The uniqueness theorem for differential pencils with the jump condition in the finite interval

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Abstract

The purpose of this paper is to investigate the inverse problem for a second order differential equation the so-called differential pencil on the finite interval \((0,1]\) when the solutions are not smooth. We establish properties of the spectral characteristics, derive the Weyl function and prove the uniqueness theorem for this inverse problem.

Keywords: Inverse problem; differential pencil; jump condition; Weyl function

1. Introduction

We consider the boundary value problem for the following differential equation in the interval \([0,1]\),

\[ y''(x) + (\rho^2 + ipq_1(x) + q_0(x))y(x) = 0, \quad (1) \]

with the boundary conditions

\[ U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(1) + Hy(1) = 0, \quad (2) \]

and the jump condition

\[ y(a + 0, \rho) = \alpha_1 y(a - 0, \rho), \quad y'(a + 0, \rho) = \alpha_2 y'(a - 0, \rho), \quad (3) \]

in an interior point \(x = a\). The functions \(q_j(x)\), \(j = 0, 1\) are complex-valued and \(q_j(x) \in W^1[0,1]\).

Also the coefficients \(h\) and \(H\) are complex numbers and \(\alpha_2 \neq \pm \alpha_1\).

Boundary value problems with discontinuities inside the interval often appear in mathematics, physics, geophysics, mechanics and other branches of natural sciences. The boundary value problem without discontinuities has been studied in (Neamaty and Mosazadeh, 2010; Neamaty and Sazgar, 2008; Yurko, 2000; Koyubakan, 2006). Some aspects for discontinuous boundary value problems in various formulations have been considered in (Carlson, 1994; Hald, 1998; Neamaty and Khalili, 2010; Amirov, 2006; Altinisik et al., 2004). In Keskin et al. (2011) and Freiling and Yurko (2001), the inverse problem for classical Sturm-Liouville operators with the jump condition is studied. The inverse problem theory for differential pencils was studied in Neamaty and Khalili (2013), Yurko (1997) and Yurko (2006). Here we will study the boundary value problem for differential pencils with discontinuities that has not been studied so far. In other words, the main goal of the present work is to study the inverse problem of reconstructing the differential pencils with discontinuous conditions by using the Weyl function. The technique employed is similar to those used in (Neamaty and Khalili, 2013).

In order to study the inverse problem in this paper, we use the Weyl function. Special fundamental system of solutions (FSS) plays an important role in this method. FSS provides an opportunity to obtain the asymptotic behavior of the so-called Weyl solution and Weyl function. Using these functions, we prove the uniqueness solution of the inverse problem. In Section 2, we determine the asymptotic form of the solutions and eigenvalues and give the Weyl function. In Section 3, we prove the uniqueness theorem and finally, Section 4 contains some conclusions.

2. The properties of the spectrum

Let the functions \(C(x, \rho)\), \(S(x, \rho)\), \(\varphi(x, \rho)\) and \(\psi(x, \rho)\) be solutions of Eq. (1) under the initial conditions \(C(0, \rho) = S(0, \rho) = \varphi(0, \rho) = \psi(0, \rho) = 1\), \(C'(0, \rho) = S(0, \rho) = 0\), \(\varphi'(0, \rho) = h\), \(\psi'(0, \rho) = -H\) and the jump condition (3).

Let \(G_0(x, \rho)\) and \(S_0(x, \rho)\) be smooth solutions of Eq. (1) on the interval \([0,1]\) under the initial
conditions $C_0(0, \rho) = S'_0(0, \rho) = 1$ and $C'_0(0, \rho) = S_0(0, \rho) = 0$. We have for $x < a$,

$$
\begin{align*}
\begin{cases}
C(x, \rho) = C_0(x, \rho), \\
S(x, \rho) = S_0(x, \rho),
\end{cases}
\end{align*}
$$

(4)

Using the jump condition (3), we get for $x > a$,

$$
\begin{align*}
\begin{cases}
C(x, \rho) = A_1(\rho)C_0(x, \rho) + B_1(\rho)S_0(x, \rho), \\
S(x, \rho) = A_2(\rho)C_0(x, \rho) + B_2(\rho)S_0(x, \rho),
\end{cases}
\end{align*}
$$

(5)

Denote

$$
\Delta(\rho) = (\psi(x, \rho), \phi(x, \rho)),
$$

(6)

where $(y, z) = yz' - y'z$ is the Wronskian of the functions $y(x)$ and $z(x)$. The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem $L$.

From Buterin and Shieh (2009) and Freiling and Yurko (2001), we know that there exists the solution of the following form for $|\rho| \to \infty$ uniformly in $x < a$,

$$
\phi(x, \rho) = \cos(px - Q(x)) + o(\rho^{-1}\exp(|\tau|x)).
$$

(7)

$$
\phi'(x, \rho) = -p\sin(px - Q(x)) + o(\exp(|\tau|x)).
$$

(8)

where $Q(x) = \frac{1}{2\rho} q_1(t)dt$, $\lambda = \rho^2$, $\tau = Imp$.

**Theorem 2.1.** By virtue of Liouville’s formula for the Wronskian, $\Delta(\rho)$ does not depend on $x$, and

$$
\Delta(\rho) = V(\phi).
$$

(9)

**Proof:** Let $y(x, \lambda)$ and $z(x, \mu)$ be solutions of Eq. (1) for parameters $\lambda$ and $\mu$, respectively. Then

$$
d\frac{d}{dx}(y, z) = (\lambda - \mu)yz.\quad \text{Since } \psi(x, \lambda) \text{ and } \phi(x, \lambda)\quad \text{are the solutions of Eq. (1), we have } d\frac{d}{dx}(y, z) = 0.
$$

Therefore $(\psi(x, \lambda), \phi(x, \lambda))$ is constant, i.e., $\Delta(\rho)$ does not depend on $x$. Now, for $x = 1$, we have

$$
\Delta(\rho) = (\psi(x, \rho), \phi(x, \rho))|_{x=1} = \psi(1, \rho)\phi'(1, \rho) - \phi(1, \rho)\psi'(1, \rho) = 1 \times \phi'(1, \rho) - (-H)\phi(1, \rho) = \phi'(1, \rho) + H\phi(1, \rho) = V(\phi).
$$

The proof is completed.

**Definition 2.2.** The values of the parameter $\rho$ for which the equation (1) has nontrivial solutions satisfying the boundary conditions (2), are called the eigenvalues of $L$ and the corresponding solutions are called the eigenfunctions.

**Theorem 2.3.** For $|\rho| \to \infty$ the following asymptotical formula holds

$$
\Delta(\rho) = \rho(-b_2\sin(\rho - Q(1))) + b_2\sin(-2Q(a) + Q(1)) + b_1(h + H)\cos(\rho - Q(1)) + b_2\left(h \cos\left(2p - (2Q(a) + Q(1))\right) + H \cos\left(-2Q(a) + Q(1)\right) + O(\rho^{-1}\exp(|\tau|))\right),
$$

(10)

where $b_1 = \frac{a_1 + a_2}{2}$, $b_2 = \frac{a_1 - a_2}{2}$.

$I_2$) For sufficiently large $k$, the function $\Delta(\rho)$ has simple zeros of the form

$$
\rho_k = 2k\pi + i\text{thn}\left(\frac{-2\pi b_1}{2k} \sin(-2Q(a) + Q(1))\right) + O(1) + O(k^{-1}).
$$

(10)

**Proof:** We have (see Buterin and Shieh (2009) and Freiling and Yurko (2001))

$$
C_0(x, \rho) = \cos(px - Q(x)) + o(\rho^{-1}\exp(|\tau|x)),
$$

(10)

$$
C'_0(x, \rho) = -\rho \sin(px - Q(x)) + o(\exp(|\tau|x)).
$$

(10)

where $Q_0(x) = \frac{1}{2\rho} \int_0^x q_0(t)dt$. Also

$$
S_0(x, \rho) = \sin(px - Q(x)) + o(\rho^{-1}\exp(|\tau|x)),
$$

(10)

$$
S'_0(x, \rho) = \cos(px - Q(x)) + o(\exp(|\tau|x)).
$$

(10)

Using (5), the solutions $C_0(x, \rho), S_0(x, \rho)$ and the jump condition (3), we have

$$
\begin{align*}
A_1(\rho) &= b_1 + b_2 \cos(2\rho a - Q(a)) + O(\rho^{-2}\exp(|\tau|)), \\
A_2(\rho) &= b_2 \sin(2\rho a - Q(a)) + O(\rho^{-2}\exp(|\tau|)), \\
B_1(\rho) &= b_2 \cos(2\rho a - Q(a)) + O(\rho^{-2}\exp(|\tau|)), \\
B_2(\rho) &= b_1 + b_2 \cos(2\rho a - Q(a)) + O(\rho^{-2}\exp(|\tau|)).
\end{align*}
$$

(10)

Substituting these coefficients and the functions $C_0(x, \rho), S_0(x, \rho)$ in (5), we obtain

$$
\begin{align*}
S(x, \rho) &= \frac{1}{\rho}(b_1 \sin(px - Q(x)) + b_2 \sin\left(\rho(2a + x) - (2Q(a) + Q(x))\right)) + O(\rho^{-2}\exp(|\tau|)), \\
C(x, \rho) &= \frac{1}{\rho}(b_1 \cos(px - Q(x)) + b_2 \cos\left(\rho(2a - x) - (2Q(a) - Q(x))\right))
\end{align*}
$$

(10)
Since
\[ \varphi(x, \rho) = C(x, \rho) + hS(x, \rho), \]
we have
\[ \varphi(x, \rho) = b_1 \cos(\rho x - Q(x)) + O(\rho^{-1} \exp(|\tau|x)), \]
\[ \text{as } x \to \infty, \]
and
\[ \varphi'(x, \rho) = -b_1 \rho \sin(\rho x - Q(x)) + O(\rho^{-1} \exp(|\tau|x)), \]
\[ \text{as } x > a. \]

Using (2) and (9), we arrive at the characteristic function \( \Delta(\rho) \).

Now suppose that the function \( \Delta(\rho) \) has simple zeros of the following form (see Conway (1995))
\[ \rho_k = \rho_k^0 + O(\rho^{-1}), \quad k \to \infty, \]
where the zeros of the function
\[ \Delta^0(\rho) = -b_1 \sin(\rho - Q(1)) + O(\rho^{-1} \exp(|\tau|x)). \]

Corollary 2.4. It follows from (7), (8) and the functions \( \varphi^{(m)}(x, \rho) \) in proof of Theorem 2.3 that
\[ |\varphi^{(m)}(x, \rho)| \leq C|m|^{m} \exp(|\text{Im}|x), \quad 0 \leq x < 1. \]

Denote
\[ \phi(x, \rho) = \frac{-\psi(x, \rho)}{\Delta(x, \rho)} \]

Let \( \phi(x, \rho) \) be the solution of Eq. (1) under the boundary conditions \( U(\phi) = 1, V(\phi) = 0 \). We set \( M(\rho) = \phi(0, \rho) \). The functions \( \phi(x, \rho) \) and \( M(\rho) \) are called the Weyl solution and Weyl function for the boundary value problem \( L \), respectively. Clearly, using the conditions at the point \( x = 0 \), we get
\[ \phi(x, \rho) = S(x, \rho) + M(\rho) \varphi(x, \rho), \]
and
\[ \langle \phi(x, \rho), \varphi(x, \rho) \rangle = 1. \]

Lemma 2.5. For \( |\rho| \to \infty \),
\[ M(\rho) = \frac{1}{\rho_1 \rho_2} \cos \left( \rho(1 - a) - (Q(1) - Q(a)) \right) + O(\rho^{-1} \exp(1 - a)). \]

Proof: Using the FSS \( \{C_0(0, \rho), S_0(0, \rho)\} \), we have for \( x \in (a, 1] \),
\[ \psi(x, \rho) = L_1(\rho)C_0(x, \rho) + L_2(\rho)S_0(x, \rho). \]

Taking the Cramer's rule and the initial conditions \( \psi(x, \rho) \) in \( x = 1 \), we obtain
\[ L_1(\rho) = \cos(\rho - Q(1)) + O(\rho^{-1} \exp(|\tau|)), \]
\[ L_2(\rho) = \rho \sin(\rho - Q(1)) - H \cos(\rho - Q(1)) + O(\rho \exp(|\tau|)). \]

Substituting these coefficients in (16), we have
\[ \psi(x, \rho) = \cos \left( \rho(1 - x) - (Q(1) - Q(x)) \right) + O(\rho^{-1} \exp(|\tau|(1 - x))), \quad a < x \leq 1. \]

Analogously taking the FSS \( \{C_0(x, \rho), S_0(x, \rho)\} \) and the jump condition (3), we have
\[ \psi(x, \rho) = \frac{a_1^{1 - a_2^2}}{4} \cos(\rho(2a - x) - 2Q(a) + Q(1)) \]
\[ + \cos \left( \rho(1 - x) - (Q(1) - Q(x)) \right) \]
\[ + \frac{a_1^{1 - a_2^2}}{4} \cos(\rho(2a - x) - 2Q(a) + Q(1)) \]
\[ - 2Q(1) - Q(a) - Q(x)) \]
\[ + \cos(\rho(a - x) - Q(a) + Q(x)) \]
\[ + O(\rho^{-1} \exp(|\tau|(2 - a + x))), \quad 0 \leq x \leq a. \]

Using (12) at the point \( x = 0 \), we arrive at (15).

Taking the functions \( \psi(x, \rho) \) in proof of Lemma 2.5, we have
\[ \psi'(x, \rho) = \rho \sin \left( \rho(1 - x) - (Q(1) - Q(x)) \right) + O(\rho \exp(|\tau|(1 - x))), \quad a < x \leq 1, \]
\[ \psi'(x, \rho) = \frac{a_1^{1 - a_2^2}}{4} \sin(\rho(2a - x) - 2Q(a) + Q(1)) \]
\[ - 2Q(1) - Q(a) - Q(x)) \]
\[ + \sin(\rho(1 - x) - (Q(1) - Q(x)) \]
\[ + \frac{a_1^{1 - a_2^2}}{4} \sin(\rho(2a - x) - 2Q(a) + Q(1)) \]
\[ - 2Q(1) - Q(a) - Q(x)) \]
\[ + \sin(\rho(a - x) - Q(a) + Q(x)) \]
\[ + O(\rho \exp(|\tau|(2 - a + x))), \quad 0 \leq x \leq a. \]

Inverse Problem 2.6. Suppose that \( a \) and \( a_j, j = 1,2 \), are known a priori. Our goal is to find
$q_0(x), q_2(x), h$ and $H$ from the given Weyl function $M(\rho)$.

3. Uniqueness theorem

Now we prove the uniqueness theorem for the solution of the inverse problem. We consider together with $L = L(q_0(x), q_1(x), h, H)$ a boundary value problem $\tilde{L} = L(\tilde{q}_0(x), \tilde{q}_1(x), \tilde{h}, \tilde{H})$ of the same form (1)-(2) but with different coefficients. If a certain symbol denotes an object related to $L$, then the same symbol will denote the analogous object related to $\tilde{L}$.

**Theorem 3.1.** If $M(\rho) = \tilde{M}(\rho)$ then $L = \tilde{L}$. Thus, the specification of the Weyl function uniquely determines the BVP($L$).

**Proof:** Let us define the matrix

$$P(x, \rho) = [p_{jk}(x, \rho)]_{j,k=1,2},$$

by the formula

$$P(x, \rho) \left[ \Phi(x, \rho) \Phi^*(x, \rho) \right] = \left[ \phi(x, \rho) \phi(x, \rho) \right].$$

By virtue of (14), this yields

$$(p_{11}(x, \rho) = \phi^{(j-1)}(x, \rho) \Phi^*(x, \rho) - \Phi^{(j-1)}(x, \rho) \phi^*(x, \rho),$$

$$(p_{12}(x, \rho) = \phi^{(j-1)}(x, \rho) \Phi^*(x, \rho) - \Phi^{(j-1)}(x, \rho) \phi^*(x, \rho).$$

Also we have

$$\left( \Phi(x, \rho) = p_{11}(x, \rho) \Phi^*(x, \rho) + p_{12}(x, \rho) \phi^*(x, \rho),$$

$$\phi(x, \rho) = p_{11}(x, \rho) \Phi^*(x, \rho) + p_{12}(x, \rho) \phi^*(x, \rho).$$

Using (13) and (22), we obtain

$$p_{11}(x, \rho) = \phi^{(j-1)}(x, \rho) S(x, \rho) - S^{(j-1)}(x, \rho) \phi^*(x, \rho)$$

$$+ \tilde{M}(x, \rho) \phi^{(j-1)}(x, \rho) \phi^*(x, \rho),$$

$$p_{12}(x, \rho) = S^{(j-1)}(x, \rho) \phi^*(x, \rho) - \phi^{(j-1)}(x, \rho) S(x, \rho)$$

$$- \tilde{M}(x, \rho) \phi^{(j-1)}(x, \rho) \phi^*(x, \rho),$$

where $\tilde{M}(\rho) = \tilde{M}(\rho) - M(\rho)$. Since $\tilde{M}(\rho) = M(\rho)$, deducting that $\tilde{M}(\rho) = 0$, and consequently, for each fixed $x$ in $[0, a]$, the functions $p_{jk}(x, \rho), k = 1, 2$, are entire in $\rho$.

Fix $\varepsilon > 0$. Denote $G_\varepsilon = \{\rho \in \mathbb{C}; |\rho - \rho_k| \geq \varepsilon\}$. It follows from (12), (17), (18), (19) and the function $\Delta(\rho)$ in Theorem 2.3 that

$$|\Phi^{(n)}(x, \rho)| \leq C|\rho|^{n-1} \exp(|\text{Im}\rho|), x \in [0, a].$$

$$|\Phi^{(n)}(x, \rho)| \leq C|\rho|^{n-1} \exp(|\text{Im}\rho|), x \in (a, 1].$$

$$|\Delta(\rho)| \geq C|\rho| \exp(|\text{Im}\rho|), \rho \in G_\varepsilon.$$
4. Conclusions

Through review of the papers, it is revealed that there is not an inverse problem for differential pencils with a discontinuity. This is a lack for such problems and it is studied. In particular, the some methods in the inverse problem theory for Sturm–Liouville operators like the transformation operator do not give reliable results for differential pencils with discontinuity. The method of the spectral mappings is appropriate for the Weyl function to play an important role in. Thus we investigated this problem and obtained some new results.

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References


