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$(A_\sigma)_\Delta$-DOUBLE SEQUENCE SPACES VIA ORLICZ FUNCTIONS AND DOUBLE STATISTICAL CONVERGENCE[*]

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Abstract – The aim of this paper is to introduce and study a new concept of strong double $(A_\sigma)_\Delta$-convergence sequences with respect to an Orlicz function, and some properties of the resulting sequence spaces were also examined. In addition, we define the $(A_\sigma)_\Delta$-statistical convergence and establish some connections between the spaces of strong double $(A_\sigma)_\Delta$-convergence sequences and the space of double $(A_\sigma)_\Delta$-statistical convergence.

Keywords – Orlicz function, invariant means, almost convergence, double statistical convergence

1. INTRODUCTION AND BACKGROUND

Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a $\sigma-$mean if and only if
(i) $\varphi(x) \geq 0$ when the sequence $x=(x_k)$ is such that $x_k \geq 0$ for all k,
(ii) $\varphi(e)=1$ where e=(1,1,1,…), and
(iii) $\varphi(x)=\varphi(x_{\sigma(k)})$ for all $x \in \ell_\infty$, where ℓ_∞ is the set of bounded sequences $x=(x_k)$.
For a certain class of mapping σ, every invariant mean φ extends the limit functional on space c, in the sense that $\varphi(x)=\lim x$ for all $x \in c$ where c is the set of convergent sequences $x=(x_k)$, (see, Schaefer [1]).

The space $[V_\sigma]$ of strongly $\sigma-$convergent sequence was introduced by Mursaleen [2] as follows: A sequence $x=(x_k)$ is said to be strongly σ -convergent if there exists a number L such that

$$\frac{1}{k}\sum_{i=1}^{k}\left|x_{\sigma^i(m)}-L\right| \to \infty \tag{1.1}$$

as $k\to\infty$ uniformly in m. We will denote $[V_\sigma]$ as the set of all strongly $\sigma-$convergent sequences. When (1.1) holds we write $[V_\sigma]$- $\lim x = L$. If we let $\sigma(m)=m+1$, then $[V_\sigma]=[\hat{c}]$, which is defined by Maddox in [3]. Thus strong $\sigma-$convergence generalizes the concept of strong almost convergence sequence space.

Recall in [4] that an Orlicz function M: $[0,\infty) \to [0,\infty)$ is a continuous, convex, non-decreasing function such that $M(0)=0$ and $M(x)>0$ for $x>0$, and $M(x) \to \infty$ as $x\to\infty$.

Subsequently, the Orlicz function was used to define sequence spaces by Parashar and Choudhary [5].

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If the convexity of Orlicz function M is replaced by $M(x+y) \leq M(x)+M(y)$ then this function is called Modulus function, which was presented and discussed by Ruckle [6] and Maddox [7].

Let s'' denote the set of all double sequences of real numbers. Before proceeding further let us recall a few concepts which we shall use throughout this paper.

Definition 1.1. Let A denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the *mn-th* term to Ax is as follows:

$$(Ax)_{mn} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\, x_{k,l}.$$

By a bounded double sequence we shall mean there exists a positive number K such that $|x_{k,l}| < K$ for all (k,l), and denote such bounded by

$$\|x\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty.$$

We shall also denote the set of all bounded double sequences by ℓ''_{∞}. We also note, in contrast to the case for a single sequence, a P-convergent double sequence need not be bounded. In 1900, Pringsheim [8] presented the following definition:

Definition 1. 2. A double sequence $x = (x_{k,l})$ has a Pringsheim limit L (denoted by $P\text{-}lim\ x=L$) provided that given $\varepsilon > 0$ there exists N$\in N$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k,l > N$. We shall describe such an x more briefly as "P-convergent."

In [9] Hardy presented the notion of regularity for two dimensional matrix transformations. The definition is as follows: a two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. In addition to the numerous theorems characterizing regularity, Hardy also presented the Silverman-Toeplitz characterization of regularity. Robison, in 1926 presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [10] and [11]. The definition of the regularity for four dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four dimensional matrices.

Definition 1. 3. The four dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 1. 1. The four dimensional matrix A is RH-regular if and only if

$RH_1:\ P-lim_{m,n}\, a_{m,n,k,l} = 0$ for each k and l;

$RH_2:\ P-lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1;$

$RH_3:\ P-lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0;$ for each l;

$RH_4:\ P-lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0;$ for each k;

RH_5: $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is* $P-$*convergent; and*

RH_6: there exist positive numbers A and B such that $\sum_{k,l>B} |a_{m,n,k,l}| < A.$

Spaces of strongly summable sequences were discussed by Kuttner [12] and others. Subsequently, the class of sequences which are strongly Cesaro summable with respect to an Orlicz function was introduced and studied in [5]. In this paper, we introduce and study the concept of strong double $(A_\sigma)_\Delta -$summable with respect to an Orlicz function and also some properties of this sequence space are examined. Before we can state our main results, first we shall present the following definition by combining a four dimensional matrix transformation *A* and Orlicz function.

2. MAIN RESULTS

Definition 2. 1. Let *M* be an Orlicz function, $p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers, and $A = (a_{m,n,k,l})$ be a nonnegative RH-regular summability matrix method. We now present the following double sequence spaces:

$$w_0''(A_\sigma, M, p)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0 \right\},$$

$$w''(A_\sigma, M, p)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0, \text{ some } L \right\},$$

and

$$w_\infty''(A_\sigma, M, p)_\Delta = \left\{ x \in s'' : sup_{m,n,k,l} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\Delta_{11} x_{\sigma^{k,l}(p,q)}}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}$$

where $\Delta_{11} = sup_{r,s=1 \text{ and} \backslash \text{or } 0} \left\{ \left| x_{mn} - x_{m-r,n-s} \right| \right\}$.

Let us consider a few special cases of the above definitions.

(1) In particular, when $\sigma(p,q) = (p+1, q+1)$, we have

$$w_0''(\hat{A}, M, p)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta_{11} x_{k+p,l+q}\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0 \right\},$$

$$w''(\hat{A}, M, p)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta_{11} x_{k+p,l+q} - L\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0, \text{ some } L \right\},$$

and

$$w''_\infty(\widehat{A}, M, p)_\Delta = \left\{ x \in s'' : sup_{m,n,k,l} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta_{11} x_{k+p,l+q}\right|}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}.$$

(2) If $M(x) = x$ then we have

$$w''_0(A_\sigma, p)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|^{p_{k,l}} = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0 \right\},$$

$$w''(A_\sigma, p)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right|^{p_{k,l}} = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0, \text{ some } L \right\},$$

and

$$w''_\infty(A_\sigma, p)_\Delta = \left\{ x \in s'' : sup_{m,n,k,l} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|^{p_{k,l}} < \infty \right\}.$$

(3) If $p_{k,l} = 1$, for all *(k,l)* we have

$$w''_0(A_\sigma, M)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|}{\rho} \right) \right] = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0 \right\},$$

$$w''(A_\sigma, M)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right|}{\rho} \right) \right] = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0, \text{ some } L \right\},$$

and

$$w''_\infty(A_\sigma, M)_\Delta = \left\{ x \in s'' : sup_{m,n,k,l} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\Delta_{11} x_{\sigma^{k,l}(p,q)}}{\rho} \right) \right] < \infty \right\}.$$

(4) If we take $M(x) = x$ and $p_{k,l} = 1$, for all *(k,l)*, then we have

$$w''_0(A_\sigma)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right| = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0 \right\},$$

$$w''(A_\sigma)_\Delta = \left\{ x \in s'' : P\text{-}lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right| = 0 \quad \text{uniformly in } (p,q), \text{ for some } \rho > 0, \text{ some } L \right\},$$

and

$$w''_\infty(A_\sigma)_\Delta = \left\{ x \in s'' : sup_{m,n,k,l} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left| \Delta_{11} x_{\sigma^{k,l}(p,q)} \right| < \infty \right\}.$$

(5) If we take *A= (C,1,1)* we have

$$(w''_\sigma, M, p)_\Delta^0 = \left\{ x \in s'' : \text{P-}\lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in (p, q), for some } \rho > 0 \right\},$$

$$(w''_\sigma, M, p)_\Delta = \left\{ x \in s'' : \text{P-}\lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in (p, q), for some } \rho > 0, \text{ some L} \right\},$$

and

$$(w''_\sigma, M, p)_\Delta^\infty = \left\{ x \in s'' : \sup_{m,n,k,l} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}.$$

(6) Let us consider the following notations and definition. The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing integers sequences such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \ \ as\ r \to \infty,$$

$$l_0 = 0, h_s = k_s - k_{s-1} \to \infty \ \ as\ s \to \infty,$$

and let $\overline{h}_{r,s} = h_r h_s$, $\theta_{r,s}$ is determined by $I_{r,s} = \{(i,j) : k_{r-1} < i \le k_r \ and\ \ l_{s-1} < j \le l_s\}$.
If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{\overline{h}_{r,s}} & if\ \ (k,l) \in I_{r,s}\ ; \\ 0 & otherwise. \end{cases}$$

we have

$$(w''_\sigma, \theta, M, p)_\Delta^0 = \left\{ x \in s'' : \text{P-}\lim_{r,s} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in (p, q), for some } \rho > 0 \right\},$$

$$(w''_\sigma, \theta, M, p)_\Delta = \left\{ x \in s'' : \text{P-}\lim_{r,s} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in (p, q), for some } \rho > 0, \text{ some L} \right\},$$

and

$$(w''_\sigma, \theta, M, p)^{\infty}_{\Delta} = \left\{ x \in s'' : \sup_{\mathrm{r,s,m,n}} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}.$$

(7) As a final illustration, let

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\bar{\lambda}_{i,j}} & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in I_j = [j - \lambda_j + 1, j] \\ 0 & \text{otherwise.} \end{cases}$$

we have

$$(w''_\sigma, \lambda, M, p)^{0}_{\Delta} = \left\{ x \in s'' : \mathrm{P}\text{-}\lim_{\mathrm{i,j}} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k\in I_i, l\in I_j} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in (p, q), for some } \rho > 0 \right\},$$

$$(w''_\sigma, \lambda, M, p)^{0}_{\Delta} = \left\{ x \in s'' : \mathrm{P}\text{-}\lim_{\mathrm{i,j}} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k\in I_i, l\in I_j} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right|}{\rho} \right) \right]^{p_{k,l}} = 0 \quad \text{uniformly in (p, q), for some } \rho > 0, \text{some L} \right\},$$

and

$$(w''_\sigma, \lambda, M, p)^{\infty}_{\Delta} = \left\{ x \in s'' : \mathrm{P}\text{-}\lim_{\mathrm{i,j}} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k\in I_i, l\in I_j} \left[M\left(\frac{\left|\Delta_{11} x_{\sigma^{k,l}(p,q)}\right|}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}.$$

The following inequalities will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup p_{k,l} = H$ and let $C = max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_k\}$ and $\{b_k\}$ in the complex plane, we have

$$\left|a_{k,l} + b_{k,l}\right|^{p_{k,l}} \leq C(\left|a_{k,l}\right|^{p_{k,l}} + \left|b_{k,l}\right|^{p_{k,l}}).$$

The following theorem can be proved by using the techniques similar to those used in Theorem 1.1. of [13].

Theorem 2.1. Let $p = (p_{k,l})$ be bounded. Then $w''_0(A_\sigma, M, p)_\Delta$, $w''(A_\sigma, M, p)_\Delta$, $w''_\infty(A_\sigma, M, p)_\Delta$ are linear spaces over the set of complex numbers C.

Definition 2. 2. An Orlicz function M is said to satisfy Δ_2-condition for all values of u if there exists a constant $K>0$ such that $M(2u) \leq KM(u)$ for all $u \geq 0$. The Δ_2-condition is equivalent to the satisfaction of the following inequality $M(lu) \leq K.lM(u)$ for all values of u and for $l > 1$.

Theorem 2. 3. Let *A* be a nonnegative RH-regular summability matrix method and *M* be a Orlicz function which satisfies the Δ_2-condition. Then $w_0''(A_\sigma, p)_\Delta \subset w_0''(A_\sigma, M, p)_\Delta$, $w''(A_\sigma, p)_\Delta \subset w''(A_\sigma, M, p)_\Delta$ and $w_\infty''(A_\sigma, p)_\Delta \subset w_\infty''(A_\sigma, M, p)_\Delta$.

Proof: Let $x \in w''(A_\sigma, p)_\Delta$, then

$$s_{m,n}^{p,q} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left| \Delta_{11} x_{\sigma^{k,l}(p,q)} - L \right|^{p_{k,l}} \to 0 \tag{2.1}$$

as $m, n \to \infty$ uniformly in *(p,q)* in the Pringsheim sense. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \frac{\varepsilon}{2}$ for $0 \le t \le \delta$. Write $\Delta_{11} y_{\sigma^{k,l}(p,q)} = \left| \Delta_{11} x_{\sigma^{k,l}(p,q)} - L \right|$ and consider

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M(\Delta_{11} y_{\sigma^{k,l}(p,q)}) \right]^{p_{k,l}} = \sum_{k,l=0,0;\Delta_{11} y_{\sigma^{k,l}(p,q)} \le \delta}^{\infty,\infty} a_{m,n,k,l} \left[M(\Delta_{11} y_{\sigma^{k,l}(p,q)}) \right]^{p_{k,l}}$$

$$+ \sum_{k,l=0,0;\Delta_{11} y_{\sigma^{k,l}(p,q)} > \delta}^{\infty,\infty} a_{m,n,k,l} \left[M(\Delta_{11} y_{\sigma^{k,l}(p,q)}) \right]^{p_{k,l}}.$$

Since M is continuous we obtain

$$\sum_{k,l=0,0;\Delta_{11} y_{\sigma^{k,l}(p,q)} \le \delta}^{\infty,\infty} a_{m,n,k,l} \left[M(\Delta_{11} y_{\sigma^{k,l}(p,q)}) \right]^{p_{k,l}} \le \varepsilon^H \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}$$

and for $\Delta_{11} y_{\sigma^{k,l}(p,q)} > \delta$ we have the fact that

$$\Delta_{11} y_{\sigma^{k,l}(p,q)} < \frac{\Delta_{11} y_{\sigma^{k,l}(p,q)}}{\delta} < \left[1 + \frac{\Delta_{11} y_{\sigma^{k,l}(p,q)}}{\delta} \right]$$

where [t] denotes the integer part of *t*, and since *M* is nondecreasing and convex we have

$$\mathrm{M}(\Delta_{11} y_{\sigma^{k,l}(p,q)}) < \mathrm{M}\left(\left| 1 + \frac{\Delta_{11} y_{\sigma^{k,l}(p,q)}}{\delta} \right| \right) < \frac{M(2)}{2} + \frac{1}{2} M \left(\frac{2\Delta_{11} y_{\sigma^{k,l}(p,q)}}{\delta} \right).$$

Since M satisfies the Δ_2-condition, there exists $K \ge 1$ such that

$$M(\Delta_{11} y_{\sigma^{k,l}(p,q)}) < K \frac{\Delta_{11} y_{\sigma^{k,l}(p,q)}}{2\rho} M(2) + \frac{K\Delta_{11} y_{\sigma^{k,l}(p,q)}}{2\rho} M(2) = K \frac{\Delta_{11} y_{\sigma^{k,l}(p,q)}}{\rho} M(2).$$

Hence

$$\sum_{k,l=0,0;y_{\sigma^{k,l}(p,q)} > \delta}^{\infty,\infty} a_{m,n,k,l} \left[M(\Delta_{11} y_{\sigma^{k,l}(p,q)}) \right]^{p_{k,l}}$$

$$< max \left\{ 1, \frac{KM(2)}{\delta} \right\}^H \sum_{k,l=0,0;y_{\sigma^{k,l}(p,q)} > \delta}^{\infty,\infty} a_{m,n,k,l} \left[M(\Delta_{11} y_{\sigma^{k,l}(p,q)}) \right]^{p_{k,l}}.$$

Thus (2.1) and RH-regularity of A grants us $w''(A_\sigma, p)_\Delta \subset w''(A_\sigma, M, p)_\Delta$. Following similar arguments we can prove the following: $w''_0(A_\sigma, p)_\Delta \subset w''_0(A_\sigma, M, p)_\Delta$ and $w''_\infty(A_\sigma, p)_\Delta \subset w''_\infty(A_\sigma, M, p)_\Delta$.

Theorem 2. 4. (1) If $0 < \inf p_{k,l} \le p_{k,l} < 1$, then $w''(A_\sigma, M, p)_\Delta \subset w''(A_\sigma, M)_\Delta$
(2) If $1 < p_{k,l} \le \sup p_{k,l} < \infty$, then $w''(A_\sigma, M)_\Delta \subset w''(A_\sigma, M, p)_\Delta$.
The proof of the Theorem is similar to that of Theorem 1.1. of [14].
The following corollary follows immediately from the above theorem.

Corollary 2.1. Let *A=(C,1,1)* double Cesaro matrix and let *M* be an Orlicz function.

(1) If $p_{k,l}$ =1 for all *(k,l)* and *M* satisfies the Δ_2 - condition, then
$(w''_\sigma)^0_\Delta \subset (w''_\sigma, M)^0_\Delta$, $(w''_\sigma)_\Delta \subset (w''_\sigma, M)_\Delta$, and $(w''_\sigma)^\infty_\Delta \subset (w''_\sigma, M)^\infty_\Delta$.
(2) If $0 < \inf p_{k,l} \le p_{k,l} < 1$, then $(w''_\sigma, M, p)_\Delta \subset (w''_\sigma, M)_\Delta$.
(3) If $1 < p_{k,l} \le \sup p_{k,l} < \infty$, then $(w''_\sigma, M)_\Delta \subset (w''_\sigma, M, p)_\Delta$.

3. DOUBLE A-STATISTICAL

Natural density was generalized by Freedman and Sember in [15] by replacing C_1 with a nonnegative regular summability matrix $A = (a_{n,k})$. Thus, if *K* is a subset of *N* then the *A* –density of *K* is given by $\delta_A(K) = lim_n \sum_{k\in K} a_{n,k}$ if the limit exists. Thus, notation was used by Kolk in [16] to extend statistical convergence. In this section we define the double (A_σ, Δ)-statistical convergence and establish some connections between the spaces of strong double $(A_\sigma)_\Delta$ – convergence sequences and the space of double (A_σ, Δ)-statistical convergence. Let $K \subset NxN$ be a two-dimensional set to positive integers and let *K(m,n)* be the numbers of *(i,j)* in *K* such that $i \le n$ and $j \le m$. The two-dimensional analogues of natural density can be defined as follows: The lower asymptotic density of a set $K \subset NxN$ is defined as

$$\delta_*^2(K) = lim\ inf_{mn} \frac{K(m,n)}{mn}.$$

In the case of the double sequence $\frac{K(m,n)}{mn}$ having a limit in the Pringsheim sense, then we say that *K* has a double natural density as

$$P - lim_{mn} \frac{K(m,n)}{mn} = \delta^2(K).$$

Let $K \subset NxN$ be a two-dimensional set of positive integers, then the *A*-density of *K* is given by

$$\delta_A^2(K) = P - lim_{m,n} \sum_{(k,l)\in K} a_{m,n,k,l}$$

provided that the limit exists. The notion of double asymptotic density for double sequence was presented by Mursaleen and Edely in [17].

Definition 3.1. A double real numbers sequence *x* is said to be (A_σ, Δ) -statistically convergent to L if for every positive ε

$$\delta_A^2\left(\left\{(k,l) : \left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right| \ge \varepsilon\right\}\right) = 0$$

uniformly in *(p,q)* .

In this case we write $x_{k,l} \to L(s''(A_\sigma,\Delta))$ or $s''(A_\sigma,\Delta) - \lim x = L$ and

$$s''(A_\sigma,\Delta) = \{x : \exists L \in IR, s''(A_\sigma,\Delta) - \lim x = L\}.$$

If *A=(C,1,1),* then ($s''(A_\sigma,\Delta)$) reduces to $s''_\sigma(\Delta)$ which is defined as follows: A double real numbers sequence *x* is said to be (σ,Δ)-statistically convergent to L, if for every positive $\varepsilon > 0$ the set

$$P - lim_{m,n} \frac{1}{m,n}\left|\left\{ k \le m \ \ and \ \ l \le n: \left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right| \ge \varepsilon\right\}\right| = 0,$$

uniformly in *(p,q)* .

In this case we write $s''_\sigma(\Delta)$ *-lim x =L*. If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{\bar{h}_{r,s}} & if \ \ (k,l) \in I_{r,s}; \\ 0 & otherwise. \end{cases}$$

where the double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ and $\bar{h}_{r,s}$ are defined above. Our definition reduces to the following: A double real numbers sequence *x* is said to be lacunary (σ,Δ)-statistically convergent to L, if for every positive $\varepsilon > 0$ the set

$$P - lim_{r,s} \frac{1}{\bar{h}_{r,s}}\left|\left\{(k,l) \in I_{r,s} : \left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right| \ge \varepsilon\right\}\right| = 0,$$

uniformly in *(p,q).*

Finally, if we write

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\bar{\lambda}_{i,j}} & if \ \ k \in I_i = [i - \lambda_i + 1, i] \ \ and \ \ l \in I_j = [j - \lambda_j + 1, j] \\ 0 & otherwise. \end{cases}$$

where $\bar{\lambda}_{i,j}$ by $\lambda_i \mu_j$. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be defined above. A double real numbers sequence *x* is said to be ($\bar{\lambda}, \sigma, \Delta$)-statistically convergent to L, if for every positive $\varepsilon > 0$ the set

$$P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}}\left|\left\{k \in I_i, l \in I_j : \left|\Delta_{11} x_{\sigma^{k,l}(p,q)} - L\right| \ge \varepsilon\right\}\right| = 0,$$

uniformly in *(p,q).*

Theorem 3.1. If M is an Orlicz function and $0 < h < inf_{k,l}\, p_{k,l} \le p_{k,l} \le sup_{k,l}\, p_{k,l} = H < \infty$, then $w''(A_\sigma, M, p)_\Delta \subset s''_\sigma(A,\Delta)$.

Proof: If $x \in w''(A_\sigma, M, p)_\Delta$ then there exists ρ >0 such that

$$P\text{-}lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|}{\rho}\right)\right]^{p_{k,l}} = 0,\ uniformly\ \ in (p,q).$$

Then given $\varepsilon > 0$ and let $\varepsilon_1 = \dfrac{\varepsilon}{\rho}$, we obtain the following for each *(p,q)*,

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|}{\rho}\right)\right]^{p_{k,l}}$$

$$= \sum_{k,l=0,0;\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|\geq\varepsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|}{\rho}\right)\right]^{p_{k,l}}$$

$$+ \sum_{k,l=0,0;\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|<\varepsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|}{\rho}\right)\right]^{p_{k,l}}$$

$$\geq \sum_{k,l=0,0;\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|\geq\varepsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|}{\rho}\right)\right]^{p_{k,l}}$$

$$\geq \sum_{k,l=0,0;\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|\geq\varepsilon}^{\infty,\infty} a_{m,n,k,l}\left[M(\varepsilon_1)\right]^H$$

$$\geq \left(\min\left\{[M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\right\}\right) \sum_{k,l=0,0;\left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|\geq\varepsilon}^{\infty,\infty} a_{m,n,k,l}$$

$$\geq \left(\min\left\{[M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\right\}\right)\delta_A^2\left(\left\{(k,l): \left|\Delta_{11}x_{\sigma^{k,l}(p,q)}-L\right|\geq\varepsilon\right\}\right).$$

Hence $x \in s''_\sigma(A,\Delta)$.
The above proof can easily be modified to prove the following theorem.

Theorem 3.2. If M is bounded an Orlicz function and $0 < h < inf_{k,l}\, p_{k,l} \leq p_{k,l} \leq sup_{k,l}\, p_{k,l} = H < \infty$, then $s''_\sigma(A,\Delta) \subset w''(A_\sigma, M, p)_\Delta$.
If we let *A = (C,1,1), M(x) = x* and $p_{k,l} = 1$ for each *(k,l)* we have the following corollary:

Corollary 3.1. (i) If $x_{k,l} \to L(w''_\sigma)_\Delta$ then $x_{k,l} \to L(s''_\sigma(\Delta))$.
(ii) If $x \in \ell_\infty(\Delta)$ and $x_{k,l} \to L(s''_\sigma(\Delta))$ then $x_{k,l} \to L(w''_\sigma)_\Delta$.
(ii) $s''_\sigma(\Delta) \cap \ell''_\infty(\Delta) = (w''_\sigma)_\Delta \cap \ell''_\infty(\Delta)$ where $\ell''_\infty(\Delta) = \left\{x : \Delta_{11}x_{k,l} \in \ell''_\infty\right\}$

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