IJST (2014) 38A3 (Special issue-Mathematics): 305-309
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

The uniqueness theorem for differential pencils with the jump condition in the finite interval

A. Neamaty[1]* and Y. Khalili[2]

[1]*Department of Mathematics, University of Mazandaran, Babolsar, Iran*
[2]*Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran*
E-mail: namaty@umz.ac.ir

Abstract

The purpose of this paper is to investigate the inverse problem for a second order differential equation the so-called differential pencil on the finite interval $[0,1]$ when the solutions are not smooth. We establish properties of the spectral characteristics, derive the Weyl function and prove the uniqueness theorem for this inverse problem.

Keywords*:* Inverse problem; differential pencil; jump condition; Weyl function

1. Introduction

We consider the boundary value problem L for the following differential equation in the interval $[0,1]$,

$$y''(x)+\left(\rho^2+i\rho q_1(x)+q_0(x)\right)y(x)=0, \quad (1)$$

with the boundary conditions

$$\begin{aligned}U(y) &:= y'(0)-hy(0)=0,\\ V(y) &:= y'(1)+Hy(1)=0,\end{aligned} \quad (2)$$

and the jump condition

$$\begin{aligned}y(a+0,\rho) &= \alpha_1 y(a-0,\rho),\\ y'(a+0,\rho) &= \alpha_2 y'(a-0,\rho),\end{aligned} \quad (3)$$

in an interior point $x=a$. The functions $q_j(x)$, $j=0,1$ are complex-valued and $q_j(x)\in W_1^j[0,1]$. Also the coefficients h and H are complex numbers and $\alpha_2\neq\pm\alpha_1$.

Boundary value problems with discontinuities inside the interval often appear in mathematics, physics, geophysics, mechanics and other branches of natural sciences. The boundary value problem without discontinuities has been studied in (Neamaty and Mosazadeh, 2010; Neamaty and Sazgar, 2008; Yurko, 2000; Koyubakan, 2006). Some aspects for discontinuous boundary value problems in various formulations have been considered in (Carlsone, 1994; Hald, 1998; Neamaty and Khalili, 2010; Amirov, 2006; Altinisik et al., 2004). In Keskin et al. (2011) and Freiling and Yurko (2001), the inverse problem for classical Sturm-Liouville operators with the jump condition is studied. The inverse problem theory for differential pencils was studied in Neamaty and Khalili (2013), Yurko (1997) and Yurko (2006). Here we will study the boundary value problem for differential pencils with discontinuities that has not been studied so far. In other words, the main goal of the present work is to study the inverse problem of reconstructing the differential pencils with discontinuous conditions by using the Weyl function. The technique employed is similar to those used in (Neamaty and Khalili, 2013).

In order to study the inverse problem in this paper, we use the Weyl function. Special fundamental system of solutions (FSS) plays an important role in this method. FSS provides an opportunity to obtain the asymptotic behavior of the so-called Weyl solution and Weyl function. Using these functions, we prove the uniqueness solution of the inverse problem. In Section 2, we determine the asymptotic form of the solutions and eigenvalues and give the Weyl function. In Section 3, we prove the uniqueness theorem and finally, Section 4 contains some conclusions.

2. The properties of the spectrum

Let the functions $C(x,\rho)$, $S(x,\rho)$, $\varphi(x,\rho)$ and $\psi(x,\rho)$ be solutions of Eq. (1) under the initial conditions $C(0,\rho)=S'(0,\rho)=\varphi(0,\rho)=\psi(1,\rho)=1$, $C'(0,\rho)=S(0,\rho)=0$, $\varphi'(0,\rho)=h$, $\psi'(1,\rho)=-H$ and the jump condition (3).

Let $C_0(x,\rho)$ and $S_0(x,\rho)$ be smooth solutions of Eq. (1) on the interval $[0,1]$ under the initial

*Corresponding author
Received: 27 July 2013 / Accepted: 24 February 2014

conditions $C_0(0,\rho) = S_0'(0,\rho) = 1$ and $C_0'(0,\rho) = S_0(0,\rho) = 0$. We have for $x < a$,

$$\begin{cases} C(x,\rho) = C_0(x,\rho), \\ S(x,\rho) = S_0(x,\rho). \end{cases} \tag{4}$$

Using the jump condition (3), we get for $x > a$,

$$\begin{cases} C(x,\rho) = A_1(\rho)C_0(x,\rho) + B_1(\rho)S_0(x,\rho), \\ S(x,\rho) = A_2(\rho)C_0(x,\rho) + B_2(\rho)S_0(x,\rho). \end{cases} \tag{5}$$

Denote

$$\Delta(\rho) := \langle \psi(x,\rho), \varphi(x,\rho) \rangle, \tag{6}$$

where $\langle y, z \rangle := yz' - y'z$ is the Wronskian of the functions $y(x)$ and $z(x)$. The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem L.

From Buterin and Shieh (2009) and Freiling and Yurko (2001), we know that there exists the solution of the following form for $|\rho| \to \infty$ uniformly in $x < a$,

$$\varphi(x,\rho) = cos(\rho x - Q(x)) + O(\rho^{-1} exp(|\tau|x)), \tag{7}$$

$$\varphi'(x,\rho) = -\rho sin(\rho x - Q(x)) + O(exp(|\tau|x)), \tag{8}$$

where $Q(x) = \frac{1}{2}\int_0^x q_1(t)dt$, $\lambda = \rho^2$, $\tau = Im\rho$.

Theorem 2.1. By virtue of Liouville's formula for the Wronskian, $\Delta(\rho)$ does not depend on x, and

$$\Delta(\rho) = V(\varphi). \tag{9}$$

Proof: Let $y(x,\lambda)$ and $z(x,\mu)$ be solutions of Eq. (1) for parameters λ and μ, respectively. Then $\frac{d}{dx}\langle y, z \rangle = (\lambda - \mu)yz$. Since $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are the solutions of Eq. (1), we have $\frac{d}{dx}\langle y, z \rangle = 0$. Therefore $\langle \psi(x,\lambda), \varphi(x,\lambda) \rangle$ is constant, i.e., $\Delta(\rho)$ dos not depend on x. Now, for $x = 1$, we have

$$\begin{aligned} \Delta(\rho) &= \langle \psi(x,\rho), \varphi(x,\rho) \rangle_{|x=1} \\ &= \psi(1,\rho)\varphi'(1,\rho) - \psi'(1,\rho)\varphi(1,\rho) \\ &= 1 \times \varphi'(1,\rho) - (-H)\varphi(1,\rho) \\ &= \varphi'(1,\rho) + H\varphi(1,\rho) = V(\varphi). \end{aligned}$$

The proof is completed.

Definition 2.2. The values of the parameter ρ for which the equation (1) has nontrivial solutions satisfying the boundary conditions (2), are called the eigenvalues of L and the corresponding solutions are called the eigenfunctions.

Theorem 2.3. i_1) For $|\rho| \to \infty$ the following asymptotical formula holds

$$\begin{aligned} \Delta(\rho) = \rho\big(&-b_1 sin(\rho - Q(1)) \\ &+ b_2 sin(-2Q(a) + Q(1))\big) \\ &+ b_1(h+H)cos(\rho - Q(1)) \\ &+ b_2\Big(h\, cos\big(2\rho - (2Q(a) + Q(1))\big) \\ &+ H\, cos(-2Q(a) + Q(1))\Big) \\ &+ O(\rho^{-1} exp(|\tau|)), \end{aligned}$$

where $b_1 = \frac{\alpha_1 + \alpha_2}{2}$, $b_2 = \frac{\alpha_1 - \alpha_2}{2}$.

i_2) For sufficiently large k, the function $\Delta(\rho)$ has simple zeros of the form

$$\begin{aligned} \rho_k = 2k\pi + i ln\left(\frac{-2b_2 i}{b_1} sin(-2Q(a) + Q(1))\right) \\ + Q(1) + O(k^{-1}). \end{aligned} \tag{10}$$

Proof: We have (see Buterin and Shieh (2009) and Freiling and Yurko (2001))

$$\begin{aligned} C_0(x,\rho) &= cos(\rho x - Q(x)) \\ &\quad + \frac{Q_0(x)\, sin(\rho x - Q(x))}{\rho} + o(\rho^{-1} exp(|\tau|x)), \\ C_0'(x,\rho) &= -\rho\, sin(\rho x - Q(x)) \\ &\quad + Q_0(x)\, cos(\rho x - Q(x)) + o(exp(|\tau|x)), \end{aligned}$$

where $Q_0(x) = \frac{1}{2}\int_0^x q_0(t)dt$. Also

$$\begin{aligned} S_0(x,\rho) &= \frac{sin(\rho x - Q(x))}{\rho} \\ &\quad - \frac{Q_0(x)cos(\rho x - Q(x))}{\rho^2} + o(\rho^{-2} exp(|\tau|x)), \\ S_0'(x,\rho) &= cos(\rho x - Q(x)) \\ &\quad + \frac{Q_0(x)sin(\rho x - Q(x))}{\rho} + o(\rho^{-1} exp(|\tau|x)). \end{aligned}$$

Using (5), the solutions $C_0(x,\rho)$, $S_0(x,\rho)$ and the jump condition (3), we have

$$\begin{aligned} A_1(\rho) &= b_1 + b_2\, cos\, 2(\rho a - Q(a)) \\ &\quad + \frac{2b_2}{\rho} Q_0(a) sin2(\rho a - Q(a)) \\ &\quad + o(\rho^{-1} exp(|\tau|a)), \\ A_2(\rho) &= b_2 \frac{sin2(\rho a - Q(a))}{\rho} + O(\rho^{-2} exp(|\tau|a)), \\ B_1(\rho) &= b_2 \rho\, sin2(\rho a - Q(a)) \\ &\quad - 2b_2 Q_0(a)\, cos2(\rho a - Q(a)) + o(exp(|\tau|a)), \\ B_2(\rho) &= b_1 + b_2\, cos2(\rho a - Q(a)) \\ &\quad + O(\rho^{-1} exp(|\tau|a)). \end{aligned}$$

Substituting these coefficients and the functions $C_0(x,\rho), S_0(x,\rho)$ in (5), we obtain

$$\begin{aligned} S(x,\rho) &= \frac{1}{\rho}\Big(b_1\, sin(\rho x - Q(x)) \\ &\quad + b_2\, sin\big(\rho(2a + x) - (2Q(a) + Q(x))\big)\Big) \\ &\quad + O(\rho^{-2} exp(|\tau|x)), \\ C(x,\rho) &= \frac{1}{\rho}\Big(b_1\, cos(\rho x - Q(x)) \\ &\quad + b_2\, cos\big(\rho(2a - x) - (2Q(a) - Q(x))\big)\Big) \end{aligned}$$

$$+o\left(\rho^{-1}exp(|\tau|x)\right).$$

Since

$$\varphi(x,\rho)=C(x,\rho)+hS(x,\rho),$$

we have

$$\begin{aligned}\varphi(x,\rho)&=b_1\cos\left(\rho x-Q(x)\right)\\&+b_2\cos\left(\rho(2a-x)-\left(2Q(a)-Q(x)\right)\right)\\&+O\left(\rho^{-1}exp(|\tau|x)\right),\qquad x>a,\\\varphi'(x,\rho)&=-b_1\rho\sin\left(\rho x-Q(x)\right)\\&+b_2\,\rho\sin\left(\rho(2a-x)-\left(2Q(a)-Q(x)\right)\right)\\&+O\left(exp(|\tau|x)\right),\qquad x>a.\end{aligned}$$

Using (2) and (9), we arrive at the characteristic function $\Delta(\rho)$.

Now suppose that the function $\Delta(\rho)$ has simple zeros of the following form (see Conway (1995))

$$\rho_k=\rho_k^0+O(k^{-1}),\quad k\to\infty,$$

where ρ_k^0 are the zeros of the function

$$\begin{aligned}\Delta^0(\rho)&=-b_1\sin\left(\rho-Q(1)\right)\\&+b_2\sin\left(-2Q(a)+Q(1)\right).\end{aligned}$$

Since

$$\rho_k^0=2k\pi+i\ln\left(\frac{-2b_2i}{b_1}\sin\left(-2Q(a)+Q(1)\right)\right)+Q(1),$$
$$k\to\infty,$$

we arrive at the eigenvalues of the form (10).

Corollary 2.4. It follows from (7), (8) and the functions $\varphi^{(m)}(x,\rho)$ in proof of Theorem 2.3 that

$$\left|\varphi^{(m)}(x,\rho)\right|\le C|\rho|^m exp(|Im\rho|x),\quad 0\le x<1.\qquad(11)$$

Denote

$$\phi(x,\rho)=\frac{-\psi(x,\rho)}{\Delta(\rho)}.\qquad(12)$$

Let $\phi(x,\rho)$ be the solution of Eq. (1) under the boundary conditions $U(\phi)=1$, $V(\phi)=0$. We set $M(\rho)=\phi(0,\rho)$. The functions $\phi(x,\rho)$ and $M(\rho)$ are called the Weyl solution and Weyl function for the boundary value problem L, respectively. Clearly, using the conditions at the point $x=0$, we get

$$\phi(x,\rho)=S(x,\rho)+M(\rho)\varphi(x,\rho),\qquad(13)$$

and

$$\langle\varphi(x,\rho),\phi(x,\rho)\rangle=1.\qquad(14)$$

Lemma 2.5. For $|\rho|\to\infty$,

$$\begin{aligned}M(\rho)&=\mp\frac{1}{i\alpha_1b_1\rho}\cos\left(\rho(1-a)-\left(Q(1)-Q(a)\right)\right)\\&+o\left(\rho^{-1}exp(1-a)\right).\end{aligned}\qquad(15)$$

Proof: Using the FSS $\{C_0(x,\rho),S_0(x,\rho)\}$, we have for $x\in(a,1]$,

$$\psi(x,\rho)=L_1(\rho)C_0(x,\rho)+L_2(\rho)S_0(x,\rho).\qquad(16)$$

Taking the Cramer's rule and the initial conditions $\psi(x,\rho)$ in $x=1$, we obtain

$$\begin{aligned}L_1(\rho)&=\cos\left(\rho-Q(1)\right)+O\left(\rho^{-1}exp(|\tau|)\right),\\L_2(\rho)&=\rho\sin\left(\rho-Q(1)\right)-H\cos\left(\rho-Q(1)\right)\\&+o\left(exp(|\tau|)\right).\end{aligned}$$

Substituting these coefficients in (16), we have

$$\begin{aligned}\psi(x,\rho)&=\cos\left(\rho(1-x)-\left(Q(1)-Q(x)\right)\right)\\&+O\left(\rho^{-1}exp\left(|\tau|(1-x)\right)\right),\ a<x\le 1.\end{aligned}\qquad(17)$$

Analogously taking the FSS $\{C_0(x,\rho),S_0(x,\rho)\}$ and the jump condition (3), we have

$$\begin{aligned}\psi(x,\rho)&=\frac{\alpha_1^{-1}+\alpha_2^{-1}}{4}(\cos(\rho(2-a+x)\\&-\left(2Q(1)-Q(a)+Q(x)\right))\\&+\cos\left(\rho(a+x)-\left(Q(a)-Q(x)\right)\right))\\&+\frac{\alpha_1^{-1}-\alpha_2^{-1}}{4}(\cos(\rho(2-a-x)\\&-\left(2Q(1)-Q(a)-Q(x)\right))\\&+\cos\left(\rho(a-x)-\left(Q(a)+Q(x)\right)\right))\\&+O\left(\rho^{-1}exp\left(|\tau|(2-a+x)\right)\right),\ 0\le x\le a.\end{aligned}\qquad(18)$$

Using (12) at the point $x=0$, we arrive at (15).

Taking the functions $\psi(x,\rho)$ in proof of Lemma 2.5, we have

$$\begin{aligned}\psi'(x,\rho)&=\rho\sin\left(\rho(1-x)-\left(Q(1)-Q(x)\right)\right)\\&+O\left(exp\left(|\tau|(1-x)\right)\right),\qquad a<x\le 1,\\\psi'(x,\rho)&=\frac{\rho(\alpha_1^{-1}+\alpha_2^{-1})}{-4}(\sin(\rho(2-a+x)\\&-\left(2Q(1)-Q(a)+Q(x)\right))\\&+\sin\left(\rho(a+x)-\left(Q(a)-Q(x)\right)\right))\\&+\frac{\rho(\alpha_1^{-1}-\alpha_2^{-1})}{-4}(\sin(\rho(2-a-x)\\&-\left(2Q(1)-Q(a)-Q(x)\right))\\&+\sin\left(\rho(a-x)-\left(Q(a)+Q(x)\right)\right))\\&+O\left(exp\left(|\tau|(2-a+x)\right)\right),\ \ 0\le x\le a.\end{aligned}\qquad(19)$$

Inverse Problem 2.6. Suppose that a and α_j, $j=1,2$, are known a priori. Our goal is to find

$q_0(x), q_1(x)$, h and H from the given Weyl function $M(\rho)$.

3. Uniqueness theorem

Now we prove the uniqueness theorem for the solution of the inverse problem. We consider together with $L = L(q_0(x), q_1(x), h, H)$ a boundary value problem $\tilde{L} = L(\tilde{q}_0(x), \tilde{q}_1(x), \tilde{h}, \tilde{H})$ of the same form (1)-(2) but with different coefficients. If a certain symbol denotes an object related to L, then the same symbol will denote the analogs object related to $\tilde{L}$.

Theorem 3.1. If $M(\rho) = \tilde{M}(\rho)$ then $L = \tilde{L}$. Thus, the specification of the Weyl function uniquely determines the BVP(L).

Proof: Let us define the matrix

$$P(x,\rho) = [P_{jk}(x,\rho)]_{j,k=1,2}, \tag{20}$$

by the formula

$$P(x,\rho)\begin{bmatrix} \tilde{\varphi}(x,\rho) & \tilde{\phi}(x,\rho) \\ \tilde{\varphi}'(x,\rho) & \tilde{\phi}'(x,\rho) \end{bmatrix} = \begin{bmatrix} \varphi(x,\rho) & \phi(x,\rho) \\ \varphi'(x,\rho) & \phi'(x,\rho) \end{bmatrix}. \tag{21}$$

By virtue of (14), this yields

$$\begin{cases} P_{j1}(x,\rho) = \varphi^{(j-1)}(x,\rho)\tilde{\phi}'(x,\rho) - \phi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho), \\ P_{j2}(x,\rho) = \phi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{\phi}(x,\rho). \end{cases} \tag{22}$$

Also we have

$$\begin{cases} \varphi(x,\rho) = P_{11}(x,\rho)\tilde{\varphi}(x,\rho) + P_{12}(x,\rho)\tilde{\varphi}'(x,\rho), \\ \phi(x,\rho) = P_{11}(x,\rho)\tilde{\phi}(x,\rho) + P_{12}(x,\rho)\tilde{\phi}'(x,\rho). \end{cases} \tag{23}$$

Using (13) and (22), we obtain

$$\begin{cases} P_{j1}(x,\rho) = \varphi^{(j-1)}(x,\rho)\tilde{S}'(x,\rho) - S^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho) \\ \qquad + \hat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho), \\ P_{j2}(x,\rho) = S^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{S}(x,\rho) \\ \qquad - \hat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho), \end{cases} \tag{24}$$

where $\hat{M}(\rho) = \tilde{M}(\rho) - M(\rho)$. Since $\tilde{M}(\rho) = M(\rho)$ deduce that $\hat{M}(\rho) = 0$, and consequently, for each fixed x in [0,1] the functions $P_{jk}(x,\rho), k = 1,2$, are entire in ρ.

Fix $\varepsilon > 0$. Denote $G_\varepsilon = \{\rho \in \mathbb{C}: |\rho - \rho_k| \geq \varepsilon\}$. It follows from (12), (17), (18), (19) and the function $\Delta(\rho)$ in Theorem 2.3 that

$$\begin{cases} |\psi^{(m)}(x,\rho)| \leq C|\rho|^m \exp(|Im\rho|(2-a+x)),\ x \in [0,a), \\ |\psi^{(m)}(x,\rho)| \leq C|\rho|^m \exp(|Im\rho|(1-x)),\ x \in (a,1]. \end{cases} \tag{25}$$

$$|\Delta(\rho)| \geq C|\rho| \exp(|Im\rho|),\ \ \rho \in G_\varepsilon, \tag{26}$$

$$\begin{cases} |\phi^{(m)}(x,\rho)| \leq C|\rho|^{m-1} \exp(|Im\rho|(1-a+x)), \\ \qquad x \in [0,a), \quad \rho \in G_\varepsilon, \\ |\phi^{(m)}(x,\rho)| \leq C|\rho|^{m-1} \exp(-|Im\rho|x), \\ \qquad x \in (a,1], \ \rho \in G_\varepsilon. \end{cases} \tag{27}$$

It follows from (11), (22) and (27) that for $0 \leq x \leq 1$, $\rho \in G_\varepsilon$,

$$|P_{11}(x,\rho)| \leq C, \quad |P_{12}(x,\rho)| \leq C|\rho|^{-1}.$$

Therefore $P_{11}(x,\rho) = P_1(x)$ and $P_{12}(x,\rho) = 0$ for each $0 \leq x \leq 1$. Together with (23), this yields

$$P_1(x)\tilde{\varphi}(x,\rho) = \varphi(x,\rho),\ P_1(x)\tilde{\phi}(x,\rho) = \phi(x,\rho). \tag{28}$$

First let $x \in (a,1]$. Taking (12), (17) and the functions $\varphi(x,\rho)$ and $\Delta(\rho)$ in Theorem 2.3, we have as $|\rho| \to \infty$, $\arg\rho \in \left(0, \frac{\pi}{2}\right)$,

$$\frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = \exp\left(i\big(Q(x) - \tilde{Q}(x)\big)\right)[1], \\ \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = \exp\left(-i\big(Q(x) - \tilde{Q}(x)\big)\right)[1], \tag{29}$$

where $[1] = 1 + O(\rho^{-1})$. It follows from (28) and (29) that

$$P_1(x) = \exp\left(i\left(Q(x) - \tilde{Q}(x)\right)\right)[1], \\ P_1(x) = \exp\left(-i\left(Q(x) - \tilde{Q}(x)\right)\right)[1], \tag{30}$$

and consequently, $Q(x) = \tilde{Q}(x)$ for $a < x \leq 1$ and $P_1(x) = 1$.

Now, let $x \in [0,a)$. Analogously taking (7), (12), (18), the function $\Delta(\rho)$ in Theorem 2.3 and equalities $Q(a) = \tilde{Q}(a)$ and $Q(1) = \tilde{Q}(1)$, we get for $|\rho| \to \infty$, $\arg\rho \in \left(0, \frac{\pi}{2}\right)$,

$$\frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = \exp\left(i\big(Q(x) - \tilde{Q}(x)\big)\right)[1], \\ \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = \exp\left(-i\big(Q(x) - \tilde{Q}(x)\big)\right)[1]. \tag{31}$$

One has from (28) and (31) that

$$P_1(x) = \exp\left(i\left(Q(x) - \tilde{Q}(x)\right)\right)[1], \\ P_1(x) = \exp\left(-i\left(Q(x) - \tilde{Q}(x)\right)\right)[1], \tag{32}$$

and consequently, $Q(x) = \tilde{Q}(x)$ for $0 \leq x < a$ and $P_1(x) = 1$.

Thus $Q(x) = \tilde{Q}(x)$ for all $0 \leq x \leq 1$, and consequently $q_1(x) = \tilde{q}_1(x)$ for any $0 \leq x \leq 1$. Also since $P_1(x) = 1$, we have $\varphi(x,\rho) = \tilde{\varphi}(x,\rho)$ and $\phi(x,\rho) = \tilde{\phi}(x,\rho)$, hence, $q_0(x) = \tilde{q}_0(x)$, $h = \tilde{h}$ and $H = \tilde{H}$. Therefore $L = \tilde{L}$. The proof is completed.

4. Conclusions

Through review of the papers, it is revealed that there is not an inverse problem for differential pencils with a discontinuity. This is a lack for such problems and it is studied. In particular, the some methods in the inverse problem theory for Sturm–Liouville operators like the transformation operator do not give reliable results for differential pencils with discontinuity. The method of the spectral mappings is appropriate for the Weyl function to play an important role in. Thus we investigated this problem and obtained some new results.

Aknowledgement

The authors would like to express their thankfulness for the valuable comments and suggestions made by the anonymous reviewers.

References

Altinisik, N., Kadakal, M., & Mukhtarov, O. S. (2004). Eigenvalues and eigenfunctions of discontinuous Sturm-Liouville problems with eigenparameter dependent boundary conditions. *Acta Mathematica Hungarica*, 102(1-2), 159–193.

Amirov, R. Kh. (2006). On Sturm-Liouville operators with discontinuity conditions inside an interval. *Journal of Mathematical Analysis and Applications*, 317(1), 163–176.

Buterin, S. A., & Shieh, C. T. (2009). Inverse nodal problems for differential pencils. *Applied Mathematics Letters*, 22(8), 1240–1247.

Carlsone, R. (1994). An inverse spectral problem for Sturm-Lioville operator with discontinues coefficients. *Proceedings of the American Mathematical Society*, 120(2), 475–484.

Conway, J .B. (1995). *Functions of One Complex Variable*. Springer-Verlag, New York.

Freiling, G., & Yurko, V. A. (2001). *Inverse Sturm-Liouville Problems and their Applications*. NOVA Science Publisher, New York.

Hald, O. H. (1998). Discontinues inverse eigenvalue problems. *Communications on Pure and Applied Mathematics*, 37(5), 539–577.

Keskin, B., Sinan, A., & Yalkin, N. (2011). Inverse spectral problems for discontinuous Sturm-Liouville operator with eigenparameter dependent boundary conditions. *Communications de la Faculté des Sciences de l'Université d'Ankara, Series A1*, 60(1), 15–25.

Koyunbakan, H. (2006). A new inverse problem for the diffusion operator. *Applied Mathematics Letters*, 19(10), 995–999.

Neamaty, A., & Sazgar, E. A. (2008). Asymptotic eigenvalues for Sturm-Liouville problem with neumann conditions in two turning points case. *Iranian Journal of Science & Technology, Transaction*, 32(A4), 289–296.

Neamaty, A., & Mosazadeh, S. (2010). On the canonical solution and dual equations of Sturm-Liouville problem with singularity and turning point. *Iranian Journal of Science & Technology, Transaction*, 34(A1), 71–88.

Neamaty, A., & Khalili, Y. (2010). The eigenvalue problems for differential equations with jump condition and turning point. *Far East Journal of Mathematical Sciences*, 37(1), 1–7.

Neamaty, A., & Khalili, Y. (2013). The differential pencils with turning point on the half line. *Arab Journal of Mathematical Sciences*, 19(1), 95–104.

Yurko, V. A. (1997). An inverse problem for systems of differential equations with nonlinear dependence on the spectral parameter. *Differential Equations*, 33(3), 388–394.

Yurko, V. A. (2000). An inverse problem for pencils of differential operators. *Sbornik: Mathematics*, 191(10), 1561–1586.

Yurko, V. (2006). Inverse spectral problems for differential pencils on the half-line with turning points. *Journal of Mathematical Analysis and Applications*, 320(1), 439–463.