Iranian Journal of Science & Technology, Transaction A, Vol. 31, No. A2
Printed in The Islamic Republic of Iran, 2007

DOUBLE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS*

E. SAVAS[1] AND R. F. PATTERSON[2**]

[1]Istanbul Ticaret University, Department of Mathematics, Uskudar, Istanbul, Turkey
Email: ekremsavas@yahoo.com
[2]Department of Mathematics and Statistics,
University of North Florida, Building 11 Jacksonville, Florida 32224, USA
Email: rpatters@unf.edu

Abstract – In this paper we introduce some new double sequence spaces using the Orlicz function and examine some properties of the resulting sequence spaces.

Keywords – Double sequences, P-convergent, Orlicz functions

1. INTRODUCTION AND BACKGROUND

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

Recall that an Orlicz function M: $[0,\infty) \to [0,\infty)$ is continuous, convex, non-decreasing function such that $M(0)=0$ and $M(x)>0$ for $x>0$, and $M(x)\to\infty$ as $x\to\infty$. If the convexity of an Orlicz function is replaced by $M(x+y)\le M(x)+M(y)$, then this function is called the modulus function, which is defined and characterized by Ruckle [1].

In 1900, Pringsheim [2] presented the following definition:

Definition 1. 1. A double sequence $x=(x_{k,l})$ has a Pringsheim limit ℓ (denoted by $P\text{-}lim\ x=\ell$) provided that given $\varepsilon>0$, there exists $N\in N$ such that $|x_{k,l}-\ell|<\varepsilon$ whenever $k,l>N$. We shall describe such an x more briefly as "P-convergent".

Let w'' denote the set of all double sequences of real numbers.

Definition 1. 2. Let M be an Orlicz function and $p=(p_{kl})$ be any factorable double sequence of strictly positive real numbers. We define the following sequence spaces:

$$L''_M(p)=\left\{x\in w'':\sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|x_{k,l}|}{\rho}\right)\right)^{p_{k,l}}<\infty,\quad for\ some\ \rho>0\right\}$$

$$W''(M,p)=\left\{x\in w'':P-\lim_{mn}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left(M\left(\frac{|x_{k,l}-\ell|}{\rho}\right)\right)^{p_{k,l}}=0,\ for\ some\ \rho>0\ and\ \ell\right\}$$

*Received by the editor March 27, 2005 and in final revised form February 19, 2007
**Corresponding author

$$W_0^{''}(M,p)=\left\{x\in w^{''}: P-\lim_{mn}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left(M\left(\frac{|x_{k,l}|}{\rho}\right)\right)^{p_{k,l}}=0,\ for\ \ some\ \ \rho>0\right\}$$

and

$$W_\infty^{''}(M,p)=\left\{x\in w^{''}: \sup_{mn}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left(M\left(\frac{|x_{k,l}|}{\rho}\right)\right)^{p_{k,l}}<\infty,\ for\ \ some\ \ \rho>0\right\}.$$

When M(x)=x, then the family of sequences defined above becomes $L^{''}(p)$, $[c,1,1,p]$, $[c,1,1,p]_0$, $[c,1,1,p]_\infty$ respectively. If we let $p_{k,l}=1$ for all k and l, then $L_M^{''}(p)$, $W^{''}(M,p)$, $W_0^{''}(M,p)$, and $W_\infty^{''}(M,p)$ reduce to $L_M^{''}$, $W^{''}(M)$, $W_0^{''}(M)$, and $W_\infty^{''}(M)$ respectively.

Theorem 1. 1. Let H=$\sup_{k,l} p_{k,l}$, then $L_M^{''}(p)$ is a linear set over the set of complex numbers C.

Proof: Let *x* and *y* be elements of $L_M^{''}(p)$ and both α and β are complex numbers. The goal of this proof is to find some ρ_3 such that

$$\sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|\alpha x_{k,l}+\beta y_{k,l}|}{\rho_3}\right)\right)^{p_{k,l}}<\infty.$$

Since *x* and *y* are in $L_M^{''}(p)$, there exist some positive ρ_1 and ρ_2 such that

$$\sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|x_{k,l}|}{\rho_1}\right)\right)^{p_{k,l}}<\infty$$

and

$$\sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|y_{k,l}|}{\rho_2}\right)\right)^{p_{k,l}}<\infty.$$

Similar to Parashar and Choudhary [3], we will define $\rho_3=max\{2|\alpha|\rho_1,2|\beta|\rho_2\}$.
Since *M* is non-decreasing and convex

$$\sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|\alpha x_{k,l}+\beta y_{k,l}|}{\rho_3}\right)\right)^{p_{k,l}}\le\sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|\alpha x_{k,l}|}{\rho_3}+\frac{|\beta y_{k,l}|}{\rho_3}\right)\right)^{p_{k,l}}$$

$$\le\sum_{k,l=1,1}^{\infty,\infty}\frac{1}{2^{p_{k,l}}}\left[M\left(\frac{|\alpha x_{k,l}|}{\rho_1}\right)+M\left(\frac{|\beta y_{k,l}|}{\rho_2}\right)\right]^{p_{k,l}}$$

$$< \sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\frac{|x_{k,l}|}{\rho_1}\right)+M\left(\frac{|y_{k,l}|}{\rho_2}\right)\right]^{p_{k,l}}$$

$$\leq C\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\frac{|x_{k,l}|}{\rho_1}\right)\right]^{p_{k,l}}+C\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\frac{|y_{k,l}|}{\rho_2}\right)\right]^{p_{k,l}}\leq\infty,$$

where $C= max\ \{1,2^{H-1}\}$. Thus $L_M^{''}(p)$ is a linear space.

Definition 1. 3. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K>0$ such that $M(2u)\leq KM(u)$ for all $u\geq 0$.
It is easy to show that K>2 always. The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu)\leq K(l)M(u)$ for all values of u and for l>1.
We shall present the following trivial lemma.

Lemma 1. 1. Let M be the Orlicz function which satisfies Δ_2-condition, and let $0<\delta<1$. Then for each $x\geq\delta$ we have $M(x)<Kx\frac{1}{\delta}M(2)$ for some constant K>0.

Theorem 1. 2. For any Orlicz function M which satisfies Δ_2-condition, we have
(1) $[c,1,1]\subset W^{''}(M)$
(2) $[c,1,1]_0\subset W_0^{''}(M)$
and
(3) $[c,1,1]_\infty\subset W_\infty^{''}(M)$.

Proof: Let $x\in[c,1,1]$, thus

$$A_{m,n}=\frac{1}{mn}\sum_{k,l=1,1}^{m,n}|x_{k,l}-\ell|\to 0 \text{ as } m,n\to\infty.$$

Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $M(t)<\varepsilon$ for $0\leq t\leq\delta$. Thus, by the above lemma we obtain the following

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n}M(\ |x_{k,l}-\ell|\ )=\frac{1}{mn}\sum_{k,l=1,1\&|x_{k,l}-\ell|\leq\delta}^{m,n}M(\ |x_{k,l}-\ell|\ )+\frac{1}{mn}\sum_{k,l=1,1\&|x_{k,l}-\ell|>\delta}^{m,n}M(\ |x_{k,l}-\ell|\ )$$

$$<\frac{1}{mn}\varepsilon(mn)+\frac{1}{mn}K\frac{1}{\delta}M(2)(mn)A_{m,n}.$$

Therefore, as m and n go to infinity, in Pringsheim's sense it follows that $x\in W^{''}(M)$. Part 2 and 3 follow similar arguments as Part 1 and are thus omitted. This completes the proof.

Theorem 1. 3. (1) Let $0<\inf\ p_{k,l}\leq p_{k,l}\leq 1$. Then

$$W^{''}(M,p)\subseteq W^{''}(M).$$

(2) Let $1\leq p_{k,l}\leq sup\ p_{k,l}<\infty.$ Then

$$W^{''}(M)\subseteq W^{''}(M,p).$$

Proof: $x \in W''(M,p)$, since $0 < inf\ p_{k,l} \le 1$, we obtain the following:

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{|x_{k,l}-\ell|}{\rho}\right) \le \frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{|x_{k,l}-\ell|}{\rho}\right)^{p_{k,l}} .$$

Thus $x \in W''(M)$. Let us establish Part (2). Let $p_{k,l} \ge 1$ for each k and l, and $sup\ p_{k,l} < \infty$. Let $x \in W''(M)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer N such that

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{|x_{k,l}-\ell|}{\rho}\right) \le \varepsilon < 1$$

for all $n,m \ge N$. This implies that

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{|x_{k,l}-\ell|}{\rho}\right)^{p_{k,l}} \le \frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{|x_{k,l}-\ell|}{\rho}\right) .$$

Therefore $x \in W''(M,p)$. This completes the proof.

Theorem 1. 4. Let $0 < p_{k,l} \le q_{k,l} < \infty$, for each k and l. Then

$$L''_M(p) \subseteq L''_M(q).$$

Proof: Let $x \in L''_M(p)$. Then there exists for some $\rho > 0$ such that

$$\sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|x_{k,l}|}{\rho}\right)\right)^{p_{k,l}} < \infty.$$

This implies

$$M\left(\frac{|x_{i,j}|}{\rho}\right) \le 1,$$

for sufficiently large values of *i* and *j*. Since M is non decreasing, we granted

$$\sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|x_{k,l}|}{\rho}\right)\right)^{q_{k,l}} \le \sum_{k,l=1,1}^{\infty,\infty}\left(M\left(\frac{|x_{k,l}|}{\rho}\right)\right)^{p_{k,l}} < \infty.$$

Thus $x \in L''_M(q)$.

2. STATISTICAL CONVERGENCE

The concept of statistical convergence was introduced by Fast [4] in 1951.

Definition 2.1. The sequence $x = (x_k)$ has statistic limit ℓ, denoted by st_1-$lim x = \ell$ or $x_k \to \ell(st_1)$ provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\left|\ \{k \le n : |x_k - \ell| \ge \varepsilon\}\ \right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

Quite recently, Mursaleen and Edely [5] defined the statistical analogue for double sequence $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be P-statistical convergence to ℓ provided that for each $\varepsilon > 0$

$$P - \lim_{mn} \frac{1}{mn}\left\{number\ of\ (k,l) : k < m\ and\ l < n : \left|x_{k,l} - \ell\right| \ge \varepsilon\right\} = 0.$$

In this case, we write $st_2 - \lim_{kl} x_{k,l} = \ell$ and we denote the set of all P-statistical convergent double sequences by st_2 and denote the set of P-statistically null sequences by $(st_2)_0$.

Theorem 2.1. If M be an Orlicz function, then $W_0''(M) \subset (st_2)_0$.

Proof: Suppose $x \in W_0''(M)$ and $\varepsilon > 0$, then we obtain the following for every *n and m*

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{|x_{k,l}|}{\rho}\right) \ge \frac{1}{mn}\sum_{k,l=1,1\&|x_{k,l}|\ge\varepsilon}^{m,n} M\left(\frac{|x_{k,l}|}{\rho}\right)$$

$$\ge M(\varepsilon)\left|\ \left\{k \le m, l \le n : |x_{k,l}| \ge \varepsilon\right\}\ \right|.$$

Hence $x \in (st_2)_0$.

Theorem 2. 2. $(st_2)_0 = W_0''(M)$ if and only if M is bounded.

Proof: Suppose that *M* is bounded and that $x \in (st_2)_0$. Since *M* is bounded there exists an integer *K* such that $M(x)<K$ for all $x \ge 0$. Then for each *m* and *n*, we have

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} M\left(\frac{|x_{k,l}|}{\rho}\right) = \frac{1}{mn}\sum_{k,l=1,1\&|x_{k,l}|\ge\varepsilon}^{m,n} M\left(\frac{|x_{k,l}|}{\rho}\right) + \frac{1}{mn}\sum_{k,l=1,1\&|x_{k,l}|<\varepsilon}^{m,n} M\left(\frac{|x_{k,l}|}{\rho}\right)$$

$$\le \frac{K}{mn}\left|\ \left\{k \le m, l \le n : |x_{k,l}| \ge \varepsilon\right\}\ \right| + M(\varepsilon)$$

and thus the Pringsheim's limit on *m* and *n* grant us the results. Conversely, suppose that *M* is unbounded so that there is a positive double sequence $s_{m,n}$ with $M(s_{mn}) = (mn)^2$ for $m,n=1, 2,\ldots$ Now the sequence *x* defined by $x_{k,l} = s_{mn}$ if $k,l = (mn)^2$ for *m, n= 1, 2,...* . and $x_{k,l} = 0$, otherwise. Then we have

$$\frac{1}{mn}\left|\ \{k \le m, l \le n : |x_{k,l}| \ge \varepsilon\}\ \right| \le \frac{\sqrt{mn}}{mn} \to 0$$

as $m,n \to \infty$, hence $x_{k,l} \to 0(st_2)_0$. But $x \notin W_0''(M)$, contradicting $(st_2)_0 = W_0''(M)$. This completes the proof.

REFERENCES

1. Ruckle, W. H. (1973). FK Spaces in which the sequence of coordinate vectors in bounded. *Canad, J. Math*. *25*,

973-978.

2. Pringsheim, A. (1900). Zur theorie der zweifach unendlichen Zahlenfolgen. *Mathematische Annalen*, *53*, 289-321.
3. Parashar, S. D. & Choudhary, B. (1994). Sequence spaces defined by Orlicz functions. *Indian J. Pure Appl. Math*. *25*(4), 419-428.
4. Fast, H. (1951). Sur la convergence statistique. *Collog. Math*. *2*, 241-244.
5. Mursaleen, M. & Edely, O. H. (2003). Statistical convergence of double sequences. *J. Math. Anal. Appl*. *288*(1), 223-231.