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"Research Note"

ON THE SOLVABILITY OF SOME OPERATOR-DIFFERENTIAL EQUATIONS IN COMPLEX DOMAIN*

A. M. AKHMEDOV** AND Y. Y. MUSTAFAYEVA

Baku State University, Department of Mechanics and Mathematics,
Z. Khalilov st., 23, P. O. Box Az-1148, Baku, Azerbaijan
Emails: ali_akhmedov@hotmail.com; helenmust@rambler.ru

Abstract – In the present paper an operator-differential equation of second order in complex domain is considered when the coefficients have singularity of pole type at the point z=0. A theorem of existence of the solution of the equation is proved and the spectral property of the solution is separately investigated when the coefficients are spectral operators.

Keywords – Banach algebra, operator-differential equation, spectral operators, Boolean algebra

1. SOLVABILITY OF AN OPERATOR-DIFFERENTIAL EQUATION OF SECOND ORDER

Let $L(H)$ be a Banach algebra of linear bounded operators, acting in H, where H is a Hilbert space. Consider the equation

$$\frac{d^2U}{dz^2} = \frac{1}{z}\left(\sum_{k=0}^{\infty} B_k z^k\right)\frac{dU}{dz} + \frac{1}{z^2}\left(\sum_{k=0}^{\infty} A_k z^k\right)U , \tag{1}$$

where z is complex variable, $A_k, B_k \in L(H)$ $(k = 0,1,2,...)$ and the series $\sum_{k=0}^{\infty} A_k z^k$ and $\sum_{k=0}^{\infty} B_k z^k$ are absolutely convergent in the circles $|z| < \rho_1$ and $|z| < \rho_2$, respectively. Let $\rho = \min(\rho_1, \rho_2)$ and later we will consider the problem in the circle $|z| < \rho$.

We seek the solution of (1) in the form:

$$U(z) = (\sum_{m=0}^{\infty} U_m z^n) z^R , \tag{2}$$

where the operators U_m and R will be determined later

Having calculated derivatives $\frac{dU}{dz}$ and $\frac{d^2U}{dz^2}$, putting them in (1) and applying the abstract analogy of the Frobenius method, we can write out formulas for coefficients U_m :

$$U_o(R^2 - R) - B_0 U_0 R - A_0 U_0 = 0 , \tag{3}$$

$$U_m\left[R^2 + 2mR + m(m-1)I\right] - mB_0 U_m - mB_0 U_m - B_0 U_m R - A_0 U_m = F_m , \; m = 1,2,..., \tag{4}$$

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**Corresponding author

where

$$F_m = \sum_{\substack{k+p=m-1 \\ p\neq m-1}} (p+1)B_k U_{p+1} + \sum_{\substack{k+p=m \\ p\neq m}} B_k U_p R + \sum_{\substack{k+p=m \\ p\neq m}} A_k U_p . \tag{5}$$

Let us choose U_0 bounded and such that U_0^{-1} exists and is bounded too. Let the operators A_0, B_0 and U_0 be commutative. If the operator $B_0^2 + 2B_0 + I + A_0$ is a spectral operator of scalar type, then from equation (4) for the desired operator R we obtain

$$R^2 - (B_0 + I)R - A_0 = 0, \tag{6}$$

and therefore

$$R = f(B_0) = \frac{B_0 + I + (B_0^2 + 2B_0 + I + A_0)^{\frac{1}{2}}}{2}, \tag{7}$$

$$\text{or } R = g(B_0) = \frac{B_0 + I - (B_0^2 + 2B_0 + I + A_0)^{\frac{1}{2}}}{2}. \tag{8}$$

Let $A_0 = B_0^k$, where *k* is some nonnegative integer numbers, then by theorem 3.1 from ([1], p.37) we obtain that for the solvability of equation (4) there must hold the next condition:

$$P(\lambda,\mu) = \mu^2 + (2m-1)\mu - \lambda\mu - \lambda^k - m\lambda + (m-1)m \neq 0 \tag{9}$$

for $\forall(\lambda,\mu) \in \sigma(B_0)\times\sigma(R)$ where $\sigma(B_0)$ and $\sigma(R)$ are spectrums of operators B_0 and R respectively.

Then the solution of equation (4) is determined by the formula

$$U_m = \frac{1}{4\pi^2} \int_{\Gamma_{B_0}} \int_{\Gamma_R} \frac{(B_0 - \lambda I)^{-1} F_m (R - \mu I)^{-1}}{P(\lambda,\mu)} d\mu d\lambda, \tag{10}$$

and Γ_{B_0}, Γ_R are piece-smooth contours, surrounding the spectrums of operators B_0 and *R*, respectively.

If *R* is defined by (7) then we have

$$(R - \mu I)^{-1} = (f(B_0) - \mu I)^{-1} = \frac{1}{2\pi} \int_{\Gamma_{B_0}} \frac{(B_0 - \nu I)^{-1} d\nu}{(f(\nu) - \mu)}. \tag{11}$$

Putting (11) into (10) we obtain:

$$U_m = \frac{1}{4\pi^2} \int_{\Gamma_{B_0}} \int_{\Gamma_R} \frac{(B_0 - \lambda I)^{-1} F_m (R - \mu I)^{-1}}{P(\lambda,\mu)} d\mu d\lambda = \frac{1}{4\pi^2} \int_{\Gamma_0} \int_{\Gamma_0} \frac{(B_0 - \lambda I)^{-1} F_m (B_0 - \nu I)^{-1}}{P(\lambda, f(\nu))} d\nu d\lambda. \tag{12}$$

Therefore, the solution of equation (4) is defined by formula (12), and the condition (9) now looks so:

$$P(\lambda, f(\nu)) = f(\nu)^2 + (2m-1)f(\nu) - \lambda f(\nu) - \lambda^k - m\lambda + (m-1)m \neq 0 \tag{13}$$

for arbitrary $(\lambda,\nu) \in \sigma(B_0)\times\sigma(B_0)$.

It is clear that $P(\lambda, f(\nu)) = O(m^2)$. Using this we obtain: $\|U_m\| \le \frac{c}{m^2}\|F_m\|$. It is not difficult to prove that for any ρ_1, such that $0 < \rho_1 < \rho$, and for any $m \ge 0$

$$\|U_m\|\rho_1^m \le const.$$

Then for any $\rho_2: \ \rho_1 < \rho_2 < \rho$ we have: $\sum_{n=1}^{\infty}\|U_n\|\rho_1^n = \sum_{n=1}^{\infty}\|U_n\|\rho_2^n\left(\frac{\rho_1}{\rho_2}\right)^n \le const\sum_{n=0}^{\infty}\left(\frac{\rho_1}{\rho_2}\right)^n < \infty$. Hence $\sum_{n=1}^{\infty}U_n\rho_1^n$ is convergent for $\forall\rho_1 : 0 < \rho_1 < \rho$, whence follows the existence of solution (2) of equation (1).

For the existence of solution (2) of equation (1) by the theorem 3.1 ([1], p.37), in the case of (8) the following condition must be satisfied.

$$P(\lambda, g(\nu)) = g(\nu)^2 + (2m-1)g(\nu) - \lambda g(\nu) - \lambda^k - m\lambda + (m-1)m \ne 0. \quad (14)$$

Let us point out that according to condition $A_0 = B_0^k$, for the commutative property of operators A_0, B_0 and U_0 it suffices to require the commutative property of operators B_0 and U_0. So we come to the theorem of existence.

Theorem 1. Let the next conditions be satisfied:

a) operator $B_0^2 + 2B_0 + I + A_0$ is an operator of scalar type; b) $A_0 = B_0^k$, where k is nonnegative integer; c) operators B_0 and U_0 are commutative and operator U_0^{-1} exists and is bounded; d) for $f(\nu)$, defined in (7), condition (13) holds (or for $g(\nu)$, defined in (8), condition (14) holds). Then there exists the solution of equation (1) in the form $U(z) = (\sum_{n=0}^{\infty}U_n z^n)z^R$, where operator R is defined by (7) (by (8)); at that series $\sum_{n=0}^{\infty}U_n z^n$ is absolutely convergent in the circle $|z| < \rho, \ \rho = \min(\rho_1, \rho_2)$.

2. THE CASE OF SPECTRAL COEFFICIENTS

Suppose that A_i, B_j, $i, j = 0,1,2,...$, are mutually commutative spectral operators.

The next relation is known for the resolvent of a spectral operator B_0 ([2], XV.5.2):

$$(B_0 - \lambda I)^{-1} = \sum_{n_1=0}^{\infty} N^{n_1} \int_{\sigma(B_0)} \frac{E(d\theta)}{(\lambda-\theta)^{n_1+1}}, \quad (15)$$

where N is a quasinilpotent part, E is a resolution of the identity operator B_0.

Let's put (15) in (12):

$$U_m = \sum_{n_1+n_2=0}^{\infty} N^{n_1+n_2} \int_{\Gamma_{B_0}}\int_{\Gamma_{B_0}} \frac{1}{P(\lambda, f(\nu))} \int_{\sigma(B_0)} \frac{E(d\theta)}{(\lambda-\theta)^{n_1+1}} \int_{\sigma(B_0)} \frac{E(d\eta)}{(\nu-\eta)^{n_2+1}} d\lambda d\nu F_m. \quad (16)$$

Denoting $P_1(\lambda,\nu) = P(\lambda, f(\nu))$, by Fubini's theorem we obtain

$$U_m = \left(\sum_{n_1+n_2=0}^{\infty} \frac{N^{n_1+n_2}}{n_1!n_2!} \int_{\sigma(B_0)}\int_{\sigma(B_0)} \frac{\partial^{n_1+n_2}}{\partial\theta^{n_1}\partial\eta^{n_2}}\left(\frac{1}{P_1(\theta,\eta)}\right) E(d\theta)E(d\eta)\right) F_m. \quad (17)$$

Let's denote the first factor of the product in (17) by V and consider separately its first addend:

$$V = \int_{\sigma(B_0)}\int_{\sigma(B_0)} \frac{E(d\theta)E(d\eta)}{P_1(\theta,\eta)} + \sum_{n_1+n_2=1}^{\infty} \frac{N^{n_1+n_2}}{n_1!n_2!} \int_{\sigma(B_0)}\int_{\sigma(B_0)} \frac{\partial^{n_1+n_2}}{\partial\theta^{n_1}\partial\eta^{n_2}}\left(\frac{1}{P_1(\theta,\eta)}\right) E(d\theta)E(d\eta). \quad (18)$$

Similar to work [3], we can prove that except for the first term in (18) the rest of the sum represents a quasinilpotent operator, and $\int\limits_{\sigma(B_0)}\int\limits_{\sigma(B_0)} \frac{E(d\theta)E(d\eta)}{P_1(\theta,\eta)}$ is an operator of scalar type.

From the general formula (4) for F_m it can be easily proved by induction that if U_0 is spectral and commutative with A_i, B_j, $i,j = 0,1,..$, then all the operators F_m and consequently U_m are spectral. So, the following theorem is proved:

Theorem 2. Let all the conditions of theorem 1 be satisfied. If operators A_i, B_j, $i,j = 0,1,..$, and U_0 are spectral and mutually commutative, then besides the statement of theorem 1, it is also true that operator coefficients U_m, $m = 1,2,..$, in (2) are spectral too.

3. ON A SPECTRAL SOLUTION

Let's consider the conditions under which an operator-differential equation (1) in Hilbert space has a solution being a spectral operator.

Let Ω be a complete algebra in N. Danford sense ([2], XVII.1), generated by the family of commutative spectral operators $\tau = \{U_0, A_i, B_j, j = 0,1,...\}$ and their resolutions of the identity operator, and closed in a uniform operator topology. It is clear that operators $U_m \in \Omega$ and, therefore, the finite sums $\sum_{m=0}^{n} U_m z^m \in \Omega$. As in paragraph 1 the convergence of series $\sum_{m=0}^{\infty} U_m z^m$ in a uniform operator topology was proved. Then, taking into account the closedness of algebra Ω in a uniform operator topology, the sum of series $\sum_{m=0}^{\infty} U_m z^m$ belongs to Ω, too. Suppose that the Boolean algebra generated by the resolutions of the identity of the operators of family $\tau = \{U_0, A_i, B_j, i, j = 0,1,...\}$ is bounded. Then by theorem XVII.2.14 from [2] any operator from Ω is spectral and, therefore, so is the sum $\sum_{m=0}^{\infty} U_m z^m$.

Since R is a spectral operator, then by the known theorem on an analytic function of spectral operator ([2], XV.5.6) the operator $e^{R \ln z}$ is spectral, too.

As a function of B_0 operator z^R commutates with all U_m, $m = 0,1,..$, and, therefore, with $\sum_{m=0}^{\infty} U_m z^m$. The product of two commutative spectral operators in Hilbert space is a spectral operator, so $U(z) = (\sum_{m=0}^{\infty} U_m z^m) z^R$ is a spectral operator too. So we proved the following

Theorem 3. If Boolean algebra, generated by the resolutions of the identity operator of spectral commutative operators of family $\tau = \{U_0, A_i, B_j, ,j = 0,1,...\}$ is bounded and the conditions of theorem 1 are satisfied, then equation (1) has a solution being a spectral operator.

It should be noted that the question of solvability of equation (1) was investigated in the partial case in the papers [3-6].

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