On generalized AIP-rings

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Abstract

In this paper, we introduce the concept of the generalized AIP-ring as a generalization of the generalized quasi-Baer rings and generalized p.p.-rings. We show that the class of the generalized AIP-rings is closed under direct products and Morita invariance. We also characterize the 2-by-2 formal upper triangular matrix rings of this new class of rings. Finally, we provide several examples to show the applicability and limitation of this class of rings.

Keywords: Baer rings; quasi-Baer rings; p.p.-rings; annihilators; idempotent; s-unital ideal

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are left unital.

Recall that a ring $R$ is (quasi-) Baer if the right annihilator of every (right ideal) nonempty subset of $R$ is generated (as a right ideal) by an idempotent of $R$. These definitions are left-right symmetric.


As a generalization of quasi-Baer rings, Birkenmeier, Kim and Park (2001) introduced a principally quasi-Baer ring and used them to generalize many results on reduced (i.e., it has no nonzero nilpotent elements) p.p.-rings. A ring $R$ is called right principally quasi-Baer (or simply right p.q.-Baer) if the right annihilator of a principal right ideal is generated by an idempotent. Similarly, left p.q.-Baer rings can be defined.

Moussavi, Haj Seyyed Javadi and Hashemi (2005) defined the concept of generalized right (principally) quasi-Baer rings. A ring $R$ is called generalized right (principally) quasi-Baer if for any (principal) right ideal $I$ of $R$, the right annihilator of $I^n$ is generated by an idempotent for some positive integer $n$, depending on $I$. A ring $R$ is called generalized right (principally) quasi-Baer if for any (principal) right ideal $I$ of $R$, the right annihilator of $I^n$ is generated by an idempotent.

As a generalization of left p.q.-Baer and right p.p.-rings, Zhongkui and Renyu (2006) introduced the concept of left APP-rings and used them to generalize several basic results. A submodule $N$ of a left $R$-module $M$ is called a pure submodule if $L \otimes_R N \rightarrow L \otimes_R M$ is a monomorphism for every right $R$-module $L$. A ring $R$ is called left APP if the left annihilator $Ra$ is pure as a left ideal of $R$ for any element $a \in R$. Right APP-rings can be defined analogously.

As a generalization of quasi-Baer rings, Clark (1967) defined quasi-Baer rings and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra.

As a generalization of Baer rings, a ring $R$ is called right (resp. left) p.p.-ring if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of any element of $R$ is generated by an idempotent of $R$). $R$ is called a p.p.-ring (also called a Ricart ring (1946)), if it is both right and left p.p.-ring. The concept of p.p.-rings is left-right symmetric according Chase (1961).

An ideal $I$ of $R$ is said to be right $s$-unital if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. By Stenstrom (1975, Proposition 11.3.13), an ideal $I$ is right $s$-unital if and only if $R/I$ is flat as a left $R$-module if and only if $I$ is pure as a right ideal of $R$. Note that if $I$ and $J$ are right $s$-unital ideals, then so is $I \cap J$. By Tominaga (1975,
Theorem 1), I is right s-unital if and only if for any finite elements $a_1, a_2, ..., a_n \in I$ there exists an element $x \in I$ such that $a_i x = a_i$, $i = 1, 2, ..., n$. Clearly every left p.q.-Baer ring and right p.p.-ring is a left APP-ring. Thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings. By Hirano (2002), a ring R is called quasi-Armendariz if whenever $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, we have $a_i R b_j = 0$ for every $i$ and $j$. Left APP-rings are quasi-Armendriz (Hirano, 2002, Theorem 3.9).

Majidinya et al. defined the concept of right AIP-rings. A ring R is called right AIP if R has the property that the right annihilator of any ideal is pure as a right ideal. A left ideal I of R is centrally s-unital if, for each $a \in I$ there exists a central element $x \in I$ such that $ax = a$. Also a ring R is called centrally left AIP if for any ideal I of R, the left annihilator of I is centrally s-unital as an ideal of R. The class of right AIP-rings is included in the class of right APP-rings.

In this paper we introduce the concept of n-generalized right AIP-rings. Given a fixed positive integer n, we say a ring R is n-generalized right AIP if for any right ideal I of R, the right annihilator of $I^n$ is pure as an ideal of R. We say a ring R is generalized right AIP if for any right ideal I of R, the right annihilator of $I^n$ is pure as an ideal of R. Left cases may be defined analogously. If R is both left and right generalized AIP, then we say R is generalized AIP-ring.

The class of n-generalized right AIP-rings includes all AIP-rings. Using Example 3.2, we can construct a class of n-generalized left AIP-rings which is neither left nor right AIP. However for a semiprime ring the definition of n-generalized right AIP coincides with that of AIP. By Examples 2.1, 4.3, we show that the concept of AIP and n-generalized AIP rings is not left-right symmetric.

In Section 3, we propose the definition of an n-generalized AIP-ring. Clearly every n-generalized quasi-Baer ring is an n-generalized AIP-ring. Using Example 3.5 the various classes of generalized AIP-rings that are not generalized quasi-Baer are provided. For a ring that satisfies the ascending chain condition on principal left ideals, generalized left AIP and generalized left quasi-Baer properties coincide (Proposition 3.10).

In Section 4, we characterize the n-generalized right AIP property of 2-by-2 generalized triangular matrix rings and full matrix rings. We also prove that, unlike the Baer or right p.p. condition, the n-generalized AIP condition is a Morita invariant property.

For notation we use $Mat_n(R)$ and $T_n(R)$ for the ring of $n \times n$ matrices and the ring of the $n \times n$ upper triangular matrices over R, respectively. For a non-empty subset $X$ of $R$, $\lambda_R(X)$ (resp. $\lambda_l(X)$) is used for the right (resp. left) annihilator of $X$ over R. Also $Z$ and $Z_n$ denote the integers and the integers modulo n, respectively. Observe that every von Neumann regular ring is a left AIP-ring. Since, if I is a left ideal of R and $a \in l_R(I)$, then $a = ab$ for some $b \in R$. Now, let $t = ba$. Then $t \in l_R(I)$ and $a \in l_R(I)$.

From (Birkenmeier et al., 2001, Example 1.6), there exist regular (hence left AIP-) rings which are neither right nor left p.q.-Baer. Also, by (Birkenmeier et al., 2001, Example 1.5), there are right p.q.-Baer (hence left AIP-) rings which are neither quasi-Baer, nor right p.p., nor left p.p.

From (Bell, 1970) a ring R is said to satisfy the IFP (insertion of factors property), if $r_R(a)$ is an ideal for all $a \in R$. A ring R is called abelian if every idempotent in it is central. It is evident that any reduced ring satisfies IFP and any ring with IFP is abelian. R is an abelian right p.p.-ring if and only if R is right p.q.-Baer and satisfies IFP. For a reduced ring R, we have $l_R(Rx) = l_R((Rx)^n) = l_R(x^n) = l_R(x^n) = l_R(x^n) = l_R(x^n)$, for every $x \in R$ and every positive integer n.

In this section we consider some properties of left AIP-rings. We begin by showing that the AIP condition is not left-right symmetric.

Example 2.1. (Lam, 1999, Example 2.34) Let S be a von Neumann regular ring with an ideal I such that, as a submodule $S_I$, I is not a direct summand. Let $R = S/I$, which is also a von Neumann regular ring. As a right S-module, R is not projective. Viewing R as an (R,S)-bimodule, we can form the triangular ring $T = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$. Then T is left semihereditary, and so it is a left p.p.-ring. Hence T is a right AIP-ring. Now, let $X = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}$. Then $l_T(X) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, which clearly it is not right s-unital, since $l_T(X)l_T(X) = 0$. Therefore T is not left AIP.

In the following example we show that the Factor ring of a left AIP-ring, is not, in general, left AIP.

Example 2.2. Let $R = Z$ and I = 4Z, then R is an AIP-ring. But, $R/I$ is neither left AIP nor right AIP-ring. Note that $R/I \cong Z_4$ and if $I = \{0, 2\}$, then $l_{Z_2}(I) = J$, and $2 \notin l_{Z_2}(J)$ but $2 \notin l_{Z_2}(J)$.

Proposition 2.3. Let R be a left AIP-ring. Then R is semiprime if and only if $l_R(I) \subseteq r_R(I)$, for any left ideal I of R.
Proof: Suppose that $R$ is semiprime and let $I$ be a left ideal of $R$, then clearly $(I_r)(I)^2 = 0$ and so $l_g(I) \subseteq r_g(I)$. Conversely, suppose that $I^2 = 0$, for some left ideal $I$ of $R$. Then $I \subseteq l_g(I)$, and since $R$ is AIP, $I \subseteq H(I) \subseteq I$ $r_g(I) = 0$.

Corollary 2.4. Commutative AIP-rings are reduced.

3. Generalized AIP-rings

In this section we introduce and study the concept of generalized AIP-rings. We show that the generalized AIP property is inherited by direct products. Clearly every AIP-ring is $n$-generalized AIP-ring.

Definition 3.1. We say a ring $R$ is $n$-generalized right AIP if for any right ideal $I \subseteq R$, the right annihilator of $I^n$ is left $s$-unital as an ideal of $R$ for a fixed positive integer $n$. Also we say a ring $R$ is generalized right AIP if for any right ideal $I \subseteq R$, the right annihilator of $I^n$ is $s$-unital for some positive integer $n$, depending on $I$. In particular, if $n = 1$, then $R$ is called right AIP.

Left cases can be defined analogously. Obviously every right AIP-ring is an $n$-generalized right AIP-ring. We provide an example of 2-generalized AIP-ring that is neither left nor right AIP.

Example 3.2. Let $\mathbb{Z} < x, y >$ be the free $\mathbb{Z}$-ring over $x, y$ and $R = \mathbb{Z} < x, y > / \langle x^2 - x, xy, y^2 \rangle$. Then it is easy to see that $I = \langle ay + bx | a, b \in \mathbb{Z} \rangle$ is an (left) ideal of $R$ such that $y^2 \notin I$ and $x, y^2 \in I$. So $R$ is not a left AIP-ring. Also, $r_g(I) = \langle axy - ay | a \in \mathbb{Z} \rangle$, which is not left $s$-unital and so $R$ is not right AIP-ring.

However, it is easy to check that $R$ is 2-generalized AIP.

The following example shows that subrings of generalized AIP-rings are not in general generalized AIP.

Proposition 3.3. Let $R$ be a semiprime ring. Then $R$ is $n$-generalized right AIP if and only if $R$ is right AIP.

Proof: Let $R$ be $n$-generalized right AIP and $x \in r_g(I)$ for right ideal $I$ of $R$. Then $x = ax$ for $a \in r_g(I^n)$ and $(Ia)^n = 0$. Since $R$ is semiprime, $a \in r_g(I)$.

The following example shows that subrings of generalized AIP-rings are not in general generalized AIP.

Example 3.4. The ring $\mathbb{Z} \oplus \mathbb{Z}$ is generalized left AIP. Let $p$ be a prime number and $R = \langle (a, b) \in \mathbb{Z} \oplus \mathbb{Z} | a \equiv b \mod p \rangle$. Then $R$ is a commutative reduced ring and a subring of $\mathbb{Z} \oplus \mathbb{Z}$ which is not left AIP, since $l_g(R(p, 0)) = R(0, p)$ and $(0, p) \notin (0, p)r_g(R(p, 0))$. The following examples show that there are generalized AIP-rings which are not generalized quasi-Baer.

Example 3.5. (i) (Birkenmeier, 2001, Example 1.6) For a field $F$, take $F_n = F$ for $n = 1, 2, \ldots$, and let $R = \left( \prod_{n=1}^{\infty} F_n, \mathfrak{a}_n \right)$, which is a subring of the $2 \times 2$ matrix ring over $\prod_{n=1}^{\infty} F_n$, where $\mathfrak{a}_n = 1 >$ is the $F$-algebra generated by $\mathfrak{a}$ and 1. Then by Goodearl (1991), the ring $R$ is a von Neumann regular ring and so $R$ is an AIP-ring (and hence is generalized AIP). Also, by (Moussavi et al., 2005, Example 2.3), $R$ is neither generalized right quasi-Baer nor generalized left quasi-Baer.

(ii) Let $A = R \oplus \text{Mat}_2(\mathbb{Z})$, where $R$ is the ring in (i). Then $A$ is generalized left AIP-ring. But $A$ is neither generalized left quasi-Baer nor generalized right quasi-Baer.

(iii) Let $R = \left( \begin{array}{cc} A & A \\ 0 & A \end{array} \right)$, where $A$ is a left p.q.-Baer ring and $A = A/P$ for a prime ideal $P$ such that if $b \in P$ then $I_A(\text{Ab})$ is not a subset of $P$. Then by (Birkenmeier, 2001, Corollary 2.4), $R$ is a left p.q.-Baer ring. Hence $R$ is a generalized right AIP-ring. But by (Moussavi et al., 2005, Example 4.5), $R$ is not generalized right p.q.-Baer (and hence not generalized right quasi-Baer).

In the following, we have a generalized right AIP-ring, which is not generalized p.p.

Example 3.6. Let $R = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) | a, b, c, d \in \mathbb{Z}; a \equiv d, b \equiv 0 \text{ and } c \equiv 0 (\text{mod } 2) \rangle$. Since $R$ is a prime ring, so it is right AIP and hence generalized right AIP. But by (Moussavi et al., 2005, Example 2.9), $R$ is neither generalized left p.p. nor generalized right p.p.

Lemma 3.7. Let $R$ be a commutative $n$-generalized right AIP-ring. Then $r_g(I^n) = r_g(I^{n+1})$ for each positive integer $m \geq n$.

Proof: It is enough to show that $r_g(I^n) = r_g(I^{n+1})$. Let $x \in r_g(I^{n+1})$. Then $Ix \subseteq r_g(I^n)$. Hence for every $a \in I$, $ax \in r_g(I^n)$. Since $R$ is an $n$-generalized right AIP-ring, there exists $b \in r_g(I^n)$ such that $ax = bax$. Thus $I^{n-1}ax = I^n - abx = 0$ and $x \in r_g(I^n)$.
Corollary 3.8. The commutative n-generalized right AIP-rings have no nilpotent right ideal of order \( m > n \).

Lemma 3.9. (Tominaga, 1975, Theorem 1) If \( I \) is a left ideal of \( R \) then the following are equivalent:
(1) \( I \) is right \( s \)-unital.
(2) For any elements \( a_1, a_2, \ldots, a_n \in I \) there exists an element \( x \in I \) such that \( a_i x = a_i \) for all \( i = 1, 2, \ldots, n \).
We include the following result to indicate the condition under which the n-generalized AIP and n-generalized quasi-Baer properties coincide.

Proposition 3.10. Let \( R \) satisfy the ascending chain condition on principal left ideals. Then the following conditions are equivalent:
(1) \( R \) is an n-generalized left AIP-ring.
(2) \( R \) is an n-generalized left quasi-Baer ring.

Proof: The proof is similar to that of Zhongkui and Renyu (2006, Proposition 2.7).

Corollary 3.11. Let \( R \) be a semiprime ring which satisfies ACC on principal left ideals. Then the following conditions are equivalent:
(i) \( R \) is a quasi-Baer ring.
(ii) \( R \) is an n-generalized left AIP-ring.
(iii) \( R \) is an n-generalized left quasi-Baer ring.
Proof: The equivalences, (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) are clear.

Proposition 4.2. Let \( R = \left( \begin{array}{c} S \\ M \end{array} ight) \). Then \( R \) is an n-generalized right AIP if and only if the following conditions are satisfied,
(i) \( R \) and \( S \) are n-generalized right AIP.
(ii) \( r_M(I^n) = r_M(M) \) for each ideal \( I \) of \( S \).
(iii) If \( \left( \begin{array}{c} I \\ N \end{array} \right) \) is an ideal of \( T = \left( \begin{array}{c} S \\ M \end{array} \right) \) then \( r_T(I^n) \cap Ann_T(I^{n+1}) \cap \ldots \cap Ann_T(I^{n+k}) \)

Corollary 3.12. Let \( R \) be a left perfect ring. Then \( R \) is an n-generalized left AIP- ring if and only if \( R \) is n-generalized quasi-Baer.

4. Generalized matrix rings

Throughout this section \( T \) will denote a 2-by-2 generalized (or formal) triangular matrix ring \( T = \left( \begin{array}{c} S \\ M \end{array} \right) \), where \( R \) and \( S \) are rings and \( M \) is an \( (S,R) \)-bimodule. If \( N \) is an \( (S,R) \)-submodule of \( M \), then \( Ann_R(N) = \{ r \in R \mid Nr = 0 \} \) and \( Ann_S(N) = \{ s \in S \mid sN = 0 \} \).

Lemma 4.1. (Birkenmeier et al., 2002, Lemma 3.1) Let \( \left( \begin{array}{c} I \\ N \end{array} \right) \) be an ideal of \( T = \left( \begin{array}{c} S \\ M \end{array} \right) \). Then\( r_T(I^n) = r_M(I^n) \cap Ann_T(I^{n+1}) \cap \ldots \cap Ann_T(I^{n+k}) \)

Proof: The proof is similar to that of Zhongkui and Renyu (2006, Proposition 2.7).

A left perfect ring \( R \) satisfies DCC on principal right ideals (Lam, 1991, Theorem 23.20).

By Lam (1999, 6.55), a left perfect ring \( R \) satisfies ACC on principal left ideals. Now we have the following corollary:

Corollary 3.13. Let \( R \) be a left perfect ring. Then \( R \) is an n-generalized left AIP-ring if and only if \( R \) is n-generalized quasi-Baer.

Corollary 3.14. Let \( R \) be a left perfect ring. Then \( r_T(I^n) \cap Ann_T(I^{n+1}) \cap \ldots \cap Ann_T(I^{n+k}) \)
and $r_T\left(\begin{array}{cc} I & M \\ 0 & 0 \end{array}\right)^n = (T_r(J^n)^n \quad (r_m(J^n)^n))$ is left s-unital. Thus $R$ and $S$ are n-generalized right AIP-rings.

(ii) Now let $I$ be an ideal of $S$. Then $\left(\begin{array}{cc} I & M \\ 0 & 0 \end{array}\right)$ is an ideal of $T$. Since $T$ is n-generalized right AIP, $r_T\left(\begin{array}{cc} I & M \\ 0 & 0 \end{array}\right)^n$ is left s-unital. Hence $r_m(I^n) = r_S(I^nM)$.

(iii) Let $\left(\begin{array}{cc} I & N \\ 0 & 0 \end{array}\right)$ be an ideal of $T$. Since $T$ is n-generalized right AIP, for $\begin{array}{cc} a & m \\ 0 & 0 \end{array}$ and $\begin{array}{cc} f_1 & m \\ 0 & f_2 \end{array} \in r_T\left(\begin{array}{cc} I & N \\ 0 & 0 \end{array}\right)^n$, we have $\begin{array}{cc} a & m \\ 0 & 0 \end{array} = f_1 \begin{array}{cc} f_2 \end{array}$ and $f_2 \\begin{array}{cc} a & m \end{array}$. Hence for $b \in r_T(I^n)$ and $Ann_R(I^{n-1}N) \cap Ann_R(I^{n-1}N) \cap \ldots \cap Ann_R(I^{n-1}N) = b = f_2b$. Thus $r_T(I^n) = r_T(I^{n-1}N) \cap \ldots \cap Ann_R(I^{n-1}N)$ is left s-unital.

There exists an n-generalized right AIP-ring which is not n-generalized left AIP, and hence the definition of n-generalized AIP-ring is not symmetric.

**Example 4.3.** (Moussavi et al., 2005, Example 4.4) Let $A = \left(\begin{array}{cc} R & S \\ 0 & 0 \end{array}\right)$, where $R = \left(\begin{array}{cc} Z & 0 \\ 0 & 0 \end{array}\right)$ is an AIP-ring and $S = \left(\begin{array}{cc} 0 & Z \\ 0 & 0 \end{array}\right)$ is a $(R,S)$-bimodule. By proposition 4.2, $A$ is 2-generalized right AIP, while for $I = \left(\begin{array}{cc} R & S \\ 0 & 0 \end{array}\right)$, we have $l_A(I^n) = \left(\begin{array}{cc} 0 & S \\ 0 & 0 \end{array}\right)$ for each positive integer $n$ and $\left(\begin{array}{cc} 0 & S \\ 0 & 0 \end{array}\right) \in (\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array})$. Hence

$$\left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right) \in$$

which is an ideal of $T_2(R)$. Since $R$ is commutative generalized right AIP, $r_{T_2(R)}\left(\begin{array}{cc} I & N \\ 0 & 0 \end{array}\right)^n$ is left s-unital for some positive integer $m$. Let $\begin{array}{cc} a & b \\ 0 & c \end{array} \in r_{T_2(R)}\left(\begin{array}{cc} I & L \\ 0 & 0 \end{array}\right)^m$. Then $a = ea$ and $b = eb$ for $e \in r_T(I^n)$, and $\begin{array}{cc} a & b \\ 0 & c \end{array}$ is an ideal of $T_2(R)$. Also, there exists $e_1 \in r_T(I^{2m}) = r_T(I^{2m})$ such that $e_1 = c$. Since $c \in Ann_R(I^{2m-1}L + \ldots + I^{2m-1})$, we have $I^{2m}R + Rf^{2m} = I^{2m-1}L + \ldots + I^{2m-1}$, where $f \in r_T(I^{2m}) \cap Ann_R(I^{2m-1}L + \ldots + I^{2m-1})$. Let $Z = \left(\begin{array}{cc} e & 0 \\ 0 & f \end{array}\right)$. It is clear that $\begin{array}{cc} a & b \\ 0 & c \end{array} = \left(\begin{array}{cc} e & 0 \\ 0 & f \end{array}\right)\left(\begin{array}{cc} a & b \\ 0 & c \end{array}\right)$. Now let $m_1(R) = \left(\begin{array}{cc} R & M \\ 0 & T_{m_1}(R) \end{array}\right)$, where $M = (R, \ldots, R)$ (n-tuple). Let $\left(\begin{array}{cc} I & N \\ 0 & 0 \end{array}\right)$ be an ideal of $T_{m_1}(R)$. Since $R$ and $T_{m_1}(R)$ are generalized right AIP, $r_R(I^{2k}) = r_R(I^{2k})$ and $r_T(m_1(R)(I^{2k})) = r_T(m_1(R)(I^{2k}))$ are left s-unital. We show that $r_{m_1}(I^{2k}) \cap Ann_{m_1}(I^{2k-1}N + I^{2k-2}N + \ldots + IN^{2k-2} + NJ^{2k-1})$ is left s-unital. Let $\left(\begin{array}{cc} a & b \\ 0 & 0 \end{array}\right) \in r_{m_1}(I^{2k})$. Then $a \in r_T(I^{2k})$ and $B \in r_{m_1}(R)(I^{2k}) \cap Ann_{m_1}(I^{2k-1}N + I^{2k-2}N + \ldots + IN^{2k-2} + NJ^{2k-1})$. So there exists $e_1 \in r_T(I^{2k})$ such that $a = e_1a$. Since $B \in r_{m_1}(R)(I^{2k})$, $B = \lambda B$ for some $\lambda \in r_{m_1}(R)(I^{2k})$. Also $B \in Ann_{m_1}(I^{2k-1}N + I^{2k-2}N + \ldots + IN^{2k-2} + NJ^{2k-1})$. Since $I^{2k}M + M^{2k} = I^{2k-1}M + \ldots + N^{2k-1}$, we have $(I^{2k}M + M^{2k})B = 0$. Hence $I^{2k}MB = 0$. Thus all entries of $B$ belong to $r_T(I^{2k})$, and by Lemma 4.2, $B = \ell B$. Let $f = \ell B$, then we have $B = fB$ and $f \in r_T(I^{2k})$. Next let $m$ be a positive integer and let $I$ be an ideal of $M_m(R)$. Then there is an ideal $J$ of $R$ such that $I = M_m(J)$. Let $A \in r_{m_1}(R)(I^{2k}) = r_{m_1}(R)(M_m(J^n)) = M_m(r_T(J^n))$. Then $a_{ij} \in r_T(J^n)$. Since $R$ is n-generalized right AIP, there exists $x \in r_T(J^n)$ such that $a_{ij} = a_{ij}x$, for $i, j = 1, 2, \ldots, n$. Therefore $A = A \times x_1M_m(R)$, where $1M_m(R)$ denotes the identity matrix of $M_m(R)$.

**Proposition 4.6.** The class of n-generalized right AIP-rings is closed under direct product.

**Proof:** The proof is obvious.
The next lemma allows us to construct numerous examples of generalized left AIP-rings that are neither generalized right quasi-Baer, nor generalized right p.p., nor generalized left p.p.

Lemma 4.7. (Birkenmeier et al., 2001, Lemma 1.4) Let $T$ be a ring with unity such that $|T| > 1$, and let $S = \prod_{\lambda \in \Lambda} T_\lambda$, where $T_\lambda = T$ and $\Lambda$ is an infinite set. If $R$ is the subring of $S$ generated by $\bigoplus_{\lambda \in \Lambda} T_\lambda$ and $1 \in S$, then $R$ is not a generalized right quasi-Baer ring. Moreover, if $T$ is a right p.q.-Baer ring, then $R$ is a generalized left AIP-ring which is not generalized right quasi-Baer.

Using Theorem 4.5 and applying Lemma 4.7, we obtain the following example of generalized AIP-rings which are not generalized right quasi-Baer.

Example 4.8. (i) Let $S$ be the 2-by-2 full matrix ring over $\mathbb{Z}[x]$ and $R$ be the 2-by-2 full matrix ring over $\mathbb{Z}$. It is clear that $S$ is isomorphic to $R[x]$ and $R$ is right p.p. by Chatters and Hajarnavis (1980, Theorem 8.17), since $\mathbb{Z}$ is right hereditary. Moreover $R$ is Baer; since $R$ is right Noetherian, $R$ is orthogonally finite and so it is Baer by Chatters and Hajarnavis (1980, Lemma 3.4). Since $\mathbb{Z}[x]$ is reduced p.p., $S$ is p.q.-Baer and hence is generalized AIP. But $S$ is not generalized right p.p. (Huh et al., 2002, Example 4).

(ii) Let $n$ be a positive integer for which there exists a prime number $p$ with $p^2 | n$. Let $T$ be the 2-by-2 full matrix ring over $\mathbb{Z}_n[x]$ and $R$ be the 2-by-2 full matrix ring over $\mathbb{Z}_n$. Then $T$ is clearly isomorphic to $R[x]$. Since $\mathbb{Z}_n$, $\mathbb{Z}_n[x]$ are generalized quasi-Baer (and hence, generalized AIP), so $R$ and $T$ are both generalized AIP, by Theorem 4.5.

(iii) Let $S$ and $T$ be the rings in (ii) and (iii) respectively. Then $S \oplus T$ is neither generalized p.p. nor p.q.-Baer, but it is generalized quasi-Baer (and hence, generalized AIP).

References


