IJST (2014) 38A3 (Special issue-Mathematics): 337-342
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

On generalized AIP-rings

M. Anzani and H. Haj Seyyed Javadi*

Department of Mathematics and Computer Science, Shahed University, Tehran, Iran
E-mail: h.s.javadi@shahed.ac.ir

Abstract

In this paper, we introduce the concept of the *generalized AIP-rings* as a generalization of the *generalized quasi-Baer rings* and *generalized p.p.-rings*. We show that the class of the generalized AIP-rings is closed under direct products and Morita invariance. We also characterize the 2-by-2 formal upper triangular matrix rings of this new class of rings. Finally, we provide several examples to show the applicability and limitation of this class of rings.

Keywords: Baer rings; quasi-Baer rings; p.p.-rings; annihilators; idempotent; s-unital ideal

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are left unital.

Recall that R *is (quasi-) Baer* if the right annihilator of every (right ideal) nonempty subset of R is generated (as a right ideal) by an idempotent of R. These definitions are left-right symmetric.

The study of Baer rings has its roots in functional analysis. Kaplansky (1965) introduced Baer rings to abstract various properties of AW*-algebras, von Neumann algebras, and complete *-regular rings. Clark (1967) defined quasi-Baer rings and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra.

As a generalization of Baer rings, a ring R is called *right (resp. left) p.p.-ring* if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of any element of R is generated by an idempotent of R). R is called a *p.p.-ring* (also called a *Ricart ring* (1946)), if it is both right and left p.p.-ring. The concept of p.p.-rings is not left-right symmetric according Chase (1961).

A ring R is said to be *generalized right p.p.* if for any $x \in R$ the right annihilator of x^n is generated by an idempotent for some positive integer n (Huh et al., 2002). Von Neumann regular rings are p.p.-rings (Goodearl, 1991, Theorem 1.1), and π-regular rings are generalized p.p.-rings.

As a generalization of quasi-Baer rings, Birkenmeier, Kim and Park (2001) introduced a principally quasi-Baer ring and used them to generalize many results on *reduced* (i.e., it has no nonzero nilpotent elements) p.p.-rings. A ring R is called right *principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated by an idempotent. Similarly, left p.q.-Baer rings can be defined.

Moussavi, Haj Seyyed Javadi and Hashemi (2005) defined the concept of generalized right (principally) quasi-Baer rings. A ring R is called *generalized right (principally) quasi-Baer* if for any (principal) right ideal I of R, the right annihilator of I^n is generated by an idempotent for some positive integer n, depending on I. Given a fixed positive integer n, a ring R is called n*-generalized right (principally) quasi-Baer* if for any (principal) right ideal I of R, the right annihilator of I^n is generated by an idempotent.

As a generalization of left p.q.-Baer and right p.p.-rings, Zhongkui and Renyu (2006) introduced the concept of left APP-rings and used them to generalize several basic results. A submodule N of a left R-module M is called a *pure submodule* if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right R-module L. A ring R is called *left APP* if the left annihilator Ra is pure as a left ideal of R for any element $a \in R$. Right APP-rings can be defined analogously.

An ideal I is said to be *right s-unital* if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. By Stenstrom (1975, Proposition 11.3.13), an ideal I is right s-unital if and only if R/I is flat as a left R-module if and only if I is pure as a left ideal of R. Note that if I and J are right s-unital ideals, then so is $I \cap J$. By Tominaga (1975,

*Corresponding author
Received: 10 September 2013 / Accepted: 24 February 2014

Theorem 1), I is right s-unital if and only if for any finite elements $a_1, a_2, \ldots, a_n \in I$ there exists an element $x \in I$ such that $a_i x = a_i$, i=1, 2,..., n. Clearly every left p.q.-Baer ring and right p.p.-ring is a left APP-ring. Thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings. By Hirano (2002), a ring R is called quasi-Armendariz if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, we have $a_i R b_j = 0$ for every i and j. Left APP-rings are quasi-Armendriz (Hirano, 2002, Theorem 3.9).

Majidinya et al. defined the concept of right AIP-rings. A ring R is called *right AIP* if R has the property that the right annihilator of any ideal is pure as a right ideal. A left ideal I of R is *centrally s-unital* if, for each $a \in I$ there exists a central element $x \in I$ such that $ax = a$. Also a ring R is called *centrally left AIP* if for any ideal I of R, the left annihilator of I is centrally s-unital as an ideal of R. The class of right AIP-rings is included in the class of right APP-rings.

In this paper we introduce the concept of n-generalized right AIP-rings. Given a fixed positive integer n, we say a ring R is *n-generalized right AIP* if for any right ideal $I \in R$, the right annihilator of I^n is pure as an ideal of R. We say a ring R is *generalized right AIP* if for any right ideal $I \in R$, the right annihilator of I^n is pure as an ideal of R. Left cases may be defined analogously. If R is both left and right generalized AIP, then we say R is generalized AIP-ring.

The class of n-generalized right AIP-rings includes all AIP-rings. Using Example 3.2, we can construct a class of n-generalized left AIP-rings which is neither left nor right AIP. However for a semiprime ring the definition of n-generalized right AIP coincides with that of AIP. By Examples 2.1, 4.3, we show that the concept of AIP and n-generalized AIP rings is not left-right symmetric.

In Section 3, we propose the definition of an n-generalized AIP-ring. Clearly every n-generalized quasi-Baer ring is an n-generalized AIP-ring. Using Example 3.5 the various classes of generalized AIP-rings that are not generalized quasi-Baer are provided. For a ring that satisfies the ascending chain condition on principal left ideals, generalized left AIP and generalized left quasi-Baer properties coincide (Proposition 3.10).

In Section 4, we characterize the n-generalized right AIP property of 2-by-2 generalized triangular matrix rings and full matrix rings. We also prove that, unlike the Baer or right p.p. condition, the n-generalized AIP condition is a Morita invariant property.

2. Preliminaries

For notation we use $Mat_n(R)$ and $T_n(R)$ for the ring of $n \times n$ matrices and the ring of the $n \times n$ upper triangular matrices over R, respectively. For a non-empty subset X of R, $r_R(X)$ (resp. $l_R(X)$) is used for the right (resp. left) annihilator of X over R. Also $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the integers and the integers modulo n, respectively. Observe that every von Neumann regular ring is a left AIP-ring. Since, if I is a left ideal of R and $a \in l_R(I)$, then $a = aba$ for some $b \in R$. Now, let $t = ba$. Then $t \in l_R(I)$ and $a \in a\, l_R(I)$.

From (Birkenmeier et al., 2001, Example 1.6), there exist regular (hence left AIP-) rings which are neither right nor left p.q.-Baer. Also, by (Birkenmeier et al., 2001, Example 1.5), there are right p.q.-Baer (hence left AIP-) rings which are neither quasi-Baer, nor right p.p., nor left p.p.

From (Bell, 1970) a ring R is said to satisfy the *IFP* (insertion of factors property), if $r_R(a)$ is an ideal for all $a \in R$. A ring R is called *abelian* if every idempotent in it is central. It is evident that any reduced ring satisfies *IFP* and any ring with *IFP* is abelian. R is an abelian right p.p.-ring if and only if R is right p.q.-Baer and satisfies *IFP*. For a reduced ring R, we have $l_R(Rx) = l_R((Rx)^n) = l_R(x^n) = l_R(x) = r_R(x) = r_R(x^n) = r_R((xR)^n) = r_R(xR)$, for every $x \in R$ and every positive integer n.

In this section we consider some properties of left AIP-rings. We begin by showing that the AIP condition is not left-right symmetric.

Example 2.1. (Lam, 1999, Example 2.34) Let S be a von Neumann regular ring with an ideal I such that, as a submodule S_s, I is not a direct summand. Let $R = S/I$, which is also a von Neumann regular ring. As a right S-module, R is not projective. Viewing R as an (R, S)-bimodule, we can form the triangular ring $T = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$. Then T is left semihereditary, and so it is a left p.p.-ring. Hence T is a right AIP-ring. Now, let $X = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}$. Then $l_R(X) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, which clearly it is not right s-unital, since $l_R(X) l_R(X) = 0$. Therefore T is not left AIP.

In the following example we show that the Factor ring of a left AIP-ring, is not, in general, left AIP.

Example 2.2. Let $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$, then R is an AIP-ring. But, R/I is neither left AIP nor right AIP-ring. Note that $R/I \cong \mathbb{Z}_4$ and if $J = \{\bar{0}, \bar{2}\}$, then $l_{\mathbb{Z}_4}(J) = J$, and $\bar{2} \in l_{\mathbb{Z}_4}(J)$ but $\bar{2} \notin \bar{2}\, l_{\mathbb{Z}_4}(J)$.

Proposition 2.3. Let R be a left AIP-ring. Then R is semiprime if and only if $l_R(I) \subseteq r_R(I)$, for any left ideal I of R.

Proof: Suppose that R is semiprime and let I be a left ideal of R, then clearly $(Il_R(I)R)^2 = 0$ and so $l_R(I) \subseteq r_R(I)$. Conversely, suppose that $I^2 = 0$, for some left ideal I of R. Then $I \subseteq l_R(I)$, and since R is AIP, $I \subseteq Il_R(I) \subseteq I\, r_R(I) = 0$.

Corollary 2.4. Commutative AIP-rings are reduced.

3. Generalized AIP-rings

In this section we introduce and study the concept of generalized AIP-rings. We show that the generalized AIP property is inherited by direct products. Clearly every AIP-ring is n-generalized AIP-ring.

Definition 3.1. We say a ring R is *n-generalized right AIP* if for any right ideal $I \in R$, the right annihilator of I^n is left s-unital as an ideal of R for a fixed positive integer n. Also we say a ring R is *generalized right AIP* if for any right ideal $I \in R$, the right annihilator of I^n is s-unital for some positive integer n, depending on I. In particular, if $n = 1$, then R is called right AIP.

Left cases can be defined analogously. Obviously every right AIP-ring is an n-generalized right AIP-ring. We provide an example of 2-generalized AIP-ring that is neither left nor right AIP.

Example 3.2. Let $\mathbb{Z} < x, y >$ be the free $\mathbb{Z}$-ring over x, y and $R = \mathbb{Z} < x, y > / < x^2 - x, xy, y^2 >$. Then it is easy to see that $I = \{ay + byx | a, b \in \mathbb{Z}\}$ is an (left) ideal of R such that $y \in l_R(I) = Ry$, and $y \notin yRy$. So R is not a left AIP-ring. Also, $r_R(I) = \{axy - ay \mid a \in \mathbb{Z}\}$, which is not left s-unital and so R is not right AIP-ring. However, it is easy to check that R is 2-generalized AIP.

The following example shows that subrings of generalized AIP-rings are not in general generalized AIP.

Proposition 3.3. Let R be a semiprime ring. Then R is n-generalized right AIP if and only if R is right AIP.

Proof: Let R be n-generalized right AIP and $x \in r_R(I)$ for right ideal I of R. Then $x = ax$ for $a \in r_R(I^n)$ and $(Ia)^n = 0$. Since R is semiprime, $a \in r_R(I)$.

The following example shows that subrings of generalized AIP-rings are not in general generalized AIP.

Example 3.4. The ring $\mathbb{Z} \oplus \mathbb{Z}$ is generalized left AIP. Let p be a prime number and $R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} | \, a \equiv b \bmod p\}$. Then R is a commutative reduced ring and a subring of $\mathbb{Z} \oplus \mathbb{Z}$ which is not left AIP, since $l_R(R(p, 0)) = R(0, p)$ and $(0, p) \notin (0, p)l_R(R(p, 0))$. The following examples show that there are generalized AIP-rings which are not generalized quasi-Baer.

Example 3.5. (i) (Birkenmeier, 2001, Example 1.6) For a field F, take $F_n = F$ for $n = 1, 2, \ldots$, and let $R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \oplus_{n=1}^{\infty} F_n \\ \oplus_{n=1}^{\infty} F_n & < \oplus_{n=1}^{\infty} F_n, 1 > \end{pmatrix}$, which is a subring of the 2×2 matrix ring over $\prod_{n=1}^{\infty} F_n$, where $< \oplus_{n=1}^{\infty} F_n, 1 >$ is the F-algebra generated by $\oplus_{n=1}^{\infty} F_n$ and 1. Then by Goodearl (1991), the ring R is a von Neumann regular ring and so R is an AIP-ring (and hence is generalized AIP). Also, by (Moussavi et al., 2005, Example 2.3), R is neither generalized right quasi-Baer nor generalized left quasi-Baer.

(ii) Let $A = R \oplus Mat_2(\mathbb{Z})$, where R is the ring in (i). Then A is generalized left AIP-ring. But A is neither generalized left quasi-Baer nor generalized right quasi-Baer.

(iii) Let $R := \begin{pmatrix} \bar{A} & \bar{A} \\ 0 & A \end{pmatrix}$, where A is a left p.q.-Baer ring and $\bar{A} = A/P$ for a prime ideal P such that if $b \in P$ then $l_A(\text{Ab})$ is not a subset of P. Then by (Birkenmeier, 2001, Corollary 2.4), R is a left p.q.-Baer ring. Hence R is a generalized right AIP-ring. But by (Moussavi et al., 2005, Example 4.5), R is not generalized right p.q.-Baer (and hence not generalized right quasi-Baer).

In the following, we have a generalized right AIP-ring, which is not generalized p.p.

Example 3.6. Let $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z};\ a \equiv d, b \equiv 0 \text{ and } c \equiv 0 (mod\ 2)\}$. Since R is a prime ring, so it is right AIP and hence generalized right AIP. But by (Moussavi et al., 2005, Example 2.9), R is neither generalized left p.p. nor generalized right p.p.

Lemma 3.7. Let R be a commutative n-generalized right AIP-ring. Then $r_R(I^n) = r_R(I^m)$ for each positive integer $m \geq n$.

Proof: It is enought to show that $r_R(I^n) = r_R(I^{n+1})$. Let $x \in r_R(I^{n+1})$. Then $Ix \subseteq r_R(I^n)$. Hence for every $a \in I$, $ax \in r_R(I^n)$. Since R is an n-generalized right AIP-ring, there exists $b \in r_R(I^n)$ such that $ax = bax$. Thus $I^{n-1}ax = I^{n-1}abx = 0$ and so $x \in r_R(I^n)$.

Corollary 3.8. The commutative n-generalized right AIP-rings have no nilpotent right ideal of order $m > n$.

Lemma 3.9. (Tominaga, 1975, Theorem 1) If I is a left ideal of R then the following are equivalent:
(1). I is right s-unital.
(2) For any elements $a_1, a_2, \ldots, a_n \in I$ there exists an element $x \in I$ such that $a_i x = a_i$ for all $i = 1, 2, \ldots, n$.

We include the following result to indicate the condition under which the n-generalized AIP and n-generalized quasi-Baer properties coincide.

Proposition 3.10. Let R satisfy the ascending chain condition on principal left ideals. Then the following conditions are equivalent:
(1) R is an n-generalized left AIP-ring.
(2) R is an n-generalized left quasi-Baer ring.

Proof: The proof is similar to that of Zhongkui and Renyu (2006, Proposition 2.7).

Note that this reasoning shows in fact that in rings satisfying ascending chain condition on principal left ideals, right s-unital ideals are generated by idempotents (as left ideal).

For a ring satisfying the conditions of Proposition 3.10, the notions generalized left AIP-rings and quasi-Baer rings coincide.

Corollary 3.11. Let R be a semiprime ring which satisfies ACC on principal left ideals. Then the following conditions are equivalent:
(i) R is a quasi-Baer ring.
(ii) R is a left AIP-ring.
(iii) R is an n-generalized left AIP-ring.
(iv) R is an n-generalized left quasi-Baer ring.
Proof: The equivalences, $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.
$(iii) \Rightarrow (iv)$ Proposition 3.10 yields this implication.
$(iv) \Rightarrow (i)$ This follows from Moussavi et al. (2005, Proposition 2.2).

A left perfect ring R satisfies DCC on principal right ideals (Lam, 1991, Theorem 23.20).

By Lam (1999, 6.55), a left perfect ring R satisfies ACC on principal left ideals. Now we have the following corollary:

Corollary 3.12. Let R be a left perfect ring. Then R is an n-generalized left AIP- ring if and only if R is n-generalized quasi-Baer.

4. Generalized matrix rings

Throughout this section T will denote a 2-by-2 generalized (or formal) triangular matrix ring $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, where R and S are rings and M is an (S, R)-bimodule. If N is an (S, R)-submodule of M, then $Ann_R(N) = \{r \in R \mid Nr = 0\}$ and $Ann_S(N) = \{s \in S \mid sN = 0\}$.

Lemma 4.1. (Birkenmeier et al., 2002, Lemma 3.1) Let $\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then

$$r_T(I) = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap Ann_R(N) \end{pmatrix}$$

and

$$l_T(I) = \begin{pmatrix} l_S(I) \cap Ann_S(N) & l_M(L) \\ 0 & l_R(L) \end{pmatrix}.$$

Proposition 4.2. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then T is n-generalized right AIP if and only if the following conditions are satisfied,
(i) R and S are n-generalized right AIP.
(ii) $r_M(I^n) = r_S(I^n)M$ for each ideal I of S.
(iii) If $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}$ is an ideal of T then $r_R(J^n) \cap Ann_R(I^{n-1}N) \cap Ann_R(I^{n-2}NJ) \cap \ldots \cap AnnR(NJ^{n-1})$ is left s-unital.

Proof: Let $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}$ be an ideal of T. Then

$$r_T\begin{pmatrix} I^n & I^{n-1}N + I^{n-2}NJ + \cdots + NJ^{n-1} \\ 0 & J^n \end{pmatrix} = \begin{pmatrix} r_S(I^n) & r_M(I^n) \\ 0 & r_R(J^n) \cap Ann_R(I^{n-1}N + I^{n-2}NJ + \cdots + NJ^{n-1}) \end{pmatrix}.$$

Suppose that

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \begin{pmatrix} r_S(I^n) & r_M(I^n) \\ 0 & r_R(J^n) \cap Ann_R(I^{n-1}N + I^{n-2}NJ + \cdots + NJ^{n-1}) \end{pmatrix}.$$

Then $a \in r_S(I^n)$ and $m \in r_M(I^n) = r_S(I^n)M$. Since S is an n-generalized right AIP-ring, there exist $e_1, e_2 \in r_S(I^n)$ such that $a = e_1 a$ and $m = e_2 m$. From Lemma 3.9, there exists $e \in r_S(I^n)$ such that $a = ea$ and $m = em$. Also, since $r_R(J^n) \cap Ann_R(I^{n-1}N) \cap Ann_R(I^{n-2}NJ) \cap \ldots \cap Ann_R(NJ^{n-1})$ is left s-unital, there exists $e_3 \in R$ so that $b = e_3 b$. Now it is easy to see that

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e_3 \end{pmatrix}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix},$$

$$\begin{pmatrix} e & 0 \\ 0 & e_3 \end{pmatrix} \in \begin{pmatrix} r_S(I^n) & r_M(I^n) \\ 0 & r_R(J^n) \cap Ann_R(I^{n-1}N + I^{n-2}NJ + \cdots + NJ^{n-1}) \end{pmatrix}.$$

Conversely, suppose that T is an n-generalized right AIP-ring. Then for any right ideal I of R, $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ is a right ideal of T and $r_T\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}^n = \begin{pmatrix} 0 & 0 \\ 0 & r_R(I^n) \end{pmatrix}$ is left s-unital, and so $r_R(I^n)$ is left s-unital. Also, for any right ideal J of S, $\begin{pmatrix} J & M \\ 0 & 0 \end{pmatrix}$ is a right ideal of T

and $r_T\begin{pmatrix} J & M \\ 0 & 0 \end{pmatrix}^n = \begin{pmatrix} r_s(J^n) & r_M(J^n) \\ 0 & Ann_R(I^{n-1}M) \end{pmatrix}$ is left s-unital. Thus R and S are n-generalized right AIP-rings.

(ii) Now let I be an ideal of S. Then $\begin{pmatrix} I & M \\ 0 & 0 \end{pmatrix}$ is an ideal of T. Since T is n-generalized right AIP, $r_T\begin{pmatrix} I & M \\ 0 & 0 \end{pmatrix}^n$ is left s-unital. Hence $r_M(I^n) = r_S(I^n)M$.

(iii) Let $\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of T. Since T is n-generalized right AIP, for $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} f_1 & m \\ 0 & f_2 \end{pmatrix} \in r_T\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}^n$, we have $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} f_1 & m \\ 0 & f_2 \end{pmatrix}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$. Hence for $b \in r_R(J^n) \cap Ann_R(I^{n-1}N) \cap Ann_R(I^{n-2}NJ) \cap \ldots \cap Ann_R(NJ^{n-1})$ we have $b = f_2 b$. Thus $r_R(J^n) \cap Ann_R(I^{n-1}N) \cap Ann_R(I^{n-2}NJ) \cap \ldots \cap Ann_R(NJ^{n-1})$ is left s-unital.

There exists an n-generalized right AIP-ring which is not n-generalized left AIP, and hence the definition of n-generalized AIP-ring is not symmetric.

Example 4.3. (Moussavi et al., 2005, Example 4.4) Let $A = \begin{pmatrix} R & S \\ 0 & R \end{pmatrix}$, where $R = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$ is an AIP-ring and $S = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ is a (R, S)-bimodule. By proposition 4.2, A is 2-generalized right AIP, while for $I = \begin{pmatrix} R & S \\ 0 & 0 \end{pmatrix}$, we have $l_A(I^n) = \begin{pmatrix} 0 & S \\ 0 & R \end{pmatrix}$ for each positive integer n and $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \notin \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & S \\ 0 & R \end{pmatrix}$.

Theorem 4.4. The n-by-n upper triangular matrix ring over a commutative generalized right AIP-ring R, is generalized right AIP.

Proof: We proceed by induction on n. Let $\begin{pmatrix} I & L \\ 0 & J \end{pmatrix}$ be an ideal of $T_2(R)$. Since R is commutative generalized right AIP, $r_{T_2(R)}\begin{pmatrix} I & L \\ 0 & J \end{pmatrix}^m$ is left s-unital for some positive integer m. Let $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in r_{T_2(R)}\begin{pmatrix} I & L \\ 0 & J \end{pmatrix}^m$. Then $a = ea$ and $b = eb$ for $e \in r_R(I^m) = r_R(I^{2m})$. Also, there exists $e_1 \in r_R(J^m) = r_R(J^{2m})$ such that $c = e_1 c$. Since $c \in Ann_R(I^{2m-1}L + \ldots + LJ^{2m-1})$, and $I^{2m}R + RJ^{2m} \subseteq I^{2m-1}L + \ldots + LJ^{2m-1}$, we have $(I^{2m}R + RJ^{2m})c = 0$. Hence $I^{2m}Rc = 0$ and $c = ec$. Therefore $c = fc$ where $f = ee_1$ and $f \in r_R(J^{2m-1}) \cap Ann_R(I^{2m-1}L + \ldots + LJ^{2m-1})$. Let $Z = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. It is clear that $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Now let $T_{n+1}(R) = \begin{pmatrix} R & M \\ 0 & T_n(R) \end{pmatrix}$, where $M = (R, \ldots, R)$ (n-tuple). Let $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}$ be an ideal of $T_{n+1}(R)$. Since R and $T_n(R)$ are generalized right AIP-ring, $r_R(I^{2k}) = r_R(I^k)$ and $r_{T_n(R)}(I^{2k}) = r_{T_n(R)}(I^k)$ are left s-unital. We show that $r_{T_n(R)}(J^k) \cap Ann_{T_n(R)}(I^{k-1}N + I^{k-2}NJ + \cdots + INJ^{k-2} + NJ^{k-1})$ is left s-unital. Let $\begin{pmatrix} a & M \\ 0 & B \end{pmatrix} \in r_{T_{n+1}(R)}\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}^{2k}$. Then $a \in r_R(I^{2k})$ and $B \in r_{T_n(R)}(J^{2k}) \cap Ann_{T_n(R)}(I^{2k-1}N + I^{2k-2}NJ + \cdots + INJ^{2k-2} + NJ^{2k-1})$. So there exists $e_1 \in r_R(I^{2k})$ such that $a = e_1 a$. Since $B \in r_{T_n(R)}(J^{2k})$, $B = \lambda B$ for $\lambda \in r_{T_n(R)}(J^{2k})$. Also $B \in Ann_{T_n(R)}(I^{2k-1}N + I^{2k-2}NJ + \cdots + INJ^{2k-2} + NJ^{2k-1})$. Since $I^{2k}M + MJ^{2k} \subseteq I^{2k-1}N + \cdots + NJ^{2k-1}$, we have $(I^{2k}M + MJ^{2k})B = 0$. Hence $I^{2k}MB = 0$. Thus all entries of B belong to $r_R(I^{2k})$, and by Lemma 4.2, $B = \ell B$. Let $f = \ell\lambda$, then we have $B = fB$ and $f \in r_{T_n(R)}(J^k) \cap Ann_{T_n(R)}(I^{k-1}N + I^{k-2}NJ + \cdots + INJ^{k-2} + NJ^{k-1})$.

The following result is the generalized AIP analog of Pollingher and Zak's result (1970, Proposition 16) for quasi-Baer rings.

Theorem 4.5. The endomorphism ring of a finitely generated projective module over an n-generalized right AIP-ring R is n-generalized right AIP. In particular, the condition n-generalized right AIP is a Morita invariant property.

Proof: Assume that R is an n-generalized right AIP-ring. First we claim that eRe is an n-generalized right AIP-ring for every nonzero idempotent e of R. Let I be a right ideal of eRe. and $x \in r_{eRe}(I^n)$. Then $x \in r_R(I^n)$ and so $x = ux$, for $u \in r_R(I^n)$. Thus $x = ex = euexe = (eue)(exe)$ and clearly $eue \in r_{eRe}(I^n)$. Next let m be a positive integer and let I be an ideal of $M_m(R)$. Then there is an ideal J of R such that $I = M_m(J)$. Let $A \in r_{M_m(R)}(I^n) = r_{M_m(R)}(M_m(J^n)) = M_m(r_R(J^n))$. Then $a_{ij} \in r_R(J^n)$. Since R is n-generalized right AIP, there exists $x \in r_R(I^n)$ such that $a_{ij} = a_{ij}x$, for $i, j = 1, 2, \ldots, n$. Therefore $A = A \times x1_{M_m(R)}$, where $1_{M_m(R)}$ denotes the identity matrix of $M_m(R)$.

Proposition 4.6. The class of n-generalized right AIP-rings is closed under direct product.

Proof: The proof is obvious.

The next lemma allows us to construct numerous examples of generalized left AIP-rings that are neither generalized right quasi-Baer, nor generalized right p.p., nor generalized left p.p.

Lemma 4.7. (Birkenmeier et al., 2001, Lemma 1.4) Let T be a ring with unity such that $|T| > 1$, and let $S = \prod_{\lambda\in\Lambda} T_\lambda$, where $T_\lambda = T$ and Λ is an infinite set. If R is the subring of S generated by $\bigoplus_{\lambda\in\Lambda} T_\lambda$ and $1 \in S$, then R is not a generalized right quasi-Baer ring. Moreover, if T is a right p.q.-Baer ring, then R is a generalized left AIP-ring which is not generalized right quasi-Baer.

Using Theorem 4.5 and applying Lemma 4.7, we obtain the following example of generalized AIP-rings which are not generalized right quasi-Baer.

Example 4.8. (i) Let S be the 2-by-2 full matrix ring over $\mathbb{Z}[x]$ and R be the 2-by-2 full matrix ring over $\mathbb{Z}$. It is clear that S is isomorphic to $R[x]$ and R is right p.p. by Chatters and Hajarnavis (1980, Theorem 8.17), since $\mathbb{Z}$ is right hereditary. Moreover R is Baer; since R is right Noetherian, R is orthogonally finite and so it is Baer by Chatters and Hajarnavis (1980, Lemma 3.4). Since $\mathbb{Z}[x]$ is reduced p.p., S is p.q.-Baer and hence is generalized AIP. But S is not generalized right p.p. (Huh et al., 2002, Example 4).

(ii) Let n be a positive integer for which there exists a prime number p with $p^2 \mid n$. Let T be the 2-by-2 full matrix ring over $\mathbb{Z}_n[x]$ and R be the 2-by-2 full matrix ring over $\mathbb{Z}_n$. Then T is clearly isomorphic to $R[x]$. Since $\mathbb{Z}_n$, $\mathbb{Z}_n[x]$ are generalized quasi-Baer (and hence, generalized AIP), so R and T are both generalized AIP, by Theorem 4.5.

(iii) Let S and T be the rings in (ii) and (iii) respectively. Then $S \oplus T$ is neither generalized p.p. nor p.q.-Baer, but it is generalized quasi-Baer (and hence, generalized AIP).

References

Bell, H. E. (1970). Near-rings in which each element is a power of itself. *Bull. Australian Math. Soc,* 2, 363–368.

Birkenmeier, G. F., Kim, J. Y., & Park, J. K. (2001). Principally quasi-Baer rings. *Comm. Algebra,* 29, 639–660.

Birkenmeier, G. F., Kim, J. Y., & Rizvi, S. T. (2002). Generalized triangular matrix rings and the fully invariant extending property. *Rocky Mt. Journal of Math,* 32(4), 1299–1319.

Chase, S. A. (1961). Generalization of triangular matrices. *Nagoya Math. Journal,* 18, 13–25, 1961.

Chatters , A. W., & Hajarnavis, C. R. (1980). *Rings with Chain Conditions*. Pitman, Boston.

Clark, W. E. (1967). Twisted Matrix Units Semigroup Algebras. *Duke Math. J.,* 34, 417–423.

Goodearl, K. R. (1991). *Von Neumann Regular rings*. Krieger: Malabar.

Hirano, Y. (2002). On annihilator ideals of a polynomial ring over a noncommutative ring. *Journal of Pure and Appled Algebra,* 168, 45–52.

Huh, C., Kim, H. K., & Lee, Y. (2002). p.p.-rings and generalized p.p.-rings. *Journal of Pure and Applied Algebra,* 167, 37–52.

Kaplansky, I. (1965). *Rings of operators*. Benjamin: New York.

Lam, T. Y. (1991). *A First Course in Noncommutative Rings*. GTM 131, Berlin-Heidelberg, New York: Springer Verlag.

Lam, T. Y. (1999). *Lectures on Modules and Rings*. GTM 189, Berlin-Heidelberg-New York: Springer Verlag.

Majidinya, A. Moussavi, A., & Paykan, K. Rings in which the annihilator of an ideal is Pure. *to appear in Algebr. Colloq*.

Moussavi, A. Haj Seyyed Javadi, H., & Hashemi, E. (2005). Generalized quasi-Baer rings. *Comm. Algebra,* 33, 2115–2129.

Pollingher, A., & Zarks, A. (1970). On Baer and quasi-Baer rings. *Duke Mathematics Journal,* 37, 127–138.

Ricart, C. E. (1946). Banach Algebras with an adjoint operation. *Annual of Math,* 47, 528–550.

Stenstrom, B. (1975). *Rings of quotients*. GTM 217, Berlin-Heidelberg-New York: Springer-Verlag.

Tominaga, H. (1975). On s-unital rings. *Mathematics Journal Okayama Univ,* 18, 2115–2129.

Zhongkui, L., & Renyu, Z. (2006). A generalization of p.p.-rings and p.q.-Baer rings. *Glasgow Mathematics Journal,* 48, 217–229.