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SEQUENTIAL ESTIMATION IN A SUBCLASS OF EXPONENTIAL FAMILY UNDER WEIGHTED SQUARED ERROR LOSS*

N. NEMATOLLAHI**, M. JAFARI JOZANI AND N. MAHLOOJI

Department of Statistics, Faculty of Economics, Allameh Tabataba'i University, Tehran, I. R. of Iran
Email: nematollahi@atu.ac.ir

Abstract – In a subclass of the scale-parameter exponential family, we consider the sequential point estimation of a function of the scale parameter under the loss function given as the sum of the weighted squared error loss and a linear cost. For a fully sequential sampling scheme, second order expansions are obtained for the expected sample size as well as for the regret of the procedure. The former researches on Gamma and Exponential distributions can be deduced from our general results.

Keywords – Sequential estimation, stopping rule, regret analysis, exponential family, transformed chi-square distribution

1. INTRODUCTION

The problem of sequential estimation refers to any estimation technique for which the total number of observations used is not a degenerate random variable. In some problems, sequential estimation must be used because no procedure that uses a preassigned nonrandom sample size can achieve the desired objective (for example, the estimation of the parameter 1/p in a sequence of Bernoulli trials). In other problems a procedure which uses a preassigned nonrandom sample size may exist, but a sequential estimation procedure may be better in some ways.

The problems of sequential analysis were first studied in the 1940s by Barnard [1] and Wald [2], who introduced the Sequential Probability Ratio Test (SPRT) independently, Wald and Wolfowitz [3] proved its optimality, and Haldane [4] and Stein [5] showed how sequential methods can be used to tackle some unsolved problems in point and interval estimation. There is a large body of literature on this subject, and it is growing rapidly. For a summary of results, as well as a list of references, see Lai [6].

Sequential estimation of the scale parameter of Exponential and Gamma distributions have been considered by Starr and Woodroofe [7], Woodroofe [8], Gosh and Mukhopadhyay [9], Isogai and Uno [10], Isogai et al. [11] and Uno et al. [12]. Under squared error loss, Starr and Woodroofe [7] considered the risk efficient estimation of the scale parameter in exponential distribution and studied some of the first order properties of the sequential procedure. Also, for the sequential estimation of a function of the exponential parameter Uno et al. [12] gave the stopping rule and a sufficient condition to get a second order approximation to the risk of the sequential procedure.

For the estimation of the scale parameter, the scale invariant squared error loss is more appropriate than the squared error loss. In addition, there are several cases for which the estimation of a function of the scale parameter is desired. So, it is natural to use $(\frac{\delta}{\gamma(\theta)}-1)^2$ as an appropriate relative squared error loss

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**Corresponding author

function, which is a special form of weighted squared error loss function. Hence, in this paper we consider the loss as the sum of the weighted squared error loss and a linear cost, which includes scale invariant squared error loss. We obtain a sequential point estimation of a function of the scale parameter in a subclass of the scale parameter exponential families of distributions which include Exponential, Weibull, Gamma, Normal, Inverse Guassian and some other distributions, and determine a stopping rule under the loss function. Also, a second order approximation to the expected sample size and the risk of the sequential procedure as the cost per observation tends to zero are given. We show that the results obtained by Uno et al. [12] in Exponential distribution and Woodroofe [8] in Gamma distribution under squared error loss are special cases of our results.

In Section 2 the subclass of distributions is introduced and the sequential estimation problem is specified. In section 3, we derive asymptotic expansions of the expected sample size and regret associated with the proposed procedure. Some special cases of our results are given in section 4.

2. SEQUENTIAL ESTIMATION IN A SUBCLASS OF EXPONENTIAL FAMILY

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables from a distribution with density $\frac{1}{\tau} g(\frac{x}{\tau})$, where g is known and τ is an unknown scale parameter. In some cases the above density reduces to

$$f(x,\theta) = c(x)\theta^{-\nu} e^{-\frac{T(x)}{\theta}}, \ \theta > 0 \qquad (1)$$

where c(*x*) is a function of *x*, $\theta = \tau^r$, ν is a known value and T(X) is a complete sufficient statistic for θ with $Gamma(\nu, \theta)$ - distribution. Examples of distributions of the form (1) are

1. *Exponential*(β) with $\theta = \beta, \nu = 1, T(X) = X$, $c(x) = 1$,
2. *Gamma*(α, β) with known α and $\theta = \beta, \nu = \alpha$, $T(X) = X$, $c(x) = \frac{x^{a-1}}{\Gamma(a)}$,
3. *Inverse Gaussian* (∞, λ) with $\theta = \frac{1}{\lambda}$, $\nu = \frac{1}{2}$, $T(X) = \frac{1}{2X}$, $c(x) = (2\pi x^3)^{-\frac{1}{2}}$,
4. *Normal* $(0, \sigma^2)$ with $\theta = \sigma^2$, $\nu = \frac{1}{2}$, $T(X) = \frac{1}{2} X^2$, $c(x) = (2\pi)^{-\frac{1}{2}}$,
5. *Weibull*(α, β) with known β and $\theta = \alpha^{\beta}$, $\nu = 1$, $T(X) = X^{\beta}$, $c(x) = \beta x^{\beta-1}$,
6. *Rayleigh* (β) with $\theta = \beta^2$, $\nu = 1$, $T(X) = \frac{1}{2} X^2$, $c(x) = x$,
7. *Generalized Gamma* (λ, p, α) with known p and α, $\theta = \frac{1}{\lambda}$, $\nu = \frac{p}{\alpha}$, $T(X) = X^{\alpha}$, $c(x) = \frac{|\alpha|}{\Gamma(p/\alpha)} x^{p-1}$,
8. Generalized Laplace (λ, k) with known k and $\theta = \lambda^k$, $\nu = \frac{1}{k}$, $T(X) = |X|^a$, $c(x) = \frac{k}{2\Gamma(1/k)}$.

Now if $X_1, X_2, \ldots, X_n$ be a random sample of size *n* from distribution (1), then the joint density of $X_1, X_2, \ldots, X_n$ is given by

$$f(\underset{\sim}{x},\theta) = c(\underset{\sim}{x},n)\theta^{-n\nu} e^{-\sum_{i=1}^{n} T(x_i)/\theta}, \ \theta > 0 \ , \qquad (2)$$

where $c(\underset{\sim}{x}, n) = \prod_{i=1}^{n} c(x_i)$ and $S(\underset{\sim}{X}) = \sum_{i=1}^{n} T(X_i) \sim Gamma(n\nu, \theta)$. Consider the estimation of a function of the scale parameter θ, say $\gamma(\theta)$, where γ is a positive valued and three times continuously differentiable function of θ. Given a sample $X_1, X_2, \ldots, X_n$ of size *n*, we want to estimate $\gamma = \gamma(\theta)$ by $\hat{\gamma}_n = \gamma(\frac{S(\underset{\sim}{X})}{n\nu})$ under the loss function

$$L(\hat{\gamma}_n, \gamma(\theta)) = w(\theta)\,(\hat{\gamma}_n - \gamma(\theta))^2 + cn, \tag{3}$$

where $c > 0$ is the known cost per unit sample and $w(\theta)$ is a positive valued and two times continuously differentiable weight function. Special cases of the loss (3) are scaled invariant squared error loss

$$L(\hat{\gamma}_n, \gamma(\theta)) = \left(\frac{\hat{\gamma}_n}{\gamma(\theta)} - 1\right)^2 + cn\ , \tag{4}$$

and squared error loss

$$L(\hat{\gamma}_n, \gamma(\theta)) = (\hat{\gamma}_n - \gamma(\theta))^2 + cn\ , \tag{5}$$

choosing $w(\theta) = \frac{1}{\gamma^2(\theta)}$ and $w(\theta) = 1$, respectively. Note that $\frac{S(\underset{\sim}{X})}{n\nu}$ is the usual estimator of θ (i.e. MLE, UMVUE), hence using invariance property of maximum likelihood estimators, it is reasonable to estimate $\gamma(\theta)$ by $\hat{\gamma}_n = \gamma(\frac{S(\underset{\sim}{X})}{n\nu})$. The risk function is given by

$$R_n = R(\hat{\gamma}_n, \gamma(\theta)) = E[L(\hat{\gamma}_n, \gamma(\theta))] = w(\theta)\,E[(\hat{\gamma}_n - \gamma(\theta))^2] + cn.$$

We want to find an appropriate sample size that will minimize the risk R_n. Using Taylor expansion of $\gamma(\frac{S(\underset{\sim}{X})}{n\nu})$ about θ we obtain

$$E[(\hat{\gamma}_n - \gamma(\theta))^2] = [\gamma'(\theta)]^2\, Var\,(\frac{S(\underset{\sim}{X})}{n\nu}) + o(\frac{1}{n})$$

as $n \to \infty$, and hence $R_n = w(\theta)[\gamma'(\theta)]^2 \frac{\theta^2}{n\nu} + cn + o(\frac{1}{n})$, as $n \to \infty$. So for sufficiently large n,

$$R_n \approx w(\theta)[\gamma'(\theta)]^2 \frac{\theta^2}{n\nu} + cn, \tag{6}$$

which leads to the following lemma.

Lemma 2-1. The risk function R_n in (6) minimized at

$$n_0 \approx \theta\sqrt{\frac{w(\theta)}{c\nu}}\ |\gamma'(\theta)| = n^*, \tag{7}$$

and for this value

$$R_{n_0} \approx 2cn^*. \tag{8}$$

Since θ is unknown, we can not use the best fixed sample size procedure n_0. Further, there is no fixed sample size procedure that will attain the minimum risk R_{n_0}. Thus it is necessary to find a sequential sampling rule.

We use the following stopping rule

$$N = N_c = \inf\left\{ n \geq m \,\middle|\, n \geq (c\nu)^{-\frac{1}{2}} \left(\frac{S(\underset{\sim}{X})}{n\nu}\right) \sqrt{w\left(\frac{S(\underset{\sim}{X})}{n\nu}\right)} \left| \gamma'\left(\frac{S(\underset{\sim}{X})}{n\nu}\right) \right| \right\}, \tag{9}$$

where m is the pilot sample size. Now if we estimate $\gamma = \gamma(\theta)$ by $\hat{\gamma}_N = \gamma\left(\frac{S(\underset{\sim}{X})}{N\nu}\right)$, then the risk is given by

$$R_N = w(\theta)\, E[(\hat{\gamma}_N - \gamma(\theta))^2] + cE(N). \tag{10}$$

In the next section we derive second order approximation to the expected sample size *E(N)* and the risk of the above sequential procedure R_N as $c \to 0$.

3. SECOND ORDER APPROXIMATION

In this section we shall investigate second order asymptotic properties of the sequential procedure. Let

$$K(t) = \frac{1}{t \mid \gamma'(t) \mid \sqrt{w(t)}} = \frac{1}{t\sqrt{\{\gamma'(t)\}^2 w(t)}}\ ,\ t > 0\ ,$$

and $Z_n = nK\left(\frac{S(\underset{\sim}{X})}{n\nu}\right) / K(\theta).$ Then from (9), the stopping rule N becomes

$$N = N_c = \inf\left\{ n \geq m : Z_n > n^* \right\}. \tag{11}$$

Using Taylor expansion of $\gamma\left(\frac{S(\underset{\sim}{X})}{n\nu}\right)$ about θ and the relations

$$t\frac{K'(t)}{K(t)} = -\left\{ 1 + t\ \frac{\gamma''(t)}{\gamma'(t)} + \frac{tw'(t)}{2w(t)} \right\}$$

and

$$K''(t) = \frac{2K(t)}{t^2}\left\{ 1 + \frac{t\gamma''(t)}{\gamma'(t)} + \frac{t^2[\gamma''(t)]^2}{[\gamma'(t)]^2} - \frac{t^2\gamma^{(3)}(t)}{2\gamma'(t)} \right.$$

$$\left. + \ \frac{t^2 w'(t)}{2w(t)}\left[\frac{1}{t} + \frac{w'(t)}{4w(t)} + \frac{\gamma''(t)}{\gamma'(t)}\right] - \frac{t^2}{4}\left[\frac{w'(t)}{w(t)}\right]' \right\}, \tag{12}$$

we obtain the following lemma.

Lemma 3-1. Let $Y_i = \frac{T(X_i)}{\nu\theta} - 1$ for $i = 1, 2, \ldots$ and $S_n = \sum_{i=1}^{n} Y_i = \frac{S(\underset{\sim}{X})}{\nu\theta} - n$, then

$$Z_n = n + \alpha S_n + \psi_n, \tag{13}$$

where

$$\alpha = -\left(1 + \theta\frac{\gamma''(\theta)}{\gamma'(\theta)} + \frac{t}{2}\frac{w'(\theta)}{w(\theta)} \right) = \frac{\theta K'(\theta)}{K(\theta)}\ ,\ \psi_n = n\left(\frac{S(\underset{\sim}{X})}{n\nu} - \theta \right)^2 \frac{K''(\eta_n)}{2K(\theta)} \tag{14}$$

and η_n is a random variable lying between θ and $\left(\frac{S(\underset{\sim}{X})}{n\nu}\right)$.

Let

$$t = \inf\{n \ge 1 : n + \alpha S_n > 0\} \text{ and } \rho = \frac{E(t + \alpha S_t)^2}{2E(t + \alpha S_t)} \tag{15}$$

Following assumptions of Aras and Woodroofe [13], namely

(A_1) $\left\{[(Z_n - \frac{n}{\varepsilon_\circ})^+]^3,\ n \ge m\right\}$ *is uniformly integrable*

for some $0 < \varepsilon_0 < 1$, *where* $x^+ = \max(x, 0)$,

(A_2) $\sum_{n=m}^{\infty} nP\{\psi_n < -\varepsilon_1 n\} < \infty$ *for some* $0 < \varepsilon_1 < 1$,

we obtain the following approximation to *E(N)* for all $\theta \in (0, \infty)$, but not uniformly in θ.

Theorem 3-1. If (A_1) and (A_2) hold true, then

$$E(N) = n^* + \rho - l + o(1) \quad as \quad c \to 0, \tag{16}$$

where

$$l = \frac{1}{\nu}\left\{1 + \frac{\theta\gamma''(\theta)}{\gamma'(\theta)} + \frac{\theta^2[\gamma''(\theta)]^2}{[\gamma'(\theta)]^2} - \frac{\theta^2\gamma^{(3)}(\theta)}{2\gamma'(\theta)}\right.$$

$$\left. + \frac{\theta^2 w'(\theta)}{2w(\theta)}\left[\frac{1}{\theta} + \frac{w'(\theta)}{4w(\theta)} + \frac{\gamma''(\theta)}{\gamma'(\theta)}\right] - \frac{\theta^2}{4}\left[\frac{w'(\theta)}{w(\theta)}\right]'\right\} = \frac{\theta^2}{\nu}\frac{K''(\theta)}{2K(\theta)}. \tag{17}$$

Proof: Obviously $\frac{S(\underset{\sim}{X})}{n\nu} \xrightarrow{P} \theta$ as $n \to \infty$, and since η_n is a random variable lying between θ and $\frac{S(\underset{\sim}{X})}{n\nu}$, therefore $\eta_n \xrightarrow{P} \theta$ and hence $\frac{K''(\eta_n)}{2K(\theta)} \xrightarrow{p} \frac{K''(\theta)}{2K(\theta)} = \frac{\nu}{\theta^2} l$. Also $\frac{\sqrt{n\nu}}{\theta}\left(\frac{S(\underset{\sim}{X})}{n\nu} - \theta\right) \xrightarrow{d} W \sim N(0,1)$ as $n \to \infty$. So,

$$\psi_n = n\left(\frac{S(\underset{\sim}{X})}{n\nu} - \theta\right)^2 \frac{K''(\eta_n)}{2K(\theta)} \xrightarrow{d} l.W^2 = \psi \quad as \quad n \to \infty.$$

The rest of the proof is the same as the proof of Theorem 1 of Uno et al. [12] with replacing σ, $\theta(\sigma)$, ξ and *h* by θ, $\gamma(\theta)$, ψ and *K* respectively, is omitted.

We shall now assess the regret $R_N - 2cn^*$. By Taylor's theorem,

$$\gamma\left(\frac{S(\underset{\sim}{X})}{n\nu}\right) - \gamma(\theta) = \gamma'(\theta)\left(\frac{S(\underset{\sim}{X})}{n\nu} - \theta\right) + \frac{\gamma''(\theta)}{2}\left(\frac{S(\underset{\sim}{X})}{n\nu} - \theta\right)^2$$

$$+ \frac{\gamma^{(3)}(\varphi_c)}{6}\left(\frac{S(\underset{\sim}{X})}{n\nu} - \theta\right)^3, \tag{18}$$

where φ_c is a random variable lying between θ and $\frac{S(\underset{\sim}{X})}{n\nu}$. Let $\bar{Y}_n = \frac{S_n}{n} = \frac{S(\underset{\sim}{X})}{n\nu\theta} - 1$ and choose $c_0 > 0$ such that $n^* \geq 1$. We impose the following assumption

(A_3): *For some* $a > 1$ *and* $u > 1$,

$$\sup_{0<c\leq c_0} E[|\sqrt{n^*}\,\bar{Y}_n|]^{4au} < \infty \qquad and \qquad \sup_{0<c\leq c_0} E[|\gamma^{(3)}(\varphi_c)|]^{\frac{2au}{u-1}} < \infty .$$

From (A_1), (A_2) and (A_3) we have the following theorem.

Theorem 3-2. If (A_1), (A_2) and (A_3) hold true, then as $c \to 0$,

$$R_N - 2cn^* = \frac{c}{\nu}\left\{3 + 2\theta\frac{\gamma''(\theta)}{\gamma'(\theta)} + \frac{7\theta^2}{4}\frac{[\gamma''(\theta)]^2}{[\gamma'(\theta)]^2} - \frac{\theta^2\gamma^{(3)}(\theta)}{\gamma'(\theta)}\right.$$

$$\left. + \theta^2\frac{w'(\theta)}{w(\theta)}[\frac{3}{\theta} + \frac{5}{4}\frac{w'(\theta)}{w(\theta)} + \frac{2\gamma''(\theta)\}}{\gamma'(\theta)}] - \theta^2[\frac{w'(\theta)}{w(\theta)}]'\right\} + o(c) \tag{19}$$

Proof: From (18) we have

$$R_N - 2cn^* = w(\theta)E\left[\gamma(\frac{S(\underset{\sim}{X})}{N\nu}) - \gamma(\theta)\right]^2 + cE(N) - 2cn^*$$

$$= w(\theta)[\gamma'(\theta)]^2 E\left(\frac{S(\underset{\sim}{X})}{N\nu} - \theta\right)^2 + cE(N) - 2cn^*$$

$$+ w(\theta)\gamma'(\theta)\gamma''(\theta)E\left(\frac{S(\underset{\sim}{X})}{N\nu} - \theta\right)^3$$

$$+ \frac{1}{4}w(\theta)[\gamma''(\theta)]^2 E\left(\frac{S(\underset{\sim}{X})}{N\nu} - \theta\right)^4$$

$$+ \frac{1}{3}w(\theta)\gamma'(\theta)E\left\{\left(\frac{S(\underset{\sim}{X})}{N\nu} - \theta\right)^4 \gamma^{(3)}(\varphi_c)\right\}$$

$$+ \frac{1}{6}w(\theta)\gamma''(\theta)E\left\{\left(\frac{S(\underset{\sim}{X})}{N\nu} - \theta\right)^5 \gamma^{(3)}(\varphi_c)\right\}$$

$$+ \frac{1}{36}w(\theta)E\left\{\left(\frac{S(\underset{\sim}{X})}{N\nu} - \theta\right)^6 \{\gamma^{(3)}(\varphi_c)\}^2\right\}. \tag{20}$$

Following the proof of Theorem 2 of Uno et al. [12], when $c \to 0$ we obtain

$$w(\theta)[\gamma'(\theta)]^2 \ E\left(\frac{S(\underset{\sim}{X})}{N\nu}-\theta\right)^2 + cE(N)-2cn^*$$

$$= cvn^{*^2} \ E(\overline{Y}_N)^2 + cE(N)-2cn^*$$

$$= \frac{c}{v}\{4lv+3\alpha^2+4\alpha\}+o(c)$$

$$= \frac{c}{v}\left\{3+6\theta\frac{\gamma''(\theta)}{\gamma'(\theta)}+7\theta^2\frac{[\gamma''(\theta)]^2}{[\gamma'(\theta)]^2}-2\theta^2\frac{\gamma^{(3)}(\theta)}{\gamma'(\theta)}\right.$$

$$\left.+\theta^2\frac{w'(\theta)}{w(\theta)}[\frac{3}{\theta}+\frac{9}{4}\frac{w'(\theta)}{w(\theta)}+\frac{5\gamma''(\theta)}{\gamma'(\theta)}]-\theta^2[\frac{w''(\theta)}{w(\theta)}]\right\}+o(c), \quad (21)$$

$$w(\theta)\gamma'(\theta)\gamma''(\theta) \ E\left(\frac{S(\underset{\sim}{X})}{N\nu}-\theta\right)^3$$

$$= cv\frac{\theta\gamma''(\theta)}{\gamma'(\theta)}n^{*^2} E\left(\overline{Y}_N^3\right)$$

$$= \frac{c}{v}\frac{\theta\gamma''(\theta)}{\gamma'(\theta)}\{6\alpha+2\}$$

$$= \frac{c}{v}\left\{-4\theta\frac{\gamma''(\theta)}{\gamma'(\theta)}-6\theta^2\frac{[\gamma''(\theta)]^2}{[\gamma'(\theta)]^2}-3\theta^2\frac{w'(\theta)\gamma''(\theta)}{w(\theta)\gamma'(\theta)}\right\}+o(c), \quad (22)$$

$$\frac{1}{4}w(\theta)[\gamma''(\theta)]^2 E\left(\frac{S(\underset{\sim}{X})}{N\nu}-\theta\right)^4 = \frac{\theta^2}{4}\frac{[\gamma''(\theta)]^2}{[\gamma'(\theta)]^2}cvn^{*^2} E(\overline{Y}_N)^4$$

$$= \frac{c}{v}\left\{\frac{3}{4}\theta^2\frac{[\gamma''(\theta)]^2}{[\gamma'(\theta)]^2}\right\}+o(c), \quad (23)$$

$$\frac{1}{36}w(\theta)E\left[(\frac{S(\underset{\sim}{X})}{N\nu}-\theta)^6\{\gamma^{(3)}(\varphi_c)\}^2\right] = \frac{\theta^4}{36[\gamma'(\theta)]^2}E\{(\sqrt{n^*}\,\overline{Y}_N)^4[\gamma^{(3)}(\varphi_c)]^2\overline{Y}_N^{\,2}\}cv = o(c), \quad (24)$$

$$\frac{1}{3}w(\theta)\gamma'(\theta)E\left[(\frac{S(\underset{\sim}{X})}{N\nu}-\theta)^4\gamma^{(3)}(\varphi_c)\right] = \frac{\theta^2}{3\gamma'(\theta)}cvE\{(\sqrt{n^*}\,\overline{Y}_N)^4\gamma^{(3)}(\varphi_c)\}$$

$$= \frac{\theta^2}{3\gamma'(\theta)}\frac{c}{v}\{E(W^4)\gamma^{(3)}(\theta)+o(1)\}$$

$$= \frac{c}{v}\{\frac{\theta^2\gamma^{(3)}(\theta)}{\gamma'(\theta)}\}+o(c), \quad (25)$$

and

$$\frac{1}{6}w(\theta)\gamma''(\theta)E\left[(\frac{S(\underset{\sim}{X})}{N\nu}-\theta)^5\gamma^{(3)}(\varphi_c)\right] = \frac{\theta^3\gamma''(\theta)}{6[\gamma'(\theta)]^2}E\{(\sqrt{n^*}\,\overline{Y}_N)^4\gamma^{(3)}(\varphi_c)\overline{Y}_N\}cv = o(c). \quad (26)$$

Substituting (21) – (26) into (20), we get

$$R_N - 2cn^* = \frac{c}{v}\left\{3 + (6-4)\theta\frac{\gamma''(\theta)}{\gamma'(\theta)} + (7-6+\frac{3}{4})\theta^2\frac{[\gamma''(\theta)]^2}{[\gamma'(\theta)]^2} + (-2+1)\theta^2\frac{\gamma^{(3)}(\theta)}{\gamma'(\theta)}\right.$$

$$\left. + \frac{\theta^2 w'(\theta)}{w(\theta)}[\frac{3}{\theta} + \frac{9}{4}\frac{w'(\theta)}{w(\theta)} + (5-3)\frac{\gamma''(\theta)}{\gamma'(\theta)}] - \theta^2\frac{w''(\theta)}{w(\theta)}\right\} + o(c)$$

$$= \frac{c}{v}\left\{3 + 2\theta\frac{\gamma''(\theta)}{\gamma'(\theta)} + \frac{7\theta^2}{4}\frac{[\gamma''(\theta)]^2}{[\gamma'(\theta)]^2} - \theta^2\frac{\gamma^{(3)}(\theta)}{\gamma'(\theta)}\right.$$

$$\left. + \theta^2\frac{w'(\theta)}{w(\theta)}[\frac{3}{\theta} + \frac{5}{4}\frac{w'(\theta)}{w(\theta)} + \frac{2\gamma''(\theta)}{\gamma'(\theta)}] - \theta^2[\frac{w'(\theta)}{w(\theta)}]'\right\} + o(c), \text{ as } c \to 0,$$

this completes the proof.

4. SPECIAL CASES

In this section, we consider some special cases of the results obtained in section 3. These special cases are

1. For *Exponential* (β)-distribution with $\theta = \beta$, $T(X) = X$ and $v = 1$, Theorem 3-1 and 3-2 with $w(\theta) = 1$, i.e. with the loss function, (5) becomes the Theorem 1 and 2 of Uno et al. [12] respectively. So, the expected sample size *E(N)* and the regret obtained by Uno et al. [12] are special cases of (16) and (19) respectively. Also when $w(\theta) = 1$ and $\gamma(\theta) = \theta$, the regret becomes $R_N - 2cn^* = 3c + o(c)$, which is the result obtained by Woodroofe [8].
2. For $Gamma(\alpha, \theta)$- distribution with known α, $T(X) = X$ and $v = \alpha$ in estimation of $\gamma(\theta) = \theta$ under the loss function (5), i.e. $w(\theta) = 1$, the regret becomes $R_N - 2cn^* = \frac{3c}{\alpha} + o(c)$, which is the result obtained by Woodroofe [8].
3. For scale invariant squared error loss (4), we have $w(\theta) = \frac{1}{\gamma^2(\theta)}$, and from (19) the regret becomes

$$R_N - 2cn^* = \frac{c}{v}\left\{3 - 2\theta^2\frac{\gamma''(\theta)}{\gamma(\theta)} + 2\theta\frac{\gamma''(\theta)}{\gamma'(\theta)} + \frac{7}{4}\theta^2\frac{\{\gamma''(\theta)\}^2}{\{\gamma'(\theta)\}^2}\right.$$

$$\left. - \theta^2\frac{\gamma^{(3)}(\theta)}{\gamma'(\theta)} + 3\theta^2\frac{\{\gamma'(\theta)\}^2}{\{\gamma(\theta)\}^2} - 6\theta\frac{\gamma'(\theta)}{\gamma(\theta)}\right\} + o(c).$$

Therefore for $\gamma(\theta) = \theta$, the regret becomes $R_N - 2cn^* = o(c)$. Note that the loss function (4) is more appropriate for estimating the scale parameter than squared error loss.

The results of section 2 and 3 can be extended to some other distributions which do not necessarily belong to a scale family, such as Pareto or Beta distributions. A family of distributions that includes these distributions as special cases is the family of transformed chi-square distributions which is originally introduced by Rahman and Gupta [14]. They considered the one parameter exponential family

$$f(x, \eta) = e^{a(x)b(\eta) + c(\eta) + h(x)}, \tag{27}$$

and showed that $-2a(X)b(\eta)$ has a $Gamma(\frac{j}{2}, 2)$- distribution if and only if

$$\frac{2c'(\eta)b(\eta)}{b'(\eta)} = j. \tag{28}$$

In case j is an integer, $-2a(X)b(\eta)$ follows a chi-square distribution with j degrees of freedom. They called the one parameter exponential family (27), which satisfies (28), the family of transformed chi-square distributions. For example Beta, Pareto, Exponential, Longnormal and some other distributions belong to this family of distributions (see Table 1 of Rahman and Gupta [14]).

Now it is easy to show that if condition (28) holds, then the one parameter exponential family (27) is in the form of the scale parameter exponential family (1) with $v = \frac{j}{2}$, $T(X) = a(X)$ and $\theta = -\frac{1}{b(\eta)}$. Hence with these substitutions, we can extend the results of section 2 and 3 to the family of transformed chi-square distributions.

REFERENCES

1. Barnard, G. A. (1944). Economy in sampling with special reference to engineering experimentation. Brith. Min. Supply Tech. Rep. QC/R/7, I.
2. Wald, A. (1945). Sequential test of statistical hypotheses. *Ann. Math. Stat, 16,* 117-186.
3. Wald, A. & Wolfowitz, J. (1948). Optimum character of the sequential probability ratio test. *Ann. Math. Stat*., *19*, 326-339.
4. Haldane, J. B. S. (1945). On a method of estimating frequencies. *Biometrika*, *33,* 222-225.
5. Stein, C. (1945). A two sample test for a linear hypothesis whose power is independent of the variance. *Ann. Math. Stat*., *16,* 243-258.
6. Lai, T. L. (2001). Sequential Analysis: Some classical problems and new challenges. *Statistica Sinica*, *11*, 303-408.
7. Starr, N. & Woodroofe, M. (1972). Farther remarks on sequential estimation: the exponential case. *Ann. Math. Stat.*, *43,* 1147-1154.
8. Woodroofe, M. (1977). Second order approximation for sequential point and interval estimation. *Ann. Statist*., *5*, 984-995.
9. Gosh, M. & Mukhopadhyay, N. (1989). Sequential estimation of the percentiles of exponential and normal distributions. *South African Statist. J.*, *23,* 117-136.
10. Isogai, E. & Uno, C. (1994). Sequential estimation of a parameter of an exponential distribution. *Ann. Inst. Statist. Math*., *46,* 77-82.
11. Isogai, E., Ali, M. & Uno, C. (2003). Sequential estimation of the powers of normal and exponential scale parameters. *Sequential Analysis*, *22,* 77-82.
12. Uno, C., Isogai, E. & Lim, D. (2004). Sequential point estimation of a function of the exponential scale parameter. *Austrian Journal of Statistics*, *33*(3), 281-291.
13. Aras, G. & Woodroofe, M. (1993). Asymptotic expansions for the moments of a randomly stopped average. *The Annals of Statistics*, *21,* 503-519.
14. Rahman, M. S. & Gupta, R. P. (1993). Family of transformed chi-square distributions. *Comm. Statist. Theory Methods*, *22,* 135-146.