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STEINER FORMULA AND HOLDITCH-TYPE THEOREMS FOR HOMOTHETIC LORENTZIAN MOTIONS*

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Abstract – The present paper is concerned with the generalization of the Holditch Theorem under one-parameter homothetic motion on Lorentzian planes. In this paper, for the homothetic Lorentzian motion, we expressed the Steiner formula. Furthermore, we present the Holditch-Type Theorems.

Keywords – Holditch Theorem, Steiner formula, lorentzian plane, homothetic motion

1. INTRODUCTION

Let L and L' be moving and fixed Lorentzian planes and $\{O;\boldsymbol{l_1},\boldsymbol{l_2}\}$ and $\{O';\boldsymbol{l'_1},\boldsymbol{l'_2}\}$ be their coordinate systems, respectively. By taking

$$\boldsymbol{OO'} = \boldsymbol{u} = u_1\boldsymbol{l_1} + u_2\boldsymbol{l_2}\text{, for } u_1,u_2 \in \mathbb{R} \tag{1}$$

the motion defined by the transformation

$$\boldsymbol{x'} = h\,\boldsymbol{x} - \boldsymbol{u} \tag{2}$$

is called one-parameter planar homothetic motion on Lorentzian plane and denoted by $H_1 = L/L'$, where h is a homothetic scale and $\boldsymbol{x},\boldsymbol{x'}$ are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X=(x_1,x_2)\in L$, respectively. Furthermore, at the initial time $t=0$ the coordinate systems coincide. Taking $\varphi=\varphi(t)$ as the rotation angle between $\boldsymbol{l_1}$ and $\boldsymbol{l'_1}$, the equations

$$\begin{aligned}\boldsymbol{l_1} &= ch\varphi\,\boldsymbol{l'_1} + sh\varphi\,\boldsymbol{l'_2}\\ \boldsymbol{l_2} &= sh\varphi\,\boldsymbol{l'_1} + ch\varphi\,\boldsymbol{l'_2}\end{aligned} \tag{3}$$

can be written, [1]. Also homothetic scale h, the rotation angle φ and the vectors $\boldsymbol{x},\boldsymbol{x'}$ and $\boldsymbol{u}$ are continuously differentiable functions of a time parameter t. In this study we assume that

$$\dot{\varphi}(t) = d\varphi/dt \neq 0,\ h(t) \neq const.$$

Differentiating the equations in (3) and (1) with respect to t, we have

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$$\begin{aligned} \dot{\boldsymbol{l}}_1 &= \dot{\varphi}\,\boldsymbol{l}_2 \\ \dot{\boldsymbol{l}}_2 &= \dot{\varphi}\,\boldsymbol{l}_1 \end{aligned} \tag{4}$$

and

$$\dot{\boldsymbol{u}} = (\dot{u}_1 + u_2\dot{\varphi})\boldsymbol{l}_1 + (\dot{u}_2 + u_1\dot{\varphi})\boldsymbol{l}_2, \text{ [2]}. \tag{5}$$

Moreover, if we differentiate the equation in (2) with respect to t, the absolute velocity of the point $X \in L$ is found as

$$\begin{aligned} \boldsymbol{V}_a &= \{-\dot{u}_1 - (u_2 - hx_2)\dot{\varphi} + \dot{h}x_1\}\boldsymbol{l}_1 + \{-\dot{u}_2 - (u_1 - hx_1)\dot{\varphi} + \dot{h}x_2\}\boldsymbol{l}_2 \\ &\quad + h(\dot{x}_1\boldsymbol{l}_1 + \dot{x}_2\boldsymbol{l}_2) \end{aligned} \tag{6}$$

From the equation in (6), we get the sliding velocity

$$\boldsymbol{V}_f = \{-\dot{u}_1 - (u_2 - hx_2)\dot{\varphi} + \dot{h}x_1\}\boldsymbol{l}_1 + \{-\dot{u}_2 - (u_1 - hx_1)\dot{\varphi} + \dot{h}x_2\}\boldsymbol{l}_2. \tag{7}$$

If $\boldsymbol{V}_f = 0$, then the rotation pole or the instantaneous rotation pole center $P = (p_1, p_2)$ is obtained as

$$\begin{aligned} p_1 &= \frac{\dot{h}(\dot{u}_1 + u_2\dot{\varphi}) - h\dot{\varphi}(\dot{u}_2 + u_1\dot{\varphi})}{\dot{h}^2 - (h\dot{\varphi})^2} \\ p_2 &= \frac{\dot{h}(\dot{u}_2 + u_1\dot{\varphi}) - h\dot{\varphi}(\dot{u}_1 + u_2\dot{\varphi})}{\dot{h}^2 - (h\dot{\varphi})^2}. \end{aligned} \tag{8}$$

Using the equations in (7) and (8), we get

$$\boldsymbol{V}_f = \{(x_1 - p_1)\dot{h} + h\dot{\varphi}(x_2 - p_2)\}\boldsymbol{l}_1 + \{(x_2 - p_2)\dot{h} + h\dot{\varphi}(x_1 - p_1)\}\boldsymbol{l}_2, \text{ [3]}. \tag{9}$$

2. THE ORBIT AREA FORMULA FOR THE PLANAR HOMOTHETIC LORENTZIAN MOTION

Let $X = (x_1, x_2)$ be a fixed point in the moving plane L and $P = (p_1, p_2)$ be the pole point of the motion at the time t. Then the sliding velocity of a fixed point $X \in L$ with respect to L' is

$$d\boldsymbol{x}' = \{(x_1 - p_1)dh + hd\varphi(x_2 - p_2)\}\boldsymbol{l}_1 + \{(x_2 - p_2)dh + hd\varphi(x_1 - p_1)\}\boldsymbol{l}_2. \tag{10}$$

We will study the surface area swept out by the segment $\mathbf{PX}$ now, which occurs by a fixed point $X = (x_1, x_2) \in L$ and the pole point P, under the motion H_1.

If H_1 is restricted to time interval $[t_1, t_2]$, the line segment $\boldsymbol{PX}$ then sweeps the surface with the orbit area

$$F_X{}^P = 1/2\int_{t_1}^{t_2}(x_1' dx_2' - x_2' dx'). \tag{11}$$

Setting the equations (2), (8) and (10) in equation (11), we have

$$2F_X{}^P = (x_1{}^2 - x_2{}^2)\int_{t_1}^{t_2} h^2 d\varphi - 2x_1\int_{t_1}^{t_2} h^2 p_1 d\varphi + 2x_2\int_{t_1}^{t_2} h^2 p_2\, d\varphi + x_1\int_{t_1}^{t_2}\{-2hp_2 dh + hdu_2 + u_2 dh\}$$

$$+x_2\int_{t_1}^{t_2}\{2hp_1dh - hdu_1 - u_1dh\} + \int_{t_1}^{t_2}\{u_1p_2dh + hu_1p_1d\varphi - u_2p_1dh - hu_2p_2d\varphi\}. \tag{12}$$

If $X = 0$ $(x_1 = x_2 = 0)$ is taken, then equation (11) for the orbit area of the initial point leads to

$$2F_O{}^P = \int_{t_1}^{t_2}\{u_1p_2dh + hu_1p_1d\varphi - u_2p_1dh - hu_2p_2d\varphi\}. \tag{13}$$

Since $\dot{\varphi}(t) \neq 0$ and $\dot{\varphi}(t)$ is a continuous function, we can say that $\dot{\varphi}(t) < 0$ or $\dot{\varphi}(t) > 0$, that is, $\dot{\varphi}(t)$ has the same sign everywhere in the interval $[t_1, t_2]$. Hence, using the mean value theorem of integral calculus for the interval $[t_1, t_2]$, there exists at least one point $t_0 \in [t_1, t_2]$ such that the following equation holds:

$$\int_{t_1}^{t_2} h^2 d\varphi = \int_{t_1}^{t_2} h^2(t)\dot{\varphi}(t)dt = h_0^2\delta, \tag{14}$$

where $\delta = \varphi(t_2) - \varphi(t_1)$ is the total rotation angle (Gesamtdrehwinkel) [4], and $h_0 := h(t_0)$. Also, the Steiner point $S = (s_1, s_2)$ for the homothetic motion H_1 can be written

$$s_j = \frac{\int_{t_1}^{t_2} h^2 p_j d\varphi}{\int_{t_1}^{t_2} h^2 d\varphi)}, \; j = 1,2, \text{[3]}. \tag{15}$$

From the equations in (14) and (15),

$$\int_{t_1}^{t_2} h^2 p_j d\varphi = h_0^2\delta\; s_j \tag{16}$$

is found. If the equations (13), (14) and (16) are replaced in equation (12), then we get

$$F_X{}^P = F_0{}^P + h_0^2\,\delta/2(x_1{}^2 - x_2{}^2 - 2s_1x_1 + 2s_2x_2) + \mu_1x_1 + \mu_2x_2, \tag{17}$$

where

$$\mu_1 = \frac{1}{2}\int_{t_1}^{t_2}\{-2hp_2dh + hdu_2 + u_2dh\},\; \mu_2 = \frac{1}{2}\int_{t_1}^{t_2}\{2hp_1dh - hdu_1 - u_1dh\}. \tag{18}$$

The equation in (17) is called the Steiner formula for the motion H_1.
Thus, using the equation in (17) we can give the following theorem.

Theorem 1. During homothetic motion H_1, all the fixed points $X = (x_1, x_2) \in L$, which pass around equal surface areas $F_X{}^P$, lie on the same Lorentzian circle with the center

$$C = \left(s_1 - \frac{\mu_1}{h^2(t_0)\delta},\; s_2 - \frac{\mu_2}{h^2(t_0)\delta}\right)$$

in the moving plane L.

Special Case 1. In the case of the homothetic scale h identically equal to 1, we get

$$F_X{}^P = F_0{}^P + \delta/2(x_1{}^2 - x_2{}^2 - 2s_1x_1 + 2s_2x_2)$$

which was given by Hacısalihoğlu, [5].

3. HOLDITCH-TYPE THEOREMS FOR THE PLANAR LORENTZIAN MOTION

I.

Let unlimited, convex curve k_o be the common orbit curve of the points A and B of moving plane L, during the motion H_1. Under H_1, points A and B tend toward infinity for $t \to \mp\infty$, where t is the time parameter.

There could be a pair of different, parallel tangents t_1, t_2 of the edge k_o of an unlimited convex region $K_o \subset L'$. Furthermore, if contact points R_i of t_i on k_o, exists half lines $h_i \subset t_i$ of the edge k_o exist. The distance Δ between t_1 and t_2 is defined as *"wide"* of K_o. If there are not parallel tangent pairs, then we assume that $\Delta = +\infty$. Under H_1, let the endpoints of s pass through whole curve k_o. This is always possible for $\overline{AB} < \Delta$. If $\overline{AB} = \Delta < \infty$, then the desired motion is possible when the contact points R_i of parallel tangents t_i exist. The motion is impossible for $\overline{AB} > \Delta$.

During H_1, the points A and B can turn back in some cases (see Fig. 1). The dead centre of an endpoint of s is an instantaneous rotation pole center at the same time. Because of our conditions, reverse motion does not happen after a definite time and the endpoint A, B of s tends toward infinity with the same orientation on k_o.

When the sign of angular velocity ω of s does not change, the straight line s tends to infinity under H_1. During motion, there exist chords s that are parallel to every tangent of k_o. Therefore, the total rotation angle $\delta \in IR^+$ of H_1 coincides with the tangent rotation angle of k_o.

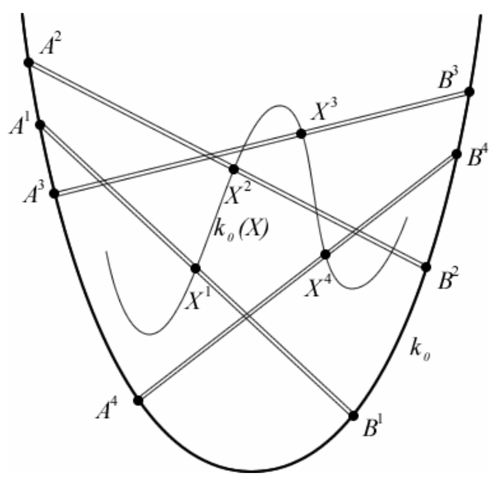

Fig. 1. The motions of a line segment AB

Theorem 2. Let k_o be an edge curve of unlimited convex region $K_o \subset IR^2$, and $\delta \in IR^+$ be its tangent rotation angle. When the endpoints A and B of the straight line s with length $a+b$ on k_o move to infinite, once in positive and then in negative, with circulation from a fixed point, the point $X \in s$ ($a = \overline{AX}$, $b = \overline{XB}$) describes a curve $k_o(X)$. Then the surface area F_s of the Holditch-Sickle $S_o \subset K_o$ bounded by k_o and $k_o(X)$ is

$$F_S = abh_0^2\delta/2.$$

Proof: Let the points $A=(0,0), B=(a+b,0)$, and $X=(a,0)$ have the position A^t, B^t, X^t in fixed system L' for $t>0$, and analog the positions A^{-t}, B^{-t}, X^{-t} for $-t$. These positions for sufficient large t do not lie on the same support line of k_o, and so can coincide with a rotation round a certain centre $D \in L'$ (see Fig. 2).

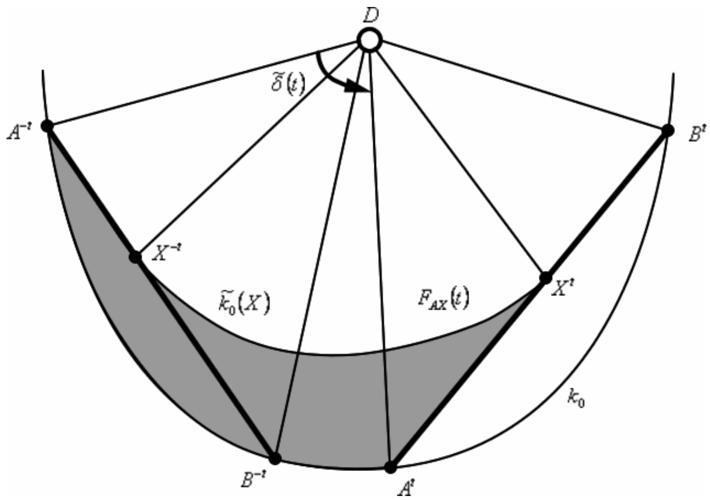

Fig. 2. The orbit curves of three collinear points

If the motion H_1 is restricted to time interval $[-t,t]$, then an open motion $\widetilde{H}_1(t)$ with total rotation angle $\widetilde{\delta}(t)$ is obtained. Under $\widetilde{H}_1(t)$, from the equation in (17), the sector of a circle (on L') determined by the center $D \in L'$ and the orbit curve piece $\widetilde{k}_o(Y)$ of the point $Y=(y_1,y_2) \in L$ has the surface area

$$F_Y^D = F_A^D + h_0^2 \frac{\widetilde{\delta}(t)}{2}(y_1^2 - y_2^2 - \lambda y_1 + \mu y_2), \tag{19}$$

where λ and μ are the motion constants.
The orbit curve pieces $\widetilde{k}_o(A)$ and $\widetilde{k}_o(X)$ of the points A and X determine a curve with the line segments $A^t X^t$ and $A^{-t} X^{-t}$. This curve has the orientated surface area

$$F_{AX}(t) = F_A^D - F_X^D \tag{20}$$

in order $A^{-t} A^t X^t X^{-t} A^{-t}$.
Similarly, we can define the area

$$F_{AB}(t) = F_A^D - F_B^D . \tag{21}$$

From the equations in (12), (13) and (14), we get

$$F_{AX}(t) = \frac{abh_0^2 \widetilde{\delta}(t)}{2} + \frac{a}{a+b} F_{AB}(t) . \tag{22}$$

For $t \to +\infty$, we have

$$\lim_{t\to\infty} F_{AX}(t) = F_s , \lim_{t\to\infty} F_{AB}(t) = 0 , \lim_{t\to\infty} \widetilde{\delta}(t) = \delta . \tag{23}$$

Then, from the equations in (22) and (23), we obtain

$$F_S = abh_0^2 \delta / 2 . \tag{24}$$

Now, we can give the following theorems as a generalization of the Holditch-type theorem.

II.

Theorem 3. Under H_1, let F_A and F_B denote the orbit areas of the orbit curves $k_A, k_B \subset L'$ of the points $A = (0,0)$, and $B = (a+b,0) \in L$ respectively. If F_X is the orbit area of the orbit curve k of the point $X = (a,0)$ which is collinear with points A and B, then

$$F_X = [aF_B + bF_A]/(a+b) - h_0^2 ab\delta/2 . \tag{25}$$

Moreover, if we choose a reference point Q' instead of the pole point P on the fixed plane L', the Holditch-Type theorem is also valid for this case.

Corollary 1. Under the homothetic motion H_1, if the line segment $\boldsymbol{AB}$, with a constant length $a+b$ moves such as its end points, A and B are mobile on the same curve $k_A = k_B$, hence from the equation in (25), this leads to

$$F_A - F_X = ab\, h_0^2 \delta/2 , \tag{26}$$

that is, in the different orbit area of the curves $k_A = k_B$, k is independent of the choice of the curves and is only dependent on the choice of point X and homothetic scale h.

Theorem 4. (General Form of Holditch Theorem [6]) During one-parameter planar Lorentzian motion L/L', let F_A, F_B and F_C be the orbit areas of the points $A = (0,0)$, $B = (b,0)$, and $C = (c,d) \in L$, respectively. Then for the orbit area of any point $X = (x,y) \in L$, we have

$$F_X = \left(1 - \frac{x}{b} + \frac{c-b}{bd}y\right)F_A + \left(\frac{x}{b} - \frac{cy}{bd}\right)F_B + \frac{y}{d}F_C +$$
$$+\left(x^2 - y^2 - bx - \frac{c^2+d^2}{d}y + \frac{bc}{d}y\right)h_0^2\delta/2 .$$

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