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PROJECTIVELY RELATED EINSTEIN FINSLER SPACES*

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Abstract – The main objective of this paper is to find the necessary and sufficient condition of a given Finsler metric to be Einstein in order to classify the Einstein Finsler metrics on a compact manifold. The considered Einstein Finsler metric in the study describes all different kinds of Einstein metrics which are pointwise projective to the given one. This study has resulted in the following theorem that needs the proof of three prepositions. Let F be a Finsler metric $(n > 2)$ projectively related to an Einstein non-projectively flat Finsler metric $\bar{F}$, then F is Einstein if and only if $F = \lambda\bar{F}$ where λ is a constant. A Schur type lemma is also proved.

Keywords – Projectively related Finsler metrics, projectively flat, Einstein Finsler metric

1. INTRODUCTION

One important study in projective geometry is to determine a relationship among geometric structures with common geodesics as the point sets. Two regular metrics on a manifold are said to be pointwise projectively related if they have the same geodesics as the point sets. Two regular metric spaces are said to be projectively related if there is a difeomorphism between them such that the pull-back metric is pointwise projective to another one. There are some quantities in the projective Finsler geometry which are projective invariant. One of the most important of them is the Weyl curvature. The Finsler metrics with $W_k^{\ i} = 0$ are called Weyl metrics. It is well-known that a Finsler metric is a Weyl metric if and only if it is of scalar flag curvature. The Ricci curvature plays an important role in the projective geometry of Riemannian-Finsler manifolds. The well-known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate Einstein's theory of gravitation [1]. A Finsler metric is Einstein if the Ricci scalar Ric is a function of x alone. Equivalently

$$Ric_{ij} = Ric(x)g_{ij}$$

In Riemannian space, if g and $\bar{g}$ are pointwise projectively related Riemannian metrics on manifolds of dimensional $n \geq 3$, then g is of constant curvature if and only if $\bar{g}$ is of constant curvature. The same statement is also true for Einstein metrics. More precisely, it can be said:

Theorem ([2, 3]) Let (M, g) be an n-dimensional Riemannian space and $\bar{g}$ another Riemannian metric pointwise projective to g. Suppose that g is Einstein, then $\bar{g}$ must be Einstein. The paper focuses on the Einstein Finsler metrics which are projectively related to the other Einstein Finsler metrics. The question that can be raised in the situations where a Finsler metric is projectively related to the Einstein one is:

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When is a Finsler metric Einstein? The contribution of this paper is to give an answer to this question. The classification of Einstein Finsler metrics in the compact case is considered. The main proposed theorem is as follows:

Theorem 1. Let F be a Finsler metric $(n > 2)$ projectively related to Einstein another Finsler metric $\bar{F}$ of non-zero Ricci scalar, then
1) F is Einstein if and only if it is a constant coefficient of $\bar{F}$ when $\bar{F}$ is not projectively flat.
2) F is Einstein if and only if it is of constant Ricci scalar when $\bar{F}$ is projectively flat.

The well-known examples of Finsler metrics are the so-called class (α,β) -metrics. A Finsler metric F on a manifold M is called (α,β) -metric if it is in the following form:

$$F=\psi(\alpha,\beta)=\alpha\varphi(\frac{\beta}{\alpha}),$$

where α is a Riemannian metric and β is a 1-form on M such that $\alpha(x,y)=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta(x,y)=b_i(x)y^i$ and φ is a positive C^∞ function on some interval $[-r,r]$. An (α,β) -metric F is projective to α if and only if the following is satisfied (1):

$$\psi_{22}\{\beta_{|i}(b_j-\frac{\beta\alpha_j}{L})-\beta_{|j}(b_i-\frac{\beta\alpha_i}{L})\}=2\psi_2 s_{ij}, \tag{1}$$

where ``$|$'' is the horizontal derivative with respect to α and

$$\psi_1=\frac{\partial\psi}{\partial\alpha},\psi_2=\frac{\partial\psi}{\partial\beta},s_{ij}=\frac{b_{i|j}-b_{j|i}}{2}.$$

The next consideration of this paper is the study of projectively related Einstein (α,β) -metrics. The Schur type lemma is stated for them.

2. PRELIMINARIES

Let M be an n-dimensional C^∞ manifold. Denote by T_xM as the tangent space at $x\in M$, and by $TM=\cup_{x\in M}T_xM$ as the tangent bundle of M. Each element of TM has the form (x, y), where $x\in M$ and $y\in T_xM$. Let $TM_0=TM\setminus\{0\}$ and the natural projection $\pi:TM\to M$ is given by $\pi(x,y)=x$. The *pull-back tangent bundle* π^*TM is a vector bundle over TM_0 whose fiber π_v^*TM at $v\in TM_0$ is just T_xM, where $\pi(v)=x$. Then

$$\pi^*TM=\{(x,y,v)\,|\,y\in T_xM_0, v\in T_xM\}.$$

A *Finsler metric* on a manifold M is a function $F:TM\to[0,\infty)$, which has the following properties:
(i) F is C^∞ on TM_0;
(ii) $F(x,\lambda y)=\lambda F(x,y)\quad \lambda>0$;
(iii) For any tangent vector $y\in T_xM$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij}(x,y)=\left[\frac{1}{2}F^2\right]_{y^iy^j}$$

is positive definite.
We obtain a symmetric tensor **C** defined by

$$\mathbf{C}(U,V,W) = C_{ijk}(y)U^i V^j W^k,$$

where $U = U^i \frac{\partial}{\partial x^i}, V = V^i \frac{\partial}{\partial x^i}, W = W^i \frac{\partial}{\partial x^i}$ and $C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}(y)$.
$\mathbf{C}$ is called the *Cartan* tensor. It is well known that **C=0** if and only if F is Riemannian. Every Finsler metric F induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\}y^i y^k = \frac{1}{4}g^{il}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}.$$

The Riemann curvature $\mathbf{R}_\mathbf{y} = R^i_k dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_pM \to T_pM$ is defined by

$$R^i_k(y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \quad (2)$$

The Riemann curvature has the following properties. For any non-zero vector $\mathbf{y} \in T_pM$,

$$\mathbf{R}_\mathbf{y}(\mathbf{y}) = 0, g_\mathbf{y}(\mathbf{R}_\mathbf{y}(\mathbf{u}),\mathbf{v}) = g_\mathbf{y}(\mathbf{u},\mathbf{R}_\mathbf{y}(\mathbf{v})), \mathbf{u},\mathbf{v} \in T_pM,$$

and

$$R^i_{kl} = \frac{1}{3}\{\frac{\partial R^i_k}{\partial y_l} - \frac{\partial R^i_l}{\partial y_k}\}. \quad (3)$$

For a two-dimensional plane $P \subset T_pM$ and a non-zero vector $\mathbf{y} \in T_pM$, the *flag curvature* $\mathbf{K}(P,y)$ is defined by [4]

$$\mathbf{K}(P,\mathbf{y}) := \frac{g_\mathbf{y}(\mathbf{u},\mathbf{R}_\mathbf{y}(\mathbf{u}))}{g_\mathbf{y}(\mathbf{y},\mathbf{y})g_\mathbf{y}(\mathbf{u},\mathbf{u}) - g_\mathbf{y}(\mathbf{y},\mathbf{u})^2},$$

where $P = \mathrm{span}\{\mathbf{y},\mathbf{u}\}$. F is said to be of *scalar curvature* $\mathbf{K} = \lambda(y)$ if for any $\mathbf{y} \in T_pM$, the flag curvature $\mathbf{K}(P,\mathbf{y}) = \lambda(\mathbf{y})$ is independent of P containing $\mathbf{y} \in T_pM$ that is equivalent to the following system in a local coordinate system (x^i, y^i) in TM,

$$R^i_k = \lambda F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}.$$

If λ is a constant, then F is said to be of *constant curvature*. The Ricci scalar function of F is given by

$$\rho := \frac{1}{F^2}R^i_i.$$

Therefore, the Ricci scalar function is positive homogeneous of degree 0 in y. This means that $\rho(x,y)$ depends on the direction of the flag pole y but not its length. The Ricci tensor of a Finsler metric F is defined by

$$Ric_{ij} := \{\frac{1}{2}R^k_k\}_{y^i y^j}.$$

Ricci-flat manifolds are Riemannian manifolds whose Ricci tensor vanishes. In physics they are important because they represent vacuum solutions to Einstein's equations.

Definition 2.1. [5] A Finsler metric is said to be an *Einstein metric* if the Ricci scalar function is a function of x alone, equivalently

$$Ric = \rho(x) g_{ij}, \text{ or } Ric_{00} = \rho(x) F^2.$$

Ricci-flat manifolds are special cases of Einstein manifolds. We now consider projectively related Finsler metrics on M, i.e. the metrics having the same geodesics as the point sets.

Definition 2.2. [6] A Finsler space F^n is projective to another Finsler space $\overline{F}^n$, if and only if there exists a one-positive homogeneous scalar field $P(x,y)$ on TM satisfying

$$\overline{G}^i(x,y) = G^i(x,y) + P(x,y) y^i.$$

Let G^i and $\overline{G}^i = G^i + Py^i$ be sprays on n-manifold M. The Riemann curvatures are related by [7]

$$\overline{R}^i_k = R^i_k + E\ \delta^i_k + \tau_k\ y^i, \tag{4}$$

where

$$E := P^2 - P_{|k} y^k,$$

$$\tau_k = 3(P_{|k} - PP_{y^k}) + E_{y^k}.$$

Definition 2.3. [6] Let (M, F) be a Finsler space. Assume that a function P on TM is C^∞ on $TM \setminus \{0\}$ satisfying

$$P(\lambda y) = \lambda P(y), \forall \lambda > 0,$$

(a) P is called a Funk function if it satisfies the following system of PDEs

$$P_{|k} = PP_{.k}.$$

(b) P is called a weak Funk function if it satisfies the following system of PDEs

$$y^k P_{|k} = P^2.$$

Lemma [8] Let (M, F) be a Finsler space. A Finsler metric F is pointwise projective to F if and only if

$$\frac{\partial \tilde{F}_{|k}}{\partial y^l} y^k - \tilde{F}_{|l} = 0.$$

Then

$$\tilde{G}^i = G^i + Py^i,$$

where

$$P = \frac{\tilde{F}_{|k} y^k}{2\tilde{F}}.$$

By the above lemma an (α, β)-metric in the form of (1.1) is pointwise projective to α if and only if

$$\varphi''(\alpha_{.l}\beta-\alpha b_l)\beta_{|k}y^k=\alpha^2\varphi'(\beta_{|k.l}y^k-\beta_{|l}). \tag{5}$$

Now, we are going to study the Weyl curvature of spray as an important projective invariant. The Weyl's projective invariant is constructed from the Riemann curvature. Define [7]

$$W^i{}_k(y)=R^i{}_k-R\delta^i{}_k-\frac{1}{n+1}\frac{\partial}{\partial y^m}(R^m{}_k-R\delta^m{}_k)y^i,$$

where $R=\frac{1}{n+1}Ric=\frac{1}{n+1}R^m{}_m$. $W_y:T_xM\to T_xM$ is a linear transformation satisfying $W_y(y)=0$. We call $W:=W_y{}_{y\in TM_0}$ the Weyl curvature. W is a projective invariant under projective transformations [9].

Theorem ([6]) A Finsler metric is of scalar curvature if and only if $W=0$.

Proof of Theorem 1.
In the following, we prove theorem1.

Proposition 3.1. Let (M,F) be a Finsler space of dimension $n>2$. F is Einstein metric if and only if

$$y_iV^i{}_k=-\frac{3(n-1)}{n+1}Ry_k,$$

where $V_k^i=W_k^i-R_k^i-\frac{3}{n+1}Ric_{0k}y^i$

Proof: (i) Assume that F is Einstein. By definition of the Weyl tensor, we have

$$y_iW^i{}_k-y_iR^i{}_k=-Ry_k-\frac{n-2}{n+1}R_{.k}F^2+\frac{3F^2}{n+1}Ric_{0k},$$

then

$$y_iV^i{}_k=-Ry_k-\frac{n-2}{n+1}R_{.k}F^2,$$

since F is Einstein

$$2Ry_k=R_{.k}F^2,$$

therefore,

$$y_iV^i{}_k=-\frac{3(n-1)}{n+1}R_{.k}y_k,$$

this completes the proof (i).
(ii) Suppose

$$y_iV^i{}_k=-\frac{3(n-1)}{n+1}R_{.k}y_k,$$

by definition of the Weyl tensor, we have

$$-y_iV^i{}_k=y_i(R\delta^i{}_k+\frac{n-2}{n+1}R_{.k}y_i),$$

and therefore

$$Ry_k + \frac{n-2}{n+1} R_{.k} F^2 = \frac{3(n-1)}{n+1} Ry_k,$$

by a simple computation and since $n > 2$, it is concluded that,

$$2Ry_k = R_{.k} F^2.$$

This implies

$$(\frac{R}{F^2})_{.k} = 0.$$

Therefore F is Einstein.

Proposition 2.3. Let F be a Finsler metric $(n > 2)$ projectively related to Einstein, another Finsler metric $\overline{F}$ with projective factor P. If F is Einstein, then $(\frac{E}{F^2})_{.k} = 0$ where $E = P^2 - P_{|k} y^k$.

Proof: Let W and $\overline{W}$ be the Weyl curvatures of F and $\overline{F}$. For Einstein Finsler metric F we have

$$y_i V^i{}_k = -\frac{3(n-1)}{n+1} R_{.k} y_k,$$

therefore,

$$y_i W^i{}_k = y_i (R^i{}_k + \frac{3Ric_{0k}}{n+1} y^i) - \frac{3(n-1)}{n+1} Ry_k,$$

but $\overline{W}^i{}_k$ is invariant under projective transformation, then

$$y_i W^i{}_k = y_i \overline{W}^i{}_k = y_i (\overline{R}^i{}_k + \frac{3\overline{Ric}_{0k}}{n+1} y^i) - \frac{3(n-1)}{n+1} \overline{R} y_k,$$

therefore

$$y_i (R^i{}_k - \overline{R}^i{}_k) + \frac{3y_i}{n+1}(Ric_{0k} y^i - \overline{Ric}_{0k} y^i) - \frac{3(n-1)}{n+1}(R - \overline{R}) y_k = 0, \tag{6}$$

but from (2, 3) we have

$$Ric = \overline{Ric} + (n-1)E,$$

and this implies that

$$R = \overline{R} + E. \tag{7}$$

Also, from (3) and (4)

$$3R^i_{kl} = 3\overline{R}^i_{kl} + (E_{.l} - \tau_l)\delta^i_k - (E_{.k} - \tau_k)\delta^i_l + (\tau_{k.l} - \tau_{l.k}) y^i,$$

therefore

$$3Ric_{0l} = 3\overline{Ric}_{0l} + (n-2)E_{.l} - (n+1)\tau_l,$$

by substituting the above with that in (4), it is concluded:

$$Ey_k + \tau_k F^2 + \frac{3y_i}{3(n+1)}((n-2)E_{.k}y^i - (n+1)\tau_k y^i) - \frac{3(n-1)}{n+1}Ey_k$$

$$= -\frac{2(n-2)}{n+1}Ey_k + \frac{(n-2)F^2}{n+1}E_{.k} = 0,$$

thus

$$\frac{n-2}{n+1}(E_{.k}F^2 - 2Ey_k) = 0,$$

since $n > 2$, $(\frac{E}{F^2})_{.k} = 0$.

Proposition 3.3. Let F be a Finsler metric $(n > 2)$ projectively related to Einstein, another Finsler metric $\overline{F}$ of non-zero Ricci scalar, then

1) F is Einstein if and only if $(\frac{F}{\overline{F}})_{.k} = 0$, when $\overline{F}$ is not projectively flat.

2) F is Einstein if and only if it is of constant Ricci scalar when $\overline{F}$ is projectively flat.

Proof: Assume F is Einstein. If $\overline{F}$ is projectively flat then F is projectively flat, too. Invoking preposition 12.1 in [10] we obtain F which is of scalar curvature, and since it is Einstein then F is of constant flag curvature, and therefore it is constant Ricci scalar.

In the other case, let $\overline{F}$ not be projectively flat. Since F is Einstein, $(\frac{R}{F^2})_{.k} = 0$ and by the above proposition $(\frac{E}{F^2})_{.k} = 0$, then there exist a function $\xi(x)$ where $\frac{R-E}{F^2} = \xi(x)$. F is projectively related to $\overline{F}$, by (7) we have $\frac{R}{F^2} = \frac{\overline{R}}{F^2} + \frac{E}{F^2}$.

But $\overline{F}$ is Einstein and Finsler metric of non-zero Ricci scalar, so there is a non-zero function $\lambda(x)$ such that $\overline{R} = \lambda(x)\overline{F}^2$. It can be concluded that $(\frac{F}{\overline{F}})_{.k} = 0$. It is clear, conversely.

F is projectively related to $\overline{F}$, so it can be said that

$$G^i = \overline{G}^i + Py^i, \tag{8}$$

where $P = \frac{F_{|k}y^k}{2F}$. By the above proposition, there is a function of x only, where $F = f(x)\overline{F}$ then $P = \frac{f_{|k}y^k}{2f}$. By using the formula of G^i mentioned previously, it can be concluded

$$G^i = \frac{1}{4f}\overline{g}^{il}\left[f\{(\overline{F}^2)_{x^k y^l}y^k - (F^2)_{x^l}\}\right] + \frac{\overline{g}^{il}}{4f}\left[f_{x^k}y^k(\overline{F}^2)_{.y^l} - f_{x^l}\overline{F}^2\right]$$

$$= \overline{G}^i + Py^i + \frac{f_{x^l}\overline{g}^{il}}{4f}\overline{F}^2.$$

By (8), f as a constant is obtained.

3. FINSLER METRICS PROJECTIVELY RELATED TO EINSTEIN RIEMANNIAN METRICS

Consider $F = \varphi(\alpha, \beta)$ is projectively related to Riemannian metric α of constant sectional curvature. It is important to find a sufficient condition of $F = \varphi(\alpha, \beta)$ in which α and β are Riemannian metric and a 1-form on M to be Einstein. By the Beltrami Theorem, any Riemannian metric of constant sectional curvature is projectively flat. Some type of (α, β) -metrics is considered in [11].

Corollary 3.1. Any (α, β) -metric projectively related to Einstein Riemannian metric α of non-constant sectional curvature is Einstein if and only if it is Riemannian.

By using Akbar-Zadeh's theorem [12], the main theorem proved in reference [13] and the above theorem 1, the Finsler metrics on a connected compact boundaryless manifold projectively related to an Einstein Riemannian manifold can be classified.

Corollary 3.2. Let F be a Finsler metric on a connected compact boundaryless manifold, projectively related to Einstein Riemannian metric $\bar{F}$ $(n > 2)$. F can be classified in the following conditions:

(a) $\bar{F}$ is of non-constant sectional curvature.

In this case, F is Riemannian if and only if F is Einstein.

(b) $\bar{F}$ is of constant sectional curvature. In this case, the following statements are valid for F:

- F is Riemannian if it is Einstein of negative Ricci scalar.
- F is Riemannian if it is a reversible Einstein metric of positive Ricci scalar.
- F is locally Minkowski if it is Einstein of zero Ricci scalar.

4. SCHUR TYPE LEMMA

Corollary 3.3. The Ricci scalar of any Einstein Finsler metric $(n > 2)$ projectively related to another one of constant Ricci scalar is necessarily constant.

In other words, *the Ricci scalar of any Einstein* (α, β) *- metric* $(n > 2)$ *projectively related to Einstein Riemannian metric* α *is necessarily constant*.

The Funk metric on a strongly convex domain Ω in R^n is non-reversible, positively complete and projectively flat with $K = -\frac{1}{4}$. The Hilbert metric on Ω is obtained from Funk metric by symmetrization. It is reversible, complete and projectively flat with $K = -1$.

Example 3.1. The pair of Funk metrics on the unit ball $B^n \subset R^n$ are given by

$$F_{\pm}(y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)} \pm <x, y>}{1 - |x|^2}, y \in T_x B^n = R^n,$$

where $|.|$ and $<,>$ are the representative of the standard Euclidean norm and inner product. The Hilbert metric is Riemannian. It is complete with constant curvature $K = -1$.

$$F_H(y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)}}{1 - |x|^2}, y \in T_x B^n = R^n$$

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