are the points obtained from the intersection of axisymmetric curves and the zero pressure curve (dotted line curve). It is obvious from Fig. 1 that the first bifurcation point is obtained from the intersection of the first curve (from the axisymmetric curves) with the dotted line curve. It was also observed that all of the bifurcation points are located in the region $0<\lambda_z<1$. Due to the complete agreement between the numerical data, it can be seen that the curves of the two methods coincided very well and provide us satisfactory results.

6. Conclusion

Deformation of the elastic materials under the various pressure leads to eigenvalue problems. Numerical methods are effective in such problems with the normal oscillations but for the higher oscillations numerical integration is generally useless. In this paper, two numerical, i.e. Compound matrix and Adams-Moulton methods are used for low oscillations. On solving the eigenvalue problems, it is obvious that the first method is better than the second one. Using Laplace expansion in the Compound matrix method improved the results of the Adams-Moulton and are close to the results obtained in the paper. We also concluded that all of the bifurcation points are located in the region $0 <\lambda_z< 1$.

References

Haughton, D. M. (2008). Evaluation of eigen functions from compound matrix variables in non-linear elasticity-II. Six order systems. *Journal of Computational Physics, 227,* 8960–8967.

Haughton, D. M., & Ogden, R. W. (1979). Bifurcation of inflated circular cylinder of Elastic material under axial loading-II. Exact theory for thick-walled tubes. *Journal of Mech. Phys. Solids., 27*, 489–512.

Haughton, D. M., & Orr, A. (1997). On the eversion of compressible elastic cylinders. *Int. j. Solids Structures., 34*, 1893–1914.

Haughton, D. M., & Orr, A. (1995). On the eversion of incompressible elastic cylinder. *Int. Journal of Non-Linear Mech., 30*, 81–95.

Metcalf, M., Reid, J., & Cohen, M. (2011). *Modern Fortran explained.* Oxford Science Publications.

Ng, B. S., & Reid, W. H. (1985). The Compound matrix method for ordinary differential systems. *Journal of Computational Physics, 58,* 209–228.

Sierakowski, R. L., Sun, C. T., & Ebcioglu, I. K. (1975). Instability of a hollow rubber-like cylinder under initial stress. *Int. Journal of Non-Linear Mech., 10,* 193.

Wolfram, S. (2008). *Wolfram Mathematica*, www.wolfram.com.

Zhu, Y., Luo, X. Y., & Ogden, R. W. (2008). Asymmetric bifurcation of thick-walled circular cylindrical elastic tubes under axial loading and external pressure. *International Journal of Solids and Structures, 45,* 3410-3429.