

## POSITIVE LAGRANGE POLYNOMIALS\*

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**Abstract** – In this paper we demonstrate the existence of a set of polynomials  $P_i$ ,  $1 \leq i \leq n$ , which are positive semi-definite on an interval  $[a, b]$  and satisfy, partially, the conditions of polynomials found in the Lagrange interpolation process. In other words, if  $a = a_1 < \dots < a_n = b$  is a given finite sequence of real numbers, then  $P_i(a_j) = \delta_{ij}$  ( $\delta_{ij}$  is the Kronecker delta symbol); moreover, the sum of  $P_i$ 's is identically 1.

**Keywords** – Positive polynomials, Lagrange polynomials

### 1. INTRODUCTION

All polynomials referred to in this paper belong to  $\mathbb{R}[x]$ . Let  $S$  be a subset of  $\mathbb{R}$  and  $P$ , a non-identically zero polynomial.  $P$  is said to be positive semi-definite (positive or "psd" for short) on  $S$  if  $P(x) \geq 0$  for all  $x$  in  $S$ . It is called positive definite ("pd" for short) on  $S$  if  $P(x) > 0$  on  $S$ . Representations of psd and pd polynomials when  $S$  is an interval exist in literature, see for example [1-3]. However, as we are interested in this paper to consider positive polynomials on an interval  $[a, b]$  from a different angle, there is no need for any such representations here. Suppose that the real numbers  $a = a_1 < \dots < a_n = b$  belong to the interval  $[a, b]$ . The polynomials

$$P_i = \prod_{j \neq i} (x - a_j) / \prod_{j \neq i} (a_i - a_j),$$

where  $1 \leq i \leq n$  is an integer, used in the Lagrange interpolation formula [4], satisfying the following conditions

- (I)  $P_i(a_j) = \delta_{ij}$ ,  $1 \leq i, j \leq n$ ,
- (II)  $P_i$  is of degree  $n-1$  for each  $i$ .

Actually, the set of polynomials  $\{P_i\}$  is uniquely determined by the conditions (I) and (II) above. If  $n > 2$ , then the polynomials  $P_i$  are not psd on  $[a, b]$ . Therefore, if we try to impose the condition of being psd on  $[a, b]$ , we have to somehow modify (I) and (II). As these conditions imply (II') below, we replace (II) by

$$(II') \sum_{i=1}^n P_i = 1.$$

Using the transformation  $x \mapsto (b-a)x/2 + (b+a)/2$ , we may just focus on  $[-1, 1]$  instead of working on  $[a, b]$ . Denote by  $\mathcal{A}$  the set of psd polynomials on  $[-1, 1]$ . Suppose that  $-1 = a_1 < a_2 < \dots < a_n = 1$  and  $\{P_i\}$ ,  $P_i \in \mathcal{A}$ ,  $1 \leq i \leq n$  is a set of polynomials. Then we say  $\{P_i\}$  is a set of positive Lagrange (PL for short) polynomials corresponding to  $(a_1, a_2, \dots, a_n)$  if  $\{P_i\}$  satisfies (I) and (II'). For example,

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$P_1 = x^2(1-x)/2$ ,  $P_2 = 1-x^2$ , and  $P_3 = x^2(1+x)/2$  form a set of PL polynomials corresponding to  $(-1, 0, 1)$ .

**2. EXISTENCE OF POSITIVE LAGRANGE POLYNOMIALS**

As in section 1, let  $-1 = a_1 < \dots < a_n = 1$  be a finite sequence of real numbers. Our purpose is to prove the following:

**Theorem 2.1.** Corresponding to  $(a_1, \dots, a_n)$  there exists at least one set of PL polynomials.

In order to prove the theorem we need two lemmas and some notation.

**Lemma 2.1.** Let  $a, b \in [-1, 1], a \neq b$ . Then there exists a polynomial  $g \in \mathcal{A}$  which attains a local maximum with value 1 at  $a$  and has a local minimum with value 0 at  $b$ .

**Proof:**  $g$  must be of the form  $g(x) = (x-b)^2 g_1(x)$ , where  $g_1 \in \mathcal{A}$  is a non-constant polynomial. Now it is possible to find constants  $c$  and  $d$  such that  $g(x) = c(x-b)^2(x-d)^2$  satisfies the desired properties. A direct calculation shows that  $c = (a-b)^{-4}$  and  $d = 2a-b$ , that is,

$$g(x) = \left( (x-a)^2 - (b-a)^2 \right)^2 / (b-a)^4.$$

**Remark 2.1.** In the above lemma the obtained polynomial was of degree 4. In some cases we can find a polynomial  $g$  such that it has the stated properties and  $\deg(g)$  equals 3. In fact, if  $\deg(g) = 3$  then we write  $g$  as:

$$g(x) = (x-b)^2 [m(1-x) + n(1+x)].$$

As  $g \in \mathcal{A}$ , we have  $g(1) \geq 0$  and  $g(-1) \geq 0$ . So  $g \in \mathcal{A}$  iff  $m, n \geq 0$ . Imposing the conditions at  $a$ , we obtain:

$$m = \frac{3a-b+2}{2(a-b)^3},$$

$$n = \frac{3a-b-2}{2(a-b)^3}.$$

Therefore,  $m \geq 0$  and  $n \geq 0$  iff in Fig. 1 the point  $(a, b)$  lies on the shaded area.

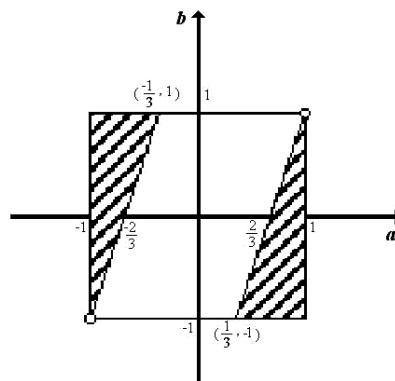


Fig. 1. Acceptable values of  $(m, n)$  lie on the shaded area

**Lemma 2.2.** For each  $i, 1 \leq i \leq n$ , there exists  $p_i \in \mathcal{A}$  having a local maximum  $p_i(a_i) = 1$  and a local minimum with value 0 at each  $a_j, 1 \leq j \leq n, j \neq i$ .

**Proof:** For each  $i$  and  $j, 1 \leq i, j \leq n, j \neq i$ , let  $g_{ij} \in \mathcal{A}$  be a polynomial such that  $g_{ij}$  has a local maximum with value 1 at  $a_i$  and a local minimum with value 0 at  $a_j$ . Define:

$$p_i = \prod_{\substack{j=1 \\ j \neq i}}^n g_{ij}.$$

Then  $p_i$  has the required properties.

**Remark 2.2.** Note that the polynomials  $g$  and  $p_i$ 's in the lemma 2.1 and the lemma 2.2 might be quite large on  $[-1, 1]$ . For example, if  $a = \frac{1}{4}, b = \frac{1}{2}$ , then  $g(x) = (16x^2 - 8x)^2$  which attains the value 576 at -1. Therefore, in the following proof of the theorem 2.1, we will multiply each  $p_i$  by a suitable power of the polynomial  $h_i$  as defined below.

For each  $i, 1 \leq i \leq n$ , let  $h_i = 1 - A_i(x - a_i)^2$ , where  $A_i = (1 + |a_i|)^{-2}$ . Note that  $h_i$  has a maximum value of 1 at  $a_i$  and  $0 \leq h_i(x) < 1$  for any  $x \in [-1, 1], x \neq a_i$ .

**The proof of Theorem 2.1.** If  $n = 2$  then  $P_1 = (1 - x)/2, P_2 = (1 + x)/2$  have the required properties. So assume  $n > 2$ . For each non-negative integer  $k$  and each integer  $i, 1 \leq i < n$ , let  $p_{i,k} = p_i h_i^k$ , where  $p_i$  is as given in the lemma 2.2 and  $h_i$  as defined just before proof of the theorem. Define the polynomial  $q_k$  by

$$q_k = \sum_{i=1}^{n-1} p_{i,k}.$$

Note that  $q_k(a_j) = 1$  for each integer  $j, 1 \leq j < n$ .

**Claim.** If  $k = k_0$  is sufficiently large, then  $q_{k_0}$  has a relative maximum with value 1 at each  $a_j$ .

**Proof of the claim:** We have  $p_i'(a_j) = 0, p_i(a_j) = \delta_{ij}$  for all  $1 \leq i, j \leq n$ , and  $h_i'(a_i) = 0$ , for all  $1 \leq i \leq n$ . Therefore,  $q_k'(a_j) = 0$  for each  $1 \leq j \leq n$ . Calculating  $q_k''(a_j)$ , we obtain

$$q_k''(a_j) = \sum_{i=1}^{n-1} p_i''(a_j) h_i^k(a_j) + k h_j''(a_j).$$

But  $0 \leq h_i(a_j) \leq 1$  and  $h_j''(a_j) < 0$  for each  $1 \leq i, j \leq n$ . So if  $k = k_0$  is sufficiently large, then  $q_k''(a_j) < 0$  for each  $1 \leq j \leq n$ . Thus the claim is proved.

Therefore, for each integer  $j, 1 \leq j < n$ , there exist real numbers  $u_j, v_j$  with  $u_j < a_j < v_j$  such that  $q_{k_0}(a_j) = 1$  is the maximum of  $q_{k_0}|_{(u_j, v_j)}$ .

For each  $x \in [-1, 1]$  with  $x \neq a_j, 1 \leq j < n$ , the sequence  $\{q_k(x)\}$  decreasingly converges to 0. On the other hand, the set

$$A = [-1, 1] \setminus \bigcup_{j=1}^{n-1} (u_j, v_j)$$

is compact. Therefore, the sequence  $\{q_k\}$  of polynomials converges uniformly to 0 on  $A$  ([5]). This means that there exists an integer  $k_1 \geq k_0$  such that  $q_{k_1}$  is bounded above by 1 on  $A$  and hence on  $[-1, 1]$ .

Now, for each  $i, 1 \leq i < n$ , let  $P_i = p_{i,k_1}$ . Then  $P_i(a_j) = \delta_{ij}$  for each  $i, j$  with  $1 \leq i < n$  and  $1 \leq j \leq n$ . Define  $P_n$  by:

$$P_n = 1 - q_{k_1} = 1 - \sum_{i=1}^{n-1} P_i.$$

Thus  $\{P_i\}_{1 \leq i \leq n}$  is a required set of polynomials.

**Example 2.1.** Let  $n = 3$ ,  $a_1 = -1$ ,  $a_2 = 1/2$ , and  $a_3 = 1$ . Then we find the polynomials  $p_1 = g_{12}g_{13}$ ,  $p_3 = g_{31}g_{32}$  which satisfy the properties stated in the lemma 2.2. Moreover, we can use the remark 2.1 so that each of  $g_{12}$ ,  $g_{13}$ ,  $g_{31}$ , and  $g_{32}$  has degree 3. In other words,

$$p_1 = g_{12}g_{13} = \frac{1}{27}(2x-1)^2(7+4x) \cdot \frac{1}{4}(x-1)^2(2+x),$$

$$p_3 = g_{31}g_{32} = \frac{1}{4}(x+1)^2(2-x) \cdot (2x-1)^2(5-4x).$$

Furthermore,

$$h_1 = 1 - \frac{1}{4}(x+1)^2,$$

$$h_3 = 1 - \frac{1}{4}(x-1)^2.$$

Then the least non-negative integer  $k$  for which we have  $p_1h_1^k + p_3h_3^k \leq 1$  on  $[-1, 1]$  is 4 (see Figs. 2 and 3).

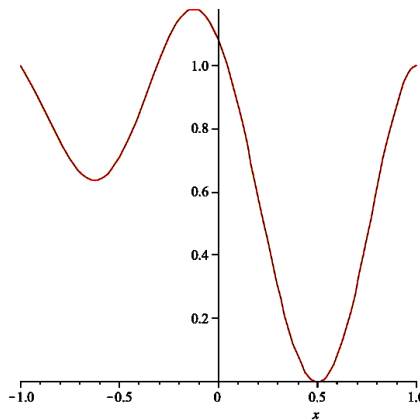


Fig. 2. The graph of  $p_1h_1^3 + p_3h_3^3$

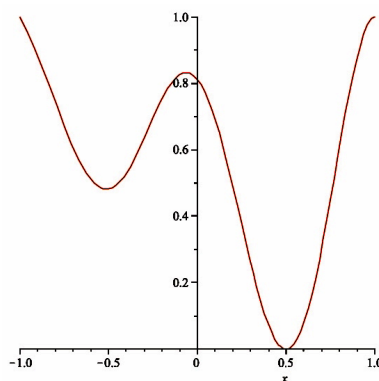


Fig. 3. The graph of  $p_1h_1^4 + p_3h_3^4$

Therefore, if we let  $P_1 = p_1 h_1^4$ ,  $P_3 = p_3 h_3^4$ , and  $P_2 = 1 - P_1 - P_3$ , then  $\{P_1, P_2, P_3\}$  is a set of PL polynomials corresponding to  $(-1, 1/2, 1)$ .

Note that corresponding to some given  $(a_1 = -1, a_2, \dots, a_n = 1)$ , it might be possible to find a set of PL polynomials with degrees less than those of the polynomials found in the theorem 2.1. For example, corresponding to  $(-1, 0, 1)$  we had a set of PL polynomials in section 1 with the degrees 2 and 3. However, if we try by using the theorem 2.1 to find a set  $\{P_i\}$  of PL polynomials, then the degrees of these polynomials would be at least 6.

**Remark 2.3.** The proof of theorem 2.1 implies that there are infinitely many sets of PL polynomials corresponding to a given  $(a_1, \dots, a_n)$ . It is possible to find a kind of minimal set: we choose the least non-negative integer  $d$  so that we have  $\deg(P_i) = d$ , for all  $i$ ,  $1 \leq i \leq n$ . However, as the polynomials  $p_i$  and  $h_i$ , for example, are not unique, there exist more than one such set  $\{P_i\}_{1 \leq i \leq n}$  of polynomials. It is an open problem to the authors as how to make a unique choice of such a set of PL polynomials in some reasonable way.

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