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# LINEAR PROGRAMMING PROBLEM WITH INTERVAL COEFFICIENTS AND AN INTERPRETATION FOR ITS CONSTRAINTS[*]

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**Abstract –** In this paper, we introduce a Satisfaction Function (SF) to compare interval values on the basis of Tseng and Klein's idea. The SF estimates the degree to which arithmetic comparisons between two interval values are satisfied. Then, we define two other functions called Lower and Upper SF based on the SF. We apply these functions in order to present a new interpretation of inequality constraints with interval coefficients in an interval linear programming problem. This problem is as an extension of the classical linear programming problem to an inexact environment. On the basis of definitions of the SF, the lower and upper SF and their properties, we reduce the inequality constraints with interval coefficients in their satisfactory crisp equivalent forms and define a satisfactory solution to the problem. Finally, a numerical example is given and its results are compared with other approaches.

**Keywords –** Interval number, inequality relation, equality relation, satisfaction function, interval linear programming

## 1. INTRODUCTION

In order to develop a good Operations Research methodology, fuzzy and stochastic approaches are frequently used to describe and treat imprecise and uncertain elements present in a decision problem. In literature, in the works of fuzzy optimization [1-8], fuzzy parameters are assumed to be with known membership functions and in stochastic optimization [9-13], parameters are assumed to have known probability distributions. However, in reality to a decision-maker (DM), it is not always easy to specify the membership function or probability distribution in an inexact environment. In some of the cases, the use of interval coefficients may serve the purpose better. An interval number can be thought of as an extension of the concept of a real number and also as a subset of the real line R [14, 15]. As a coefficient, an interval signifies the extent of tolerance (or a region) that the parameter can possibly take. However, in decision problems, its use doesn't receive the attention it merits.

In the formulation of realistic problems, a set of intervals may appear as coefficients in the inequality (or equality) constraints of an optimization problem or in the selection of the best alternative in a decision-making problem. Hence, the problem of crisp interval numbers comparison is of perennial interest because of its direct relevance in practical modeling and the optimization of real-world processes under uncertainty.

Theoretically, the crisp intervals can only be partially ordered and hence cannot be compared. However, when the interval numbers are used in practical applications or when a choice has to be made among alternatives, a comparison is needed [16-27].

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Numerous definitions of the comparison relation on crisp intervals exist [15, 19, 20, 24, 28-38]. In most cases, the authors use some quantitative indices. The values of such indices present the degree to which one interval crisp is greater/less than another interval. In some cases, even several indices are used simultaneously. Although some of these methods have shown more consistency and better performance in difficult cases, none of them may be put forward as the best one.

In this field, we find the foremost work in [14, 15], where the arithmetic of interval numbers was studied. Here, we find two transitive order relations defined on intervals; one as an extension of '<' on the real line and the other as an extension of '$\subseteq$', the concept of set inclusion. But these order relations cannot explain the ranking between two partially or fully overlapping intervals. Ishibuchi and Tanaka [20], as an advance over Moore, suggested two order relations '$\leq_{LR}$' and '$\leq_{mw}$' in '*either or survivor*' basis. However, there exist a set of a pair of intervals for which both order relations do not hold. Moreover, these order relations did not discuss anything on '*how much greater*' when one interval is known to be greater than another. If it is argued that the interval numbers are generated through the inexactness of the problem environment, there should be an attempt to satisfy (most likely subjectively) the query 'how much greater one is than another?' From these points of view, there exist numerous papers about this topic. But now we can cite only a few works [7, 24, 29-31, 33-38] which are based on the probabilistic approach. The idea of using the probability interpretation of the interval is not, in principle, a novel idea. The attraction of such an approach is based on the possibility of obtaining a completed set of probability $P(A < B)$, $P(A > B)$, and $P(A = B)$ in order to compare intervals $A$ and $B$ with only one assumption that the intervals are the supports of uniform distributions of random values $a \in A, b \in B$. Nevertheless, different expressions for probabilities estimation were obtained in the papers [7, 24, 29-31, 33-38]. Recently, Sevastjanov [34] proposed an approach which can derive the results of comparison as a probability interval. To do this, he used the Dempster-Shafer theory of evidence with its probabilistic interpretation [39-43].

From another point of view, Sengupta et al. [32] defined an index to order two intervals in terms of value. In Ref. [22], the index has been used to interpret an interval-valued inequality constraint and to define its satisfactory equivalent transformation to a crisp set of inequalities for an interval linear programming problem. In this paper, we concentrate on the linear programming problem with interval coefficients. However, we cannot apply the techniques of the classical linear programming for these problems directly. Many researchers have worked on linear programming problems with an interval objective function on the basis of order relations between two interval numbers and various techniques [19-23, 25-27]. Here, we mention two instances of these works. Tong [21] introduced an interval number linear programming problem where all coefficients are interval numbers. He reduced the interval number linear programming into two classical linear programming problems by introducing a maximum-value of range and a minimum- value of range of inequality and obtained an interval number optimal solution. Sengupta et al. [22] explained existing difficulties in using the union and intersection operators in defining the maximum- and the minimum-value of range of inequalities in Tong's approach, respectively. They presented another interpretation of inequality constraints with interval coefficients. They also gave an interpretation of the interval objective function with respect to 'Minimization' according to $\Im$-index [32]. Furthermore, they obtained a satisfactory solution for the problem. As Sengupta and Pal [38] expressed, the main disadvantage of using the comparison relations based on conditional probabilities is the complexity of calculation due to the presence of decision variables in the constraints, as their crisp satisfactory equivalent structure will be complex. In this paper, other interpretations from the constraints and the objective function of the interval linear programming problem are presented. To do this, a Satisfaction Function (SF) is proposed in order to compare the interval numbers on the basis of Tseng and Klein's idea [44]. The SF is a measure which estimates the satisfaction degree of the arithmetic comparison relations between two interval values. The SF also enables one to compare a real number with an interval number. Furthermore, if two crisp values are given, the arithmetic comparisons of those values are clear. But in the case of interval values, the SF generates a value in [0, 1]. The value shows the satisfaction degree of the DM from the result of the comparison. Continuing, we present some properties of the SF. Moreover, two new concepts called Lower and Upper SF based on the SF are defined and some of their properties are studied. We also apply these functions in order to interpret the inequality constraints with interval coefficients of an interval linear programming problem. In fact, this problem is an extension of the classical linear programming problem to an inexact environment. According to the definitions of SF, lower and upper SF and their properties, the inequality constraints with interval coefficients are reduced in their satisfactory crisp equivalent forms, and we also present the interpretation and realization of objective 'Minimization' with respect to an inexact environment and the SF concept. Finally, we define a satisfactory solution to the problem and illustrate it by a numerical example. Its results are then compared with the result of Sengupta et al.'s work [22] and Tong's work [21]. This paper is organized as follows: in Section 2, notations of the interval numbers and the interval arithmetic are briefly explained. Section 3 gives an elaborate study on the preference relations between two interval numbers. Section 4 along with its three subsections define a Satisfaction Function and explain some of its properties. Section 5 presents the definitions of Upper and Lower Satisfaction Function and some of their properties. Section 6 introduces an interval linear programming problem and presents an interpretation of inequality constraints with interval coefficients based on the SF, the lower and upper SF. Section 7 describes the solution of the interval linear programming problem. Section 8 gives a numerical example from [21] and shows the efficiency of our methodology compared with other approaches. Section 9 includes conclusions.

## 2. THE BASIC INTERVAL ARITHMETIC

All lower case letters denote the real numbers and the upper case letters denote the interval numbers or the closed intervals on $R$ . Notation $I$ denotes the set of interval numbers on $R$ .

2.1. $A=[\underline{a},\overline{a}]=\{a\in R \mid \underline{a}\le a\le \overline{a}\}$, where $\underline{a}$ and $\overline{a}$ are the left and right limit of the interval $A$ on the real line $R$ , respectively. If $\underline{a}=\overline{a}$ , then $A=[a,a]$ is a real number. Also, $m(A)$, $w(A)$, and $\mu(A)$ respectively denote the mid-point, half-width, and length of interval $A$ which are defined as follows:

$$m(A)=\frac{1}{2}(\underline{a}+\overline{a}),\qquad w(A)=\frac{1}{2}(\overline{a}-\underline{a}),\text{ and }\mu(A)=\overline{a}-\underline{a}\ .$$

2.2. Let $*'\in\{+',-',\times',\div'\}$ be a binary operation on the set of closed intervals. Then, the binary operation $*'$ is defined for each $A,B\in I$ based on the binary operation $*$ as: $A*'B=\{a*b \mid a\in A, b\in B\}$. In case of division, it is assumed that $0\notin B$ .

## 3. PREFERENCE RELATIONS BETWEEN INTERVAL NUMBERS

An extensive research and wide coverage on the interval arithmetic and its applications can be found in Moore [14]. Here, we find two transitive order relations defined on intervals: the first one as an extension of '<' on the real line as: $\mathrm{A}<\mathrm{B}$ if and only if $\overline{a}<\underline{b}$, and the other as an extension of the concept of set inclusion, i.e., $\mathrm{A}\subseteq\mathrm{B}$ if and only if $\underline{a}\ge\underline{b}$ and $\overline{a}\le\overline{b}$. These order relations cannot explain ranking between two overlapping intervals. The extension of the set inclusion here only describes the condition that the interval $A$ is nested in $B$ ; but it cannot order $A$ and $B$ in terms of value. We need to develop a definition of comparison of two interval numbers.

Ishibuchi and Tanaka [20] approached the problem of ranking two interval numbers more prominently. In their approach, in a maximization problem, if intervals $A$ and $B$ are two, say, profit intervals, then the maximum of $A$ and $B$ can be defined by an order relation $\leq_{LR}$ between $A$ and $B$ as follows:

$$A \leq_{LR} B \quad \text{if and only if} \quad \underline{a} \leq \underline{b} \quad \text{and} \quad \overline{a} \leq \overline{b}$$
$$A <_{LR} B \quad \text{if and only if} \quad A \leq_{LR} B \quad \text{and} \quad A \neq B$$

Ishibuchi and Tanaka [20] suggested another order relation $\leq_{mw}$ where, $\leq_{LR}$ cannot be applied, as follows:

$$A \leq_{mw} B \qquad \text{if and only if} \quad m(A) \leq m(B) \quad \text{and} \quad w(A) \geq w(B),$$
$$A <_{mw} B \qquad \text{if and only if} \quad A \leq_{mw} B \quad \text{and} \quad A \neq B.$$

Both of the above order relations $\leq_{LR}$ and $\leq_{mw}$ are anti-symmetric, reflexive, and transitive, and hence, define a partial order relation between intervals. Ishibuchi and Tanaka [20] showed that both of the order relations never conflict in the sense that there exists no such pair of $A$ and $B$ $(A \neq B)$ so that $A \leq_{LR} B$ and $B \leq_{mw} A$ hold. However, in a recent work, Sengupta and Pal [32] showed that there exists a set of pairs of intervals for which both $\leq_{LR}$ and $\leq_{mw}$ do not hold. Also, Sengupta et al. [22] proposed a method in order to compare two interval numbers based on an acceptability index (see Appendix A).

According to Sengupta et al.'s approach [22], there exists an important point about the acceptability index to be noted: the concept of the acceptability index for the comparison of intervals can in no way be treated as analogous to the concept of 'difference' of real analysis. For this reason, considering a superior reference interval $D^*$ for choosing a preferred maximizing alternative between two equi-centred intervals $B_1$ and $B_2$, but not identical, or considering an inferior reference $D^*$ for choosing a preferred minimizing alternative between $B_1$ and $B_2$ makes no sense and yields nothing. Now, we propose a satisfaction function to compare any two interval numbers on the real line by the decision-maker's satisfaction. This function does not have the drawback of the acceptability function of $\Im$.

## 4. SATISFACTION FUNCTION

In this section, we use the applied notions by Tseng and Klein [44] (see Appendix B) and define a Satisfaction Function (SF) in order to compare two interval values in Subsection a. Then, we study the comparison between an interval and a crisp value by the SF. Finally in Subsection c, we consider the comparison between two crisp values and show that the SF retains the comparison results of the existing crisp values in R.

### *a) Comparison between two interval values*

The satisfaction function $S$ is defined as follows:

**Definition 1**. The satisfaction function $S$ for the interval numbers $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ $(S(A < B),$ $S(A > B), \textit{and } S(A = B))$ is defined as follows:

$$S(A < B) = \frac{\mu(\{x \in A | x < y \quad \forall y \in B\}) + \mu(\{x \in B | x > y \quad \forall y \in A\})}{\mu(A) + \mu(B)}, \tag{1}$$

$$S(A > B) = \frac{\mu(\{x \in A | x > y \quad \forall y \in B\}) + \mu(\{x \in B | x < y \quad \forall y \in A\})}{\mu(A) + \mu(B)}, \tag{2}$$

$$S(A = B) = \frac{\mu(\{x \in A | x = y \quad \exists y \in B\}) + \mu(\{x \in B | x = y \quad \exists y \in A\})}{\mu(A) + \mu(B)}, \tag{3}$$

where $\mu(A)+\mu(B)\neq 0$ .

For example, the satisfaction degree of the DM of $A<B$ may be interpreted on the basis of the subset geometric length of elements $A(B)$ such that those are less (greater) than all the elements of $B(A)$. A geometric interpretation of $S(A<B)$ has been illustrated in Fig. 1. In fact, Fig. 1 illustrates $S(A<B)$ as the sum of geometric lengths of sets $A-B$ and $B-A$, divided by the sum of the geometric lengths of sets $A$ and $B$ (symbol '-' denotes the difference between two sets). We can also present similar geometric interpretations for $S(A>B)$ and $S(A=B)$.

**Remark 1.** If $S(A<B)>0$, then for a maximization problem (say $A$ and $B$ are two interval profits and the problem is to choose the maximum profit), interval $B$ is preferred to $A$ and for a minimization problem (say $A$ and $B$ are two interval costs), $A$ is preferred to $B$ in terms of value.

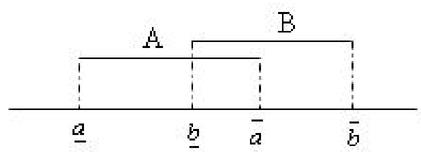



Fig. 1. $\mu(\{x\in A | x<y \quad \forall y\in B\})=\underline{b}-\underline{a}$, $\mu(\{x\in A | x=y \quad \exists y\in B\})=\bar{a}-\underline{b}$, $\mu(\{x\in B | x>y \quad \forall y\in A\})=\bar{b}-\bar{a}$

**Corollary 1.** $S(A=B)=\dfrac{2\times\mu(\{x\in A | x=y \quad \exists y\in B\})}{\mu(A)+\mu(B)}=\dfrac{2\times\mu(\{x\in B | x=y \quad \exists y\in A\})}{\mu(A)+\mu(B)}$.

**Proof:** Let $X=\{x\in A | x=y \quad \exists y\in B\}$ and $Y=\{x\in B | x=y \quad \exists y\in A\}$. We firstly prove that $X\subseteq Y$. Let $x\in X$. Then, $x\in A$ and $\exists y\in B$ such that $x=y$. Since $x=y$, we can write that $x\in B$ and $\exists y\in A$ such that $x=y$. Hence, $x\in Y$. Therefore, $X\subseteq Y$. Similarly, we can prove $Y\subseteq X$. Therefore, $X=Y$. With attention to the definition of $S(A=B)$, the proof of this corollary is completed.

$S(A<B)$ is the satisfaction degree to which the arithmetic comparison of $A<B$ is satisfied; $S(A>B)$ is to which $A>B$ is satisfied; and $S(A=B)$ is to which $A=B$. The function of $S$ has the following properties.

**Property 1.** $0\leq S(A<B)\leq 1, 0\leq S(A>B)\leq 1$, and $0\leq S(A=B)\leq 1$.

**Proof:** It is obvious from the definition of $S$.

**Property 2.** $S(A<B)=S(B>A), S(A>B)=S(B<A)$, and $S(A=B)=S(B=A)$.

**Proof:** It is obvious from the definition of $S$.

**Property 3.** $S(A=B)+S(A<B)+S(A>B)=1$.

**Proof:** It is easily seen that the following relations are true (in the following, symbol '-' denotes the difference between two sets):

$$\{x\in A | x<y \quad \forall y\in B\}\cup\{x\in A | x>y \quad \forall y\in B\}=A-B,$$

$$\{x\in A | x<y \quad \forall y\in B\}\cap\{x\in A | x>y \quad \forall y\in B\}=\varnothing,$$

$$\{x\in B | x>y \quad \forall y\in A\}\cup\{x\in B | x<y \quad \forall y\in A\}=B-A,$$

$$\{x \in B | x > y \quad \forall y \in A\} \cap \{x \in B | x < y \quad \forall y \in A\} = \varnothing,$$

$$\{x \in A | x = y \quad \exists y \in B\} = A \cap B, \text{ and } A \cup B = (A - B) \cup (A \cap B) \cup (B - A).$$

Using the above relations and properties $\mu$, we can write:

$$S(A = B) + S(A < B) + S(A > B) =$$

$$\frac{2 \times \mu(\{x \in A | x = y \quad \exists y \in B\}) + \mu(\{x \in A | x < y \quad \forall y \in B\}) + \mu(\{x \in B | x > y \quad \forall y \in A\})}{\mu(A) + \mu(B)}$$

$$+ \frac{\mu(\{x \in A | x > y \quad \forall y \in B\}) + \mu(\{x \in B | x < y \quad \forall y \in A\})}{\mu(A) + \mu(B)}$$

$$= \frac{\mu(A - B) + \mu(B - A) + 2 \times \mu(A \cap B)}{\mu(A) + \mu(B)} = \frac{\mu(A) + \mu(B)}{\mu(A) + \mu(B)} = 1.$$

**Property 4.** For $\forall x \in A$ and $\forall y \in B$, $x < y$ if and only if $S(A < B) = 1$.

**Proof:** At first, suppose $x < y$ for $\forall x \in A$ and $\forall y \in B$. Then, $\{x \in A | x < y \quad \forall y \in B\} = A$, $\{x \in B \mid x > y \quad \forall y \in A\} = B$, and $A \cap B = \varnothing$. Hence, according to Definition 1, we obtain: $S(A < B) = \frac{\mu(A) + \mu(B)}{\mu(A) + \mu(B)} = 1$. Conversely, suppose $S(A < B) = 1$. Then, $\mu(\{x \in A | x < y \quad \forall y \in B\}) + \mu(\{x \in B | x > y \quad \forall y \in A\}) = \mu(A) + \mu(B)$. On the other hand, since $\{x \in A | x < y \quad \forall y \in B\} \subseteq A$ and $\{x \in B | x > y \quad \forall y \in A\} \subseteq B$, hence, $\mu(\{x \in A | x < y \quad \forall y \in B\}) \le \mu(A)$ and $\mu(\{x \in B | x > y \quad \forall y \in A\}) \le \mu(B)$. Thus, $\mu(\{x \in A | x < y \quad \forall y \in B\}) = \mu(A)$ and $\mu(\{x \in B | x > y \quad \forall y \in A\}) = \mu(B)$. So, we conclude that $x < y$ for all $x \in A$ and $y \in B$.

**Property 5.** For $\forall x \in A$ and $\forall y \in B$ $x > y$ if and only if $S(A > B) = 1$.

**Proof:** The proof of this property is similar to the proof of property 4.

**Property 6.** $A = B$ if and only if $S(A = B) = 1$.

**Proof:** Since $A = B$, then $\{x \in A | x = y \quad \exists y \in B\} = A \cap B = A = B$. Thus, $S(A = B) = 1$. Conversely, if $S(A = B) = 1$, then $\mu(\{x \in A | x = y \quad \exists y \in B\}) + \mu(\{x \in B | x = y \quad \exists y \in A\}) = \mu(A) + \mu(B)$ and since $\{x \in A | x = y \quad \exists y \in B\} \subseteq A$ and $\{x \in B | x = y \quad \exists y \in A\} \subseteq B$, hence, $\mu(\{x \in A | x = y \quad \exists y \in B\}) = \mu(A)$ and $\mu(\{x \in B | x = y \quad \exists y \in A\}) = \mu(B)$. Since $A$ and $B$ are two closed intervals, hence, $A = B$.

**Property 7.** If $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$, and $\underline{b} - \underline{a} = \overline{a} - \overline{b}$, then $S(A > B) = S(A < B)$.

**Proof:** If $\underline{b} - \underline{a} = \overline{a} - \overline{b} \ge 0$, then $\underline{b} \ge \underline{a}$ and $\overline{a} \ge \overline{b}$. Therefore, $A \supseteq B$. Hence, $\{x \in B | x > y \quad \forall y \in A\} = \{x \in B \mid x < y \quad \forall y \in A\} = \varnothing$, $\mu(\{x \in A | x < y \quad \forall y \in B\}) = \underline{b} - \underline{a}$, and $\mu(\{x \in A | x > y \quad \forall y \in B\}) = \overline{a} - \overline{b}$. Therefore, according to Definition 1, $S(A < B) = S(A > B)$. If $\underline{b} - \underline{a} = \overline{a} - \overline{b} < 0$, then $\underline{b} < \underline{a}$ and $\overline{a} < \overline{b}$. Therefore, $B \supseteq A$. Hence, $\{x \in A | x < y \quad \forall y \in B\} = \{x \in A | x > y \quad \forall y \in B\} = \varnothing$, $\mu(\{x \in B | x < y \quad \forall y \in A\}) = \underline{a} - \underline{b} > 0$, and $\mu(\{x \in B | x > y \quad \forall y \in A\}) = \overline{b} - \overline{a} > 0$. Consequently, from Definition 1, we get $S(A < B) = S(A > B)$.

This property explains that if there exists a sequence of equi-centred interval numbers as $\{B_i\}_{i=1}^m$, then for $\forall r, t \in \{1,2,...,m\}$ and $r \neq t$, we will have: $S(B_r < B_t) = S(B_r > B_t)$. But for some $r_1, r_2, t_1, t_2 \in \{1,2,...,m\}$ which are distinct, it is not necessary to hold the following equalities:
$S(B_{r_1} < B_{t_1}) = S(B_{r_2} < B_{t_2})$, $S(B_{r_1} < B_{t_1}) = S(B_{r_2} > B_{t_2})$, $S(B_{r_1} > B_{t_1}) = S(B_{r_2} < B_{t_2})$, and $S(B_{r_1} > B_{t_1}) = S(B_{r_2} > B_{t_2})$.

**Remark 2.** If $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, then

$$S(A < B) = \begin{cases} \dfrac{\overline{b} - \overline{a} + \underline{b} - \underline{a}}{(\overline{b} - \underline{b}) + (\overline{a} - \underline{a})} & \textit{if } \underline{b} \geq \underline{a} \textit{ and } \overline{b} \geq \overline{a} \textit{ and } \underline{b} \leq \overline{a}, \\ \dfrac{\underline{b} - \underline{a}}{(\overline{b} - \underline{b}) + (\overline{a} - \underline{a})} & \textit{if } \underline{b} \geq \underline{a} \textit{ and } \overline{b} \leq \overline{a}, \\ 1 & \textit{if } \underline{b} \geq \overline{a}, \\ \dfrac{\overline{b} - \overline{a}}{(\overline{b} - \underline{b}) + (\overline{a} - \underline{a})} & \textit{if } \overline{b} \geq \overline{a} \textit{ and } \underline{b} \leq \underline{a}, \\ 0 & \textit{otherwise}(\textit{i.e.}, \overline{b} \leq \underline{a}). \end{cases}$$

**Proof:** According to different situations between the intervals $A$ and $B$ *and* the definition of $S(A < B)$, we can easily calculate $S(A < B)$.

Now, we introduce two functions $S(A \leq B)$ and $S(A \geq B)$ as follows:

$$S(A \leq B) = S(A < B) + S(A = B) \text{ and } S(A \geq B) = S(A > B) + S(A = B).$$

For $S(A \leq B)$ and $S(A \geq B)$, the properties 1-7 are rewritten as follows (the following properties are proven similar to the properties 1-7):

**Property 8.** $0 \leq S(A \leq B) \leq 1$ and $0 \leq S(A \geq B) \leq 1$.

**Proof:** According to Property 3, $S(A \leq B) + S(A > B) = 1$. Hence, $S(A \leq B) = 1 - S(A > B)$. Since $0 \leq S(A > B) \leq 1$, then $0 \leq S(A \leq B) \leq 1$. Similarly, we can prove $0 \leq S(A \geq B) \leq 1$.

**Property 9.** $S(A \leq B) = S(B \geq A)$ and $S(A \geq B) = S(B \leq A)$.

**Proof:** It is obvious from the definitions of $S(\cdot \geq \cdot)$ and $S(\cdot \leq \cdot)$.

**Property 10.** $S(A \leq B) + S(A > B) = 1$ and $S(A < B) + S(A \geq B) = 1$.

**Proof:** This is a direct result of Property 3.

**Property 11.** For $\forall x \in A$ and $\forall y \in B$, $x \leq y$ if and only if $S(A \leq B) = 1$.

**Proof:** This property is proven similar to Property 4.

**Property 12.** For $\forall x \in A$ and $\forall y \in B$, $x \geq y$ if and only if $S(A \geq B) = 1$.

**Proof:** This property is proven similar to Property 4.

**Property 13.** If $A=[\underline{a},\overline{a}], B=[\underline{b},\overline{b}]$, and $\underline{b}-\underline{a}=\overline{a}-\overline{b}$, then $S(A\geq B)=S(A\leq B)$ .

**Proof:** This property is proven similar to Property 7.
If $\neg S(A<B)$ is defined as $S(A\geq B)$, then we conclude that: $S(\neg(A<B))=S(A\geq B)=\neg S(A<B)$.

**Remark 3.** If $A=[\underline{a},\overline{a}]$ and $B=[\underline{b},\overline{b}]$, then

$$S(A\leq B)=\begin{cases}1 & \text{if } \underline{a}\leq\underline{b} \text{ and } \overline{a}\leq\overline{b},\\ \dfrac{(\underline{b}-\underline{a})+2(\overline{b}-\underline{b})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})} & \text{if } \underline{a}\leq\underline{b} \text{ and } \overline{a}\geq\overline{b},\\ \dfrac{(\overline{b}-\overline{a})+2(\overline{a}-\underline{a})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})} & \text{if } \overline{a}\leq\overline{b} \text{ and } \underline{a}\geq\underline{b},\\ \dfrac{2(\overline{b}-\underline{a})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})} & \text{if } \underline{b}\leq\underline{a}\leq\overline{b}\leq\overline{a} \text{ and } \overline{a}\neq\underline{b},\\ 0 & \text{otherwise}(i.e., \overline{b}<\underline{a}).\end{cases}$$

**Proof:** $S(A\leq B)$ can be calculated in a similar way to Remark 2.

**Remark 4.** Suppose $A=[\underline{a},\overline{a}]$ and $B=[\underline{b},\overline{b}]$, then:

1) $S(A\geq B)>0 \iff \overline{a}>\underline{b}$, 2) $S(A>B)>0 \iff \underline{a}>\underline{b}$ *or* $\overline{a}>\overline{b}$,

3) $S(A\leq B)>0 \iff \underline{a}<\overline{b}$, 4) $S(A<B)>0 \iff \underline{a}<\underline{b}$ *or* $\overline{a}<\overline{b}$.

**Proof:** From Remarks 2 and 3, these conditions are easily proven.

**Remark 5.** (1) If $A=[\underline{a},\overline{a}]$ and $B=[\underline{b},\overline{b}]$, then

$$S(A=B)=\begin{cases}0 & \text{if } \overline{a}\leq\underline{b} \text{ or } \overline{b}\leq\underline{a},\\ \dfrac{2(\overline{a}-\underline{b})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})} & \text{if } \underline{a}\leq\underline{b}<\overline{a}\leq\overline{b},\\ \dfrac{2(\overline{b}-\underline{b})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})} & \text{if } \underline{a}\leq\underline{b}<\overline{b}\leq\overline{a},\\ \dfrac{2(\overline{b}-\underline{a})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})} & \text{if } \underline{b}\leq\underline{a}<\overline{b}\leq\overline{a},\\ \dfrac{2(\overline{a}-\underline{a})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})} & \text{if } \underline{b}\leq\underline{a}<\overline{a}\leq\overline{b}.\end{cases}$$

(2) $S(A=B)>0 \Leftrightarrow \underline{b}\leq\underline{a}<\overline{b}$ *or* $\underline{a}\leq\underline{b}<\overline{a}$ .

**Proof:** (1) $S(A=B)$ can be calculated in a similar way to Remark 2. (2) With attention to part (1) of Remark 5, this condition is easily proven.

**Remark 6.** $S(A<B)=1$ or $S(A>B)=1$ if and only if $S(A=B)=0$ .

**Proof:** This is a direct result of Property 3.

**Remark 7.** Suppose $A=[\underline{a},\overline{a}]$ and $B=[\underline{b},\overline{b}]$. Then, we have: 1) If $\underline{b}-\underline{a}=\overline{a}-\overline{b}>0$, then $S(A=B)=\frac{\overline{b}-\underline{b}}{\overline{a}-\underline{b}}=\frac{\overline{b}-\underline{b}}{\overline{b}-\underline{a}}$. 2) If $\underline{b}-\underline{a}=\overline{a}-\overline{b}<0$, then $S(A=B)=\frac{\overline{a}-\underline{a}}{\overline{a}-\underline{b}}=\frac{\overline{a}-\underline{a}}{\overline{b}-\underline{a}}$.

**Proof:** It is easily proven by Remark 5.

**Remark 8.** For any two intervals $A,B\in I$, we have: $S(A<B)>0$, or $S(A>B)>0$, or $S(A=B)>0$.

**Proof:** It is a direct result of Property 3.
Geometric interpretations of parts (1) and (2) of Remark 4 are presented in Fig. 2 and Fig. 3, respectively.

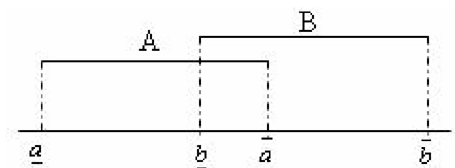
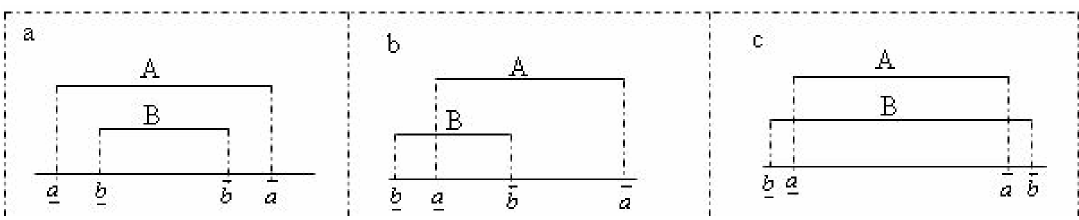


Fig. 2. $S(A\geq B)=S(A=B)=\frac{2(\overline{a}-\underline{b})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})}>0$



$\overline{a}>\overline{b}$ *and* $\underline{a}\not>\underline{b}$ $\qquad$ $\overline{a}>\overline{b}$ *and* $\underline{a}>\underline{b}$ $\qquad$ $\overline{a}\not>\overline{b}$ *and* $\underline{a}>\underline{b}$

Fig. 3. In Figs. (a), (b), and (c) according to definition $S(.>.)$: $S(A>B)>0$

For parts (3) and (4) of Remark 4, we can present similar geometric interpretations to parts (1) and (2) of Remark 4.

***b) Comparison between an interval and a crisp value***

In this subsection, we use the defined SF of the previous subsection in order to compare an interval value and a crisp value.

**Proposition 1.** If $k\in R$ and $A=[\underline{a},\overline{a}]$ where $\underline{a},\overline{a}\in R$ and $\underline{a}\neq\overline{a}$, then

$$S(k>A)=S(A<k)=\begin{cases}1 & k\geq\overline{a},\\ \frac{k-\underline{a}}{\overline{a}-\underline{a}} & \underline{a}\leq k<\overline{a},\\ 0 & k<\underline{a},\end{cases}\quad S(k<A)=S(A>k)=\begin{cases}0 & k\geq\overline{a},\\ \frac{\overline{a}-k}{\overline{a}-\underline{a}} & \underline{a}\leq k<\overline{a},\\ 1 & k<\underline{a},\end{cases}$$

and $S(k=A)=S(A=k)=0$.

**Proof:** This proposition is proven by Property 2 and the definitions of $S(A<k), S(A>k)$, and $S(A=k)$. To prove this proposition, the crisp value $k\in R$ can be written as $k=[k,k]$. First, we prove the first relation. According to Property 2, we have: $S(k>A)=S(A<k)$. Hence, we calculate $S(A<k)$. If $\underline{a}\leq k<\overline{a}$, then according to the definition of $S(A<k)$, we can write:

$$S(A<k)=\frac{\mu(\{x\in A|x<k \quad \forall k\in[k,k]\})+\mu(\{x\in[k,k]|x>y \quad \forall y\in A\})}{\mu(A)+\mu([k,k])}$$

$$=\frac{\mu(\{x\in A|x<k\})+\mu(\{x\in[k,k]|x>y \quad \forall y\in A\})}{\mu(A)}=\frac{\mu([\underline{a},k[)+\mu(\varnothing)}{\overline{a}-\underline{a}}=\frac{k-\underline{a}}{\overline{a}-\underline{a}}\ .$$

The other cases of $S(A<k)$ are similarly proven. The other relations can be proven similar to the first relation.

***c) Comparison between two crisp values***

For two crisp values $k,l\in R$, $\mu(k)+\mu(l)=0$. Hence, the defined SF in the previous section cannot be applied to compare two crisp values directly. For this reason, the crisp values $k$ and $l$ are represented by two interval values having lengths limited to 0, that is, two crisp values are considered as two interval values. Then, we can apply the function of $S$ for those values.

**Definition 2.** For two crisp values $k$ and $l$, the function of $S$ for the comparison of $k$ and $l$ are defined as follows:

$$S(k<l)=\lim_{\Delta h\to 0}S(K_\Delta<L_\Delta),\quad S(k>l)=\lim_{\Delta h\to 0}S(K_\Delta>L_\Delta),\ and\ S(k=l)=\lim_{\Delta h\to 0}S(K_\Delta=L_\Delta),$$

where for $\Delta h>0$, we have: $K_\Delta=[k-\Delta h,k+\Delta h]\ and\ L_\Delta=[l-\Delta h,l+\Delta h]$.

**Proposition 2.** For two crisp values $k$ and $l$, we have:

$$S(k<l)=\begin{cases}1 & k<l,\\ 0 & otherwise,\end{cases}\quad S(k>l)=\begin{cases}1 & k>l,\\ 0 & otherwise,\end{cases}\ and\ S(k=l)=\begin{cases}1 & k=l,\\ 0 & otherwise.\end{cases}$$

**Proof:** This proposition is proven by Definition 2. Now, we prove the first relation. From Definition 2, we can write:

$$\begin{aligned}&S(k<l)=\lim_{\Delta h\to 0}S(K_\Delta<L_\Delta)\\&=\lim_{\Delta h\to 0}\frac{\mu(\{x\in K_\Delta|x<y \quad \forall y\in L_\Delta\})+\mu(\{x\in L_\Delta|x>y \quad \forall y\in K_\Delta\})}{\mu(K_\Delta)+\mu(L_\Delta)}.\end{aligned}\tag{4}$$

With attention to the assumption $k<l$, $\exists\Delta h>0$ such that $K_\Delta\cap L_\Delta=\varnothing$. Then, we can write:

$$S(k<l)=\lim_{\Delta h\to 0}\frac{2\Delta h+2\Delta h}{4\Delta h}=1.$$

Now, if $k=l$, then according to (4) we conclude: $S(k<l)=\lim_{\Delta h\to 0}\frac{0+0}{4\Delta h}=0$,
and if $k>l$, then we have: $S(k<l)=\lim_{\Delta h\to 0}\frac{0+0}{4\Delta h}=0$.
The other relations can be proven similar to the first relation.

Proposition 2 considers the crisp values as interval values. Thus, the comparison results using Proposition 2 are the same as crisp values comparison results in $R$.

## 5. DEFINITIONS OF UPPER AND LOWER SATISFACTION FUNCTIONS

In this section, two new concepts are introduced. Also, some of their properties are studied. In the next section, we will apply these two concepts in order to interpret inequality constraints of a linear programming problem with interval coefficients.

**Definition 3.** Suppose $A$ and $B$ are two interval numbers. Then, the upper satisfaction functions $(i.e., S_U(A>B), S_U(A<B), S_U(A\geq B), \text{ and } S_U(A\leq B))$ are defined as follows:

$$S_U(A>B)=\frac{\mu(\{x\in A|x>y \quad \forall y\in B\})}{\mu(A)+\mu(B)},\ S_U(A\geq B)=\frac{\mu(\{x\in A|x>y \quad \forall y\in B\})+\mu(\{x\in A|x=y \quad \exists y\in B\})}{\mu(A)+\mu(B)},$$

$$S_U(A<B)=\frac{\mu(\{x\in B|x>y \quad \forall y\in A\})}{\mu(A)+\mu(B)},\ S_U(A\leq B)=\frac{\mu(\{x\in B|x>y \quad \forall y\in A\})+\mu(\{x\in B|x=y \quad \exists y\in A\})}{\mu(A)+\mu(B)}.$$

Also, the lower satisfaction functions $(i.e., S_L(A>B), S_L(A<B), S_L(A\geq B)$, and $S_L(A\leq B))$ are defined as follows:

$$S_L(A>B)=\frac{\mu(\{x\in B|x<y \quad \forall y\in A\})}{\mu(A)+\mu(B)},\ S_L(A\geq B)=\frac{\mu(\{x\in B|x<y \quad \forall y\in A\})+\mu(\{x\in B|x=y \quad \exists y\in A\})}{\mu(A)+\mu(B)},$$

$$S_L(A<B)=\frac{\mu(\{x\in A|x<y \quad \forall y\in B\})}{\mu(A)+\mu(B)},\ S_L(A\leq B)=\frac{\mu(\{x\in A|x<y \quad \forall y\in B\})+\mu(\{x\in A|x=y \quad \exists y\in B\})}{\mu(A)+\mu(B)}.$$

According to Definition 3, we can simply obtain the following corollaries.

**Corollary 2.** $S_U(A>B)=S_U(B<A)$ and $S_U(A\geq B)=S_U(B\leq A)$.

**Proof:** It is obvious from Definition 3.

**Corollary 3.** $0\leq S_U(A>B)\leq 1$ and $0\leq S_U(A\geq B)\leq 1$.

**Proof:** It is obvious that $\{x\in A|x>y \quad \forall y\in B\}\subseteq A$, so $\mu(\{x\in A|x>y \quad \forall y\in B\})\leq \mu(A)$. Hence, $0\leq S_U(A>B)\leq 1$. In a similar way to the proof of Property 3, we can write $\mu(\{x\in A|x=y \quad \exists y\in B\})=\mu(A\cap B)$. Since $A\cap B\subseteq B$, then $\mu(A\cap B)\leq \mu(B)$. On the other hand, $\mu(\{x\in A|x>y \quad \forall y\in B\})\leq \mu(A)$. According to the recent two relations, we deduce $0\leq S_U(A\geq B)\leq 1$.

**Corollary 4.** $S_L(A>B)=S_L(B<A)$ and $S_L(A\geq B)=S_L(B\leq A)$.

**Proof:** It is obvious from Definition 3.

**Corollary 5.** $0\leq S_L(A>B)\leq 1$ and $0\leq S_L(A\geq B)\leq 1$.

**Proof:** The proof of this corollary is similar to the proof of Corollary 3.
Geometric interpretations of the concepts of upper and lower satisfaction functions are presented in Fig. 4 and Fig. 5. The following two properties are direct results of the definitions of $S_L$, $S_U$, and SF.

**Property 14.** $S(A>B)=S_U(A>B)+S_L(A>B)$ and $S(A\geq B)=S_U(A\geq B)+S_L(A\geq B)$.

**Proof:** It is a direct result of definitions $S_L$, $S_U$, and SF.

**Property 15.** $S(A<B)=S_U(A<B)+S_L(A<B)$ and $S(A\leq B)=S_U(A\leq B)+S_L(A\leq B)$.

**Proof:** It is a direct result of definitions $S_L$, $S_U$, and SF.

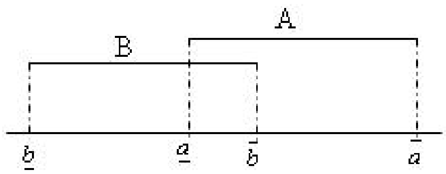



Fig. 4. $S_U(A>B)=\dfrac{\bar{a}-\bar{b}}{(\bar{a}-\underline{a})+(\bar{b}-\underline{b})}$,

$S_L(A>B)=\dfrac{\underline{a}-\underline{b}}{(\bar{a}-\underline{a})+(\bar{b}-\underline{b})}$.

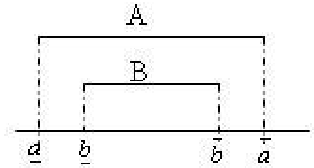



Fig. 5. $S_U(A\geq B)=\dfrac{\bar{a}-\underline{b}}{(\bar{a}-\underline{a})+(\bar{b}-\underline{b})}$,

$S_L(A\geq B)=\dfrac{\bar{b}-\underline{b}}{(\bar{a}-\underline{a})+(\bar{b}-\underline{b})}$.

**Property 16.** The functions of $S_U(.>.)$ and $S_L(.>.)$ are transitive, i.e.,
1) $\forall A,C\in I \qquad S_U(A>C)\geq Sup_{B\in I}Min\{S_U(A>B),S_U(B>C)\}$.
2) $\forall A,C\in I \qquad S_L(A>C)\geq Sup_{B\in I}Min\{S_L(A>B),S_L(B>C)\}$.
Here, $I$ is a set of overlapping intervals.

**Proof:** (1) Suppose $A=[\underline{a},\bar{a}], C=[\underline{c},\bar{c}]\in I$, and $S_U(A>C)=0$. Then, according to the definition of $S_U$, we conclude that $\bar{a}\leq\bar{c}$. On the other hand, for each $B=[\underline{b},\bar{b}]\in I$, and according to the definition of $S_U$, the following three cases are considered:
(1) If $\bar{a}\leq\bar{c}\leq\bar{b}$, then $S_U(A>B)=0$ and $S_U(B>C)\geq 0$.
(2) If $\bar{a}\leq\bar{b}\leq\bar{c}$, then $S_U(A>B)=0$ and $S_U(B>C)=0$.
(3) If $\bar{b}\leq\bar{a}\leq\bar{c}$, then $S_U(A>B)\geq 0$ and $S_U(B>C)=0$.
With attention to cases (1), (2), and (3), we conclude that $S_U(A>C)\geq Sup_{B\in I}Min\{S_U(A>B),S_U(B>C)\}$. Now, suppose $A,C\in I$ and $S_U(A>C)>0$. Then, from the definition of $S_U$, we conclude that $S_U(A>C)=\frac{\bar{a}-\bar{c}}{f}$, where $f=(\bar{a}-\underline{a})+(\bar{c}-\underline{c})$ and $\bar{a}>\bar{c}$. On the other hand, for each $B\in I$, and according to the definition of $S_U$, the following three cases are considered:
(1) If $\bar{a}>\bar{c}\geq\bar{b}$, then $S_U(B>C)=0\leq S_U(A>C)$.
(2) If $\bar{a}\geq\bar{b}\geq\bar{c}$, then $S_U(A>B)=\frac{\bar{a}-\bar{b}}{g}\leq S_U(A>C)=\frac{\bar{a}-\bar{c}}{f}$, where $g=(\bar{a}-\underline{a})+(\bar{b}-\underline{b})$, or $S_U(B>C)=\frac{\bar{b}-\bar{c}}{h}\leq S_U(A>C)=\frac{\bar{a}-\bar{c}}{f}$, where $h=(\bar{b}-\underline{b})+(\bar{c}-\underline{c})$. Because otherwise, we will have: $S_U(A>B)>S_U(A>C)$ and $S_U(B>C)>S_U(A>C)$. According to the recent relations, we have:

$$S_U(A>B)>S_U(A>C)\Rightarrow(\bar{a}-\bar{b})(\bar{a}-\underline{a})+(\bar{a}-\bar{b})(\bar{c}-\underline{c})>(\bar{a}-\bar{c})(\bar{a}-\underline{a})+(\bar{a}-\bar{c})(\bar{b}-\underline{b}),$$

$$\Rightarrow\bar{a}\bar{c}-\bar{a}\bar{b}+\underline{a}\bar{b}-\bar{a}\underline{c}+\bar{b}\underline{c}>-\bar{c}\bar{a}+\bar{c}\underline{a}+\bar{a}\bar{b}-\bar{a}\underline{b}+\bar{c}\underline{b}. \tag{5}$$

Similarly,

$$S_U(B>C)>S_U(A>C)\Rightarrow-\bar{b}\underline{a}-\bar{c}\bar{a}+\bar{c}\underline{a}+\bar{b}\bar{c}-\bar{b}\underline{c}>-\bar{a}\underline{b}-\bar{c}\bar{b}+\bar{c}\underline{b}+\bar{a}\bar{c}-\bar{a}\underline{c}. \tag{6}$$

Now, summing two inequalities (5) and (6), we get: $(\bar{b}-\underline{b})(\bar{c}-\bar{a})>0$. This contradicts the fact that $\bar{a}\geq\bar{b}\geq\bar{c}$ and $\bar{b}\geq\underline{b}$. Therefore, $S_U(A>B)\leq S_U(A>C)$ or $S_U(B>C)\leq S_U(A>C)$.
(3) If $\bar{b}\geq\bar{a}>\bar{c}$, then $S_U(A>B)=0\leq S_U(A>C)$.
From cases (1), (2), and (3), we have $\forall A,C\in I, S_U(A>C)\geq Sup_{B\in I}Min\{S_U(A>B),S_U(B>C)\}$.
(2) This part is proven similar to part (1).

**Corollary 6.** For any three intervals $A,B$, and $C$ on $R$:
1) If $S_U(A>B)>0$ *and* $S_U(B>C)>0$, then $S_U(A>C)>0$.
2) If $S_L(A>B)>0$ *and* $S_L(B>C)>0$, then $S_L(A>C)>0$.

**Proof:** (1) Since $S_U(A>B)>0$ and $S_U(B>C)>0$, $\bar{a}>\bar{b}$ and $\bar{b}>\bar{c}$. Therefore, $\bar{a}>\bar{c}$. This shows that $S_U(A>C)>0$. (2) This part is proven similar to part (1).

**Property 17.** The functions of $S_U(.>.)$ and $S_L(.>.)$ are anti-reflexive, i.e.,
1) $\forall A\in I \quad S_U(A>A)=0$. 2) $\forall A\in I \quad S_L(A>A)=0$.

**Proof:** This property is obvious from the definitions of $S_U(\cdot>\cdot)$ and $S_L(\cdot>\cdot)$.

**Property 18.** The functions of $S_U(.\geq.)$ and $S_L(.\geq.)$ are reflexive, i.e.,
1) $\forall A\in I \quad S_U(A\geq A)=1$. 2) $\forall A\in I \quad S_L(A\geq A)=1$.

**Proof:** This property is obvious from the definitions of $S_U(\cdot\geq\cdot)$ and $S_L(\cdot\geq\cdot)$.

**Property 19.** The functions of $S_U(.>.)$ and $S_L(.>.)$ are anti-symmetric, i.e., for any two interval numbers $A,B\in I$, we have:
1) If $S_U(A>B)>0$, then $S_U(B>A)=0$. 2) If $S_L(A>B)>0$, then $S_L(B>A)=0$.

**Proof:** (1) Suppose $A=[\underline{a},\bar{a}], B=[\underline{b},\bar{b}]\in I$, and $S_U(A>B)>0$. Then, from the definition of $S_U(\cdot>\cdot)$, we have: $\bar{a}>\bar{b}$. Again, applying the definition of $S_U(\cdot>\cdot)$ and $\bar{b}<\bar{a}$, we conclude that $S_U(B>A)=0$.
(2) This part is proven similar to part (1).

**Property 20.** The functions of $S_U(.>.)$ and $S_L(.>.)$ are strict order relations on the set of overlapping intervals, i.e., the functions of $S_U(.>.)$ and $S_L(.>.)$ have the properties of anti-reflexive, transitive, and anti-symmetry on the set of overlapping intervals.

**Proof:** According to Properties 16, 17, and 19, it is concluded that $S_U(.>.)$ and $S_L(.>.)$ are strict order relations on the set of overlapping intervals.

**Property 21.** For any two interval numbers $A,B\in I$ we have:
1) $S(A<B)>0 \quad\Leftrightarrow\quad S_U(A<B)>0 \quad or \quad S_L(A<B)>0$.
2) $S(A\leq B)>0 \quad\Leftrightarrow\quad S_U(A\leq B)>0 \quad or \quad S_L(A\leq B)>0$.
3) $S(A>B)>0 \quad\Leftrightarrow\quad S_U(A>B)>0 \quad or \quad S_L(A>B)>0$.
4) $S(A\geq B)>0 \quad\Leftrightarrow\quad S_U(A\geq B)>0 \quad or \quad S_L(A\geq B)>0$.

**Proof:** (1) Suppose $A=[\underline{a},\bar{a}], B=[\underline{b},\bar{b}]$, and $S(A<B)>0$. Then, according to part (2) of Remark 4, we have $\underline{b}>\underline{a}$ or $\bar{b}>\bar{a}$. If $\underline{b}>\underline{a}$, then according to the definition of $S_L(\cdot<\cdot)$, $S_L(A<B)>0$. Now, if $\bar{b}>\bar{a}$, then according to the definition of $S_U(\cdot<\cdot)$, $S_U(A<B)>0$. Conversely, if $S_U(A<B)>0$ or $S_L(A<B)>0$, then with attention to Property 15, it is easily proven that $S(A<B)>0$. Parts (2), (3), and (4) are proven similar to part (1).

Now, using the properties of SF, $S_U$, and $S_L$, we present a satisfactory crisp equivalent structure for an inequality constraint with interval coefficients. Of course, Tong [21] and Sengupta et al. [22] have presented two interpretations for the inequality constraints with interval coefficients, separately. Also, the difficulties of their approaches have been expressed in Appendix C.

## 6. LINEAR PROGRAMMING PROBLEM WITH INTERVAL COEFFICIENTS

In this section, the linear programming problem with interval coefficients is introduced as follows:

$$\begin{aligned} &\textit{Minimize} \quad Z = \sum_{j=1}^{n} \left[\underline{c}_j, \overline{c}_j\right] x_j, \\ &\textit{subject to} \quad \sum_{j=1}^{n} \left[\underline{a}_{ij}, \overline{a}_{ij}\right] x_j \geq \left[\underline{b}_i, \overline{b}_i\right] \quad \forall i = 1,\ldots,m, \\ &\qquad\qquad x_j \geq 0 \qquad \forall j = 1,\ldots,n. \end{aligned} \tag{7}$$

Two interpretations of the inequality constraints of problem (7) have been presented by Tong [21] and Sengupta et al. [22]. We briefly explain the interpretations in Appendix C. The reasons for these approaches have been given in [21, 22], respectively. Now, we will interpret the inequality constraints of problem (7) by *SF*, $S_U$ *, and* $S_L$ .
Let $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$, and $x$ be a singleton variable. According to the SF, the acceptability condition of $Ax \leq B$ may be defined as: $S(Ax \leq B) > 0$ .

Now, we will give two examples in order to illustrate our purpose before we present the interpretations of $Ax \leq B$ and $Ax \geq B$ .

1) Fig. 6 definitely satisfies the original interval inequality for $x < 4$ because $S(Ax \leq B) > 0$ . However, an optimistic DM may remain under-satisfied with the condition of $x < 4$ and for getting a greater satisfaction, it may be preferable to increase the value of $x$ to some extent that $S_U(B < Ax)$ does not pass over an assumed and fixed threshold.
2) In Fig. 7, the original interval inequality condition for $Ax \leq B$ is not denied for $x < 4.5$ because $S(Ax \leq B) > 0$ . But a pessimistic DM may not be satisfied if the right limit of $Ax$ spills over the right limit of $B$ . To attain the required level of satisfaction, the DM may even like to reduce the value of $x$ such that $\overline{a}x \leq \overline{b}$ .

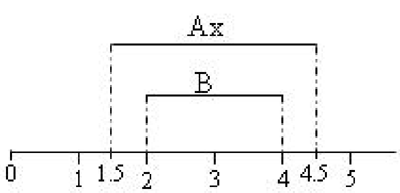



Fig. 6. $A = [1,3]$ and $B = [2,4]$
This Fig. denotes $Ax \leq B$ for $x = 1.5$ .

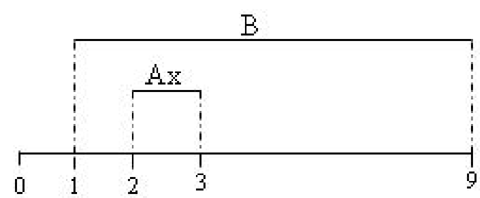



Fig. 7. $A = [2,3]$ and $B = [1,9]$
This Fig. denotes $Ax \leq B$ for $x = 1$ .

With attention to the above two remarks and the optimistic case, we propose an equivalent form of the interval inequality relation as follows:

$$Ax \leq B \Rightarrow \begin{cases} S(Ax \leq B) > 0, \\ S_U(B < Ax) \leq \alpha \in [0,1], \end{cases} \tag{8}$$

where $\alpha$ may be interpreted as an assumed and fixed optimistic threshold by the DM. Now, according to the definitions of SF, $S_U$ , relation (8), and Remark 4, we obtain a satisfactory crisp equivalent form of the interval inequality relation as follows:

$$Ax \leq B \Rightarrow \begin{cases} \underline{a}x < \overline{b} \quad \left(\textit{or equivalently} \quad \overline{b} - \underline{a}x \geq \varepsilon\right), \\ \overline{a}x - \overline{b} \leq \alpha\left(\overline{b} - \underline{b}\right) + \alpha\left(\overline{a} - \underline{a}\right)x, \end{cases}$$

where $\varepsilon > 0$ is a small positive value.

Similarly, for $Ax \geq B$ and the optimistic case, we propose an equivalent form of the interval inequality relation as follows:

$$Ax \geq B \Rightarrow \begin{cases} S(Ax \geq B) > 0, \\ S_L(B > Ax) \leq \alpha \in [0,1], \end{cases}$$

where $\alpha$ may be interpreted as an assumed and fixed optimistic threshold by the DM.
Similarly, we obtain a satisfactory crisp equivalent form of $Ax \geq B$ by the following pair:

$$\begin{cases} \overline{a}x > \underline{b} \quad \left(or\ equivalently\ \overline{a}x - \underline{b} \geq \varepsilon\right), \\ \underline{b} - \underline{a}x \leq \alpha\left(\overline{b} - \underline{b}\right) + \alpha\left(\overline{a} - \underline{a}\right)x, \end{cases}$$

where $\varepsilon > 0$ is a small positive value.

## 7. AN INTERVAL LINEAR PROGRAMMING PROBLEM AND ITS SOLUTON

Let us consider problem (7) in the previous section. As was described in the previous section, a satisfactory crisp equivalent system of the constraints of problem (7), in the optimistic case, can be generated as follows:

$$\sum_{j=1}^{n} \overline{a}_{ij} x_j > \underline{b}_i \quad \left(or\ equivalently \sum_{j=1}^{n} \overline{a}_{ij} x_j - \underline{b}_i \geq \varepsilon\right) \quad \forall i = 1,..., m,$$
$$\underline{b}_i - \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \alpha\left(\overline{b}_i - \underline{b}_i\right) + \alpha \sum_{j=1}^{n}\left(\overline{a}_{ij} - \underline{a}_{ij}\right)x_j \qquad \forall i = 1,..., m,$$

where $\alpha$ may be interpreted as an assumed and fixed optimistic threshold by the DM, and $\varepsilon > 0$ is a small positive value.
The working of SF, $S_U$ , and $S_L$ may be summarized by the following principles:

- **S-Function:** The positions (of beginning and end) of an interval and its length compared with those of another interval specify the grade to which the DM is satisfied with the superiority (inferiority) of the former compared with the latter.
- $S_U(S_L)$**-Function:** The position of end (beginning) of an interval and its length compared with those of another interval specifies the grade to which the DM is satisfied with the superiority (inferiority) of the former compared with latter.

The objective of a conventional linear programming problem is to maximize or minimize the value of its (one only, single-valued) objective function satisfying a given set of restrictions. But, a single-objective interval linear programming problem contains an interval-valued objective function. The objective function of problem (7), paying attention to the function of S, can be reduced into a linear three-objective programming problem as follows:

Min {left limit of the interval objective function},
Min {right limit of the interval objective function},
Max {length of the interval objective function},
Sub. To {set of feasibility constraints}.

The principle of function S indicates that for the minimization problem, an interval with a smaller left and right limit value is inferior to an interval with a greater left and right limit value. Hence, in order to obtain the minimum of the interval objective function, considering the left and right limit value of the interval-valued objective function is our primary concern. We reduce the interval objective function into a linear bi-objective function by its left and right limit value, i.e., the linear programming problem with an interval objective function can be reduced into a linear programming problem with a linear bi-objective function as follows:

$$\begin{gathered}\text{Min \{left limit of the interval objective function\}},\\ \text{Min \{right limit of the interval objective function\}},\\ \text{Sub. To \{set of feasibility constraints\}}.\end{gathered}\tag{9}$$

We consider the length as a secondary attribute, only to confirm whether it is within the acceptable limit of the DM. If it is not, one has to increase the extent of length (uncertainty) according to his satisfaction and thus to obtain a longer interval among non-dominated alternatives. We can obtain the non-dominated solutions via problem (9). Problem (9) can be expressed as simultaneously minimizing the left and right limit of the interval objective function. Here, a weighted function $\lambda_1\left(\sum_{j=1}^{n}\underline{c}_j x_j\right)+\lambda_2\left(\sum_{j=1}^{n}\overline{c}_j x_j\right)$ is introduced to obtain some non-dominated solutions, where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ are the weights of the left and right endpoints of $Z$, respectively, with $\lambda_1+\lambda_2=1$. Taking $\lambda_1=1$ is regarded as an optimistic opinion of minimizing $Z$ because the best situation is considered, whereas taking $\lambda_2=1$ is regarded as a pessimistic opinion because it is concerned with the worst situation. Considering that the DM is optimistic or pessimistic, we can reduce the linear bi-objective programming problem (9) into a linear programming problem, i.e., if the DM is optimistic, we will consider the following problem:

$$\begin{gathered}\text{Min \{left limit of the interval objective function\}},\\ \text{Sub. To \{set of feasibility constraints\}}.\end{gathered}$$

The above linear programming problem is the necessary equivalent form of the original problem with attention to the presented point of view. Mathematically, we have:

$$\begin{aligned}
&\textit{Minimize} && \underline{Z}=\sum_{j=1}^{n}\underline{c}_j x_j,\\
&\textit{Subject to} && \sum_{j=1}^{n}\overline{a}_{ij}x_j>\underline{b}_i \quad \left(\textit{or equivalently } \sum_{j=1}^{n}\overline{a}_{ij}x_j-\underline{b}_i\geq\varepsilon\right) \quad \forall i=1,\dots,m,\\
& && \underline{b}_i-\sum_{j=1}^{n}\underline{a}_{ij}x_j\leq\alpha\left(\overline{b}_i-\underline{b}_i\right)+\alpha\sum_{j=1}^{n}\left(\overline{a}_{ij}-\underline{a}_{ij}\right)x_j \qquad \forall i=1,\dots,m,\\
& && x_j\geq 0 \qquad \forall j=1,\dots,n,
\end{aligned}$$

where $\alpha$ may be interpreted as an assumed and fixed optimistic threshold by the DM, and $\varepsilon>0$ is a small positive value.

It is only when there exists the possibility of multiple solutions, the comparative lengths are required to be calculated and then in favor of a maximum available length, we get the solution.

## 8. NUMERICAL EXAMPLE

Let's refer to Ref. [21]. Here, there is a very good example of using interval numbers in an optimization problem:

There are 1000 raised chickens in a chicken farm and they are raised with two kinds of forages –soya and millet. It is known that each chicken eats 1.000-1.130 kg of forage every day and that for good weight gain, at least 0.21-0.23 kg of protein and 0.004-0.006 kg of calcium are needed every day. Per kg, soya contains 48-52% protein and 0.3-0.8% calcium at a price of 0.38-0.42 Yuan. Millet contains 8.5-11.5% protein and 0.3% calcium per kg at a price of 0.20 Yuan. How should the two kinds of forages be mixed in order to pay the least expense for the mixed forage?

Most of the used parameters in this problem are inexact and we can appropriately display the parameters in terms of simple intervals. Let $x_1$ kg of soya and $x_2$ kg of millet be needed in the whole chicken farm every day. Then, the optimization problem can be formulated as follows:

$$\begin{aligned}
&\textit{Minimize} \quad Z = [0.38, 0.42]x_1 + 0.2x_2,\\
&\textit{subject to} \quad x_1 + x_2 = [1, 1.130] \times 1000,\\
&\quad [0.48, 0.52]x_1 + [0.085, 0.115]x_2 \geq [0.21, 0.23] \times 1000,\\
&\quad [0.005, 0.008]x_1 + 0.003x_2 \geq [0.004, 0.006] \times 1000,\\
&\quad x_1, x_2 \geq 0.
\end{aligned}$$

This problem is solved using Tong's approach, Sengupta et al.'s approach, and the presented approach in this paper. Finally, we compare the obtained solutions by $\Im$-index and $S$-function.

1) Tong's approach gives the following solution for this problem:

$$x_1 = [x_1', x_1''] = [234.57, 1050],\ x_2 = [x_2', x_2''] = [765.43, 250],\ \text{and}\ Z^{Tong} = [242.22, 491].$$

2) The crisp equivalent form of this problem, with attention to the presented point of view by Sengupta et al. is as follows:

$$\begin{aligned}
&\textit{Min}\ m(Z) = 0.4x_1 + 0.2x_2,\\
&\textit{sub. to} \quad 1000 \leq x_1 + x_2 \leq 1130,\\
&\quad 0.48x_1 + 0.085x_2 \geq 210,\\
&\quad (1 + 0.04\alpha)x_1 + (0.2 + 0.03\alpha)x_2 \geq 440 - 20\alpha,\\
&\quad 0.005x_1 + 0.003x_2 \geq 4,\\
&\quad (0.013 + 0.003\alpha)x_1 + 0.006x_2 \geq 10 - 2\alpha,\\
&\quad x_1, x_2 \geq 0.
\end{aligned} \tag{11}$$

3) The crisp equivalent form of the problem with attention to the presented point of view in this paper is as follows:

$$\begin{aligned}
&\textit{Min}\ \underline{z} = 0.38x_1 + 0.2x_2,\\
&\textit{sub. to} \quad 1000 \leq x_1 + x_2 \leq 1130,\\
&\quad 0.52x_1 + 0.115x_2 \geq 210 + \varepsilon,\\
&\quad (0.48 + 0.04\alpha)x_1 + (0.085 + 0.03\alpha)x_2 \geq 210 - 20\alpha,\\
&\quad 0.008x_1 + 0.003x_2 \geq 4 + \varepsilon,\\
&\quad (0.005 + 0.003\alpha)x_1 + 0.003x_2 \geq 4 - 2\alpha,\\
&\quad x_1, x_2 \geq 0.
\end{aligned} \tag{12}$$

In problems (11) and (12), $\alpha \in [0,1]$ is an assumed and fixed optimistic threshold by the DM. The obtained results from solving problems (11) and (12) are presented in Tables 1 and 2 (problems (11) and (12) are solved at $\alpha = 0.1 \times h,$ where $h = 0,1,...,10$, and in problem (12), the DM assumes that $\varepsilon = 0.1$. See Tables 1 and 2).

Table 1. The obtained results from Sengupta et al.'s approach (optimistic case)

| $\alpha$ | $x_1$ | $x_2$ | $Z_\alpha^{sengupta} = \left[\underline{z}^*, \overline{z}^*\right]$ |
|---|---|---|---|
| 0 | 571.4286 | 428.5714 | [302.8571,325.7143] |
| 0.1 | 520.548 | 479.452 | [293.6986,314.5206] |
| 0.2 | 444.4445 | 592.5926 | [287.4074,305.1852] |
| 0.3 | 358.9743 | 735.0428 | [283.4188,297.7778] |
| 0.4 | 305 | 825 | [280.9,293.1] |
| 0.5 | 305 | 825 | [280.9,293.1] |
| 0.6 | 305 | 825 | [280.9,293.1] |
| 0.7 | 305 | 825 | [280.9,293.1] |
| 0.8 | 305 | 825 | [280.9,293.1] |
| 0.9 | 305 | 825 | [280.9,293.1] |
| 1 | 305 | 825.0001 | [280.9,293.1] |

Table 2. The obtained results from the presented approach in this paper (optimistic case)

| $\alpha$ | $x_1$ | $x_2$ | $Z_\alpha = \left[\underline{z}^*, \overline{z}^*\right]$ |
|---|---|---|---|
| 0 | 305 | 825.0001 | [280.9,293.1] |
| 0.1 | 293.8312 | 747.5649 | [261.1688,272.92] |
| 0.2 | 289.6725 | 710.3275 | [252.1411,263.73] |
| 0.3 | 276.3819 | 723.6181 | [249.7487,260.8] |
| 0.4 | 263.1579 | 736.8421 | [247.3684,257.89] |
| 0.5 | 250 | 750 | [245,255] |
| 0.6 | 236.9077 | 763.0923 | [242.6434,252.12] |
| 0.7 | 234.8148 | 765.1852 | [242.2667,251.66] |
| 0.8 | 234.8148 | 765.1852 | [242.2667,251.66] |
| 0.9 | 234.8148 | 765.1851 | [242.2666,251.66] |
| 1 | 234.8148 | 765.1852 | [242.2667,251.66] |

Table 3. Comparison of the obtained solutions from the three approaches using $\Im$-index

| $\alpha$ | $\Im\left(Z_\alpha^{sengupta} < Z^{Tong}\right)$ | $\Im\left(Z_\alpha < Z^{Tong}\right)$ | $\Im\left(Z_\alpha < Z_\alpha^{sengupta}\right)$ | $\Im\left(Z_\alpha^{sengupta} < Z_\alpha\right)$ |
|---|---|---|---|---|
| 0 | 0.3853 | 0.6101 | 1.5566 | - |
| 0.1 | 0.4636 | 0.7643 | 2.2758 | - |
| 0.2 | 0.5276 | 0.8348 | 2.6125 | - |
| 0.3 | 0.5777 | 0.8570 | 2.7803 | - |
| 0.4 | 0.6101 | 0.8791 | 3.0254 | - |
| 0.5 | 0.6101 | 0.9012 | 3.3333 | - |
| 0.6 | 0.6101 | 0.9233 | 3.6554 | - |
| 0.7 | 0.6101 | 0.9269 | 3.7082 | - |
| 0.8 | 0.6101 | 0.9269 | 3.7082 | - |
| 0.9 | 0.6101 | 0.9269 | 3.7082 | - |
| 1 | 0.6101 | 0.9269 | 3.7082 | - |

Since problem (10) is a minimization problem, our solutions give better expected values at $\alpha = 0.1\times h,$ where $h = 0,1,...,10$. Also, the obtained solutions from the recent two models and Tong's approach are compared using $\Im$-index and $S$-function (see Tables 3 and 4). In column 5 of Table 3, symbol '-' shows $m\left(Z_\alpha^{sengupta}\right) > m\left(Z_\alpha\right)$ for $\alpha = 0.1\times h,$ where $h = 0,1,...,10$. In this case, $\Im$-index is not defined.

Tables 3 and 4 indicate our solutions are better than both Tong's solution and Sengupta et al.'s solutions for all the values of $\alpha = 0.1\times h,$ where $h = 0,1,...,10$.

Table 4. Comparison of the obtained solutions from the three approaches using $S$ -function

| $\alpha$ | $S\left(Z_\alpha^{sengupta} < Z^{Tong}\right)$ | $S\left(Z_\alpha < Z^{Tong}\right)$ | $S\left(Z_\alpha < Z_\alpha^{sengupta}\right)$ | $S\left(Z_\alpha^{sengupta} < Z_\alpha\right)$ |
|---|---|---|---|---|
| 0 | 0.6085 | 0.7583 | 1 | 0 |
| 0.1 | 0.6546 | 0.8371 | 1 | 0 |
| 0.2 | 0.6971 | 0.8729 | 1 | 0 |
| 0.3 | 0.7343 | 0.8860 | 1 | 0 |
| 0.4 | 0.7583 | 0.8990 | 1 | 0 |
| 0.5 | 0.7583 | 0.9120 | 1 | 0 |
| 0.6 | 0.7583 | 0.9250 | 1 | 0 |
| 0.7 | 0.7583 | 0.9271 | 1 | 0 |
| 0.8 | 0.7583 | 0.9271 | 1 | 0 |
| 0.9 | 0.7583 | 0.9271 | 1 | 0 |
| 1 | 0.7583 | 0.9271 | 1 | 0 |

## 9. CONCLUSIONS

In this paper, we defined a satisfaction function to compare interval numbers. The SF was defined based on the measure of length of an interval and Tseng and Klein's idea [44]. Also, some of its properties were studied. Then, we defined the upper and lower satisfaction functions based on the SF. These functions constitute two strict order relations on the set of overlapping intervals. Also, an interval linear programming problem was introduced. Then, the functions of SF, $S_U$ , and $S_L$ were applied in order to interpret inequality constraints with interval coefficients. According to the definitions of SF, $S_U$ , and $S_L$ and their properties, the inequality constraints with interval coefficients were reduced in their satisfactory crisp equivalent forms. Furthermore, we explained the interpretation and realization of the objective of "Minimization" with respect to an inexact environment and the SF concept. Finally, a satisfactory solution of the problem for every grade $\alpha \in [0,1]$ was obtained. Also, the presented approach in this paper and the presented approaches by Sengupta et al. [22] and Tong [21], with a described good example in [21, 22] were compared.

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## APPENDIX A

Sengupta et al. [22] proposed a method in order to compare two interval numbers based on the acceptability index as follows:

**Definition A1.** Let $\prec$ be an extended order relation between the intervals of $A=[\underline{a},\bar{a}]$ and $B=[\underline{b},\bar{b}]$ on the real line $R$, then for $m(A)\le m(B)$, we construct a premise $A \prec B$ which implies that $A$ is inferior to $B$ (or $B$ is superior to $A$). Here, the term ‘inferior to’ (‘superior to’) is analogous to ‘less than’ (‘greater than’).

**Definition A2.** Let $I$ be the set of all closed intervals on the real line $R$. Here, we further define an acceptability function $\Im : I \times I \to [0,\infty)$ such that $\Im(A \prec B)$ or $\Im_{\prec}(A,B)$, or, in short, $\Im_{\prec} = \dfrac{(m(B)-m(A))}{(w(B)+w(A))}$,
where $w(A)+w(B)\neq 0$. $\Im_{\prec}$ may be interpreted as the grade of acceptability of the ‘first interval to be inferior to the second interval’.

The grade of acceptability of $A \prec B$ may be classified and interpreted further on the basis of comparative position of mean and width of interval $B$ with respect to those of interval $A$. If $\Im(A \prec B)=0$, then the premise ‘$A$ is inferior to $B$’ is not accepted. If $0<\Im(A \prec B)<1$, then the interpreter accepts the premise $(A \prec B)$ with different grades of satisfaction ranging from zero to one (excluding zero and one). If $\Im(A \prec B)\ge 1$, then the interpreter is absolutely satisfied with the premise $(A \prec B)$ or in other words, he accepts that $(A \prec B)$ is true.

## APPENDIX B

Tseng and Klein [44] used two notions of indifference and dominance in order to compare two fuzzy numbers as follows:

**Definition B1.**[44] If $A$ and $B$ are two fuzzy numbers, then $R(A,B)$ and $R(B,A)$ are two fuzzy preference relations and are defined as follows:

$$R(A,B)=\frac{(\textit{areas where } A \text{ dominates } B)+(\textit{area where } A \textit{ and } B \textit{ are indifferent})}{(\textit{area of } A)+(\textit{area of } B)},$$

$$R(B,A)=\frac{(\textit{areas where } B \text{ dominates } A)+(\textit{area where } A \textit{ and } B \textit{ are indifferent})}{(\textit{area of } A)+(\textit{area of } B)}.$$

$R(A,B)$ and $R(B,A)$ are interpreted as the degree to which $A$ is preferred to or indifferent to $B$ and $B$ is preferred to or indifferent to $A$, respectively.

## APPENDIX C

### *C1. Tong’s Approach [21]*

Tong deals with the interval inequality relations in a separate way. For the minimization problem (7), Tong first transforms each inequality constraint into $2^{n+1}$ crisp inequalities to yield: $D_i = \{D_i^k | k = 1,2,...,2^{n+1}\}$, which are the solutions of the $i$ th set of $2^{n+1}$ inequalities. On the other hand, Tong defines a characteristic formula (CF): $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$ of the $i$ th inequality relation, where $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ and $b_i \in [\underline{b}_i, \overline{b}_i]$. Now, if the $i$ th CF generates solution $D_i$ such that $D_i = \bigcup_{k=1}^{2^{n+1}} D_i^k$, then the CF is called a maximum-value of range of inequality and if the CF generates solution $D_i$ such that $D_i = \bigcap_{k=1}^{2^{n+1}} D_i^k$, then it is called a minimum-value of range of inequality. Tong then defines the minimum and the maximum of the optimal objective value of the problem using the maximum- and minimum-value of inequalities, respectively. In fact, Tong uses the union and intersection operators to define the maximum- and minimum-value of range of inequalities, respectively. The reasons of Tong's approach have been given in [21].

Sengupta et al. [22] explained the existing difficulties in using the union and intersection operators in defining the maximum- and minimum-value of range of inequalities, respectively.

### C2. Sengupta et al.'s Approach [22]

Sengupta et al. use the acceptability index [22], the Moore's concept of set-inclusion, and the points of view of optimistic and pessimistic in order to interpret the inequality constraints with interval coefficients. With attention to these concepts, they proposed a satisfactory crisp equivalent form of the interval inequality relation as follows:

$$Ax \leq B \Rightarrow \begin{cases} \overline{a}x \leq \overline{b}, \\ \Im(B \prec Ax) \leq \alpha \in [0,1], \end{cases}$$

where $\alpha$ may be interpreted as an assumed and fixed optimistic threshold by the DM. Similarly, for $Ax \geq B$, they proposed the satisfactory crisp equivalent form by the following pair:

$$\begin{aligned} &\underline{a}x \geq \underline{b}, \\ &\Im(Ax \prec B) \leq \alpha \in [0,1]. \end{aligned}$$

As Sengupta et al. [22] explained, the relation of $\Im$-index does not make any sense among equi-centred interval numbers. The reasons for Sengupta et al.'s approach have been given in [22].