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Actions of S on $C_0(X)$ and ideals of $C_0(X) \times_\alpha S$

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Abstract

Some partial action properties of a group G on a C^*-algebra A are extended to an action of a unital inverse semigroup S on $C_0(X)$. Also, invariant and quotient ideals of $C_0(X) \times_\alpha S$ are considered.

Keywords: Partial action; partial homeomorphism; partial automorphism and partial crossed product

1. Introduction

The notion of monogenic inverse semigroups and their C^*-algebras was introduced by Conway, Duncan and Paterson in 1984 (Conway et al., 1984). In 1985 Duncan and Paterson considered C^*-algebra of inverse semigroups (Duncan and Paterson, 1985). During the last four decads, many authors have discussed C^*-algebras of inverse semigroups from different aspects. Among them J. Cuntz and W. Krieger discussed the C^*-algebras generated by families of partial isometries whose initial and range projections satisfy a certain condition (Cuntz and Krieger, 1980). Also, semigroup crossed products and the Toeplitz algebras of nonabelian groups (Laca and Raeburn, 1996), and a semigroup crossed product arising in number theory (Laca and Raeburn, 1999) are given by M. Laca and I. Raeburn. Non-unital semigroup crossed products (Larsen, 2000) was considered by N. Larsen while the crossed product of C^*-algebras by a unital inverse semigroup which is introduced by N. Sieben is a kind of generalization of crossed product of a C^*-algebra with a group, (Siben, 1997). Our approach is based on Sieben's theory of crossed products.

The reference (Howie, 1976) is an excellent source of information about semigroups.

Let A be a C^*-algebra. By a *partial automorphism* of A we mean a triple (α, I, J) where I and J are closed two-sided ideals in A and $\alpha: I \longrightarrow J$ is a $*$-isomorphism. If (α, I, J) and (β, K, L) are two partial automorphisms of A, then $\alpha\beta$ is nothing but the composition of α and β with the largest possible domain. Using the fact that, ideals of ideals of a C^*-algebra are themselves, ideals of that algebra, we see that the set $PAut(A)$ of partial automorphisms of A is a unital inverse semigroup.

Example 1.1. Let $\mathbb{C}^2$ be the set of all pairs with complex coordinates. It is not hard to see that, $\mathbb{C}^2$ is a C^*-algebra with the norm, multiplication and involution as follow

$$\|(c_1, c_2)\| = max\{\,|c_1|, |c_2|\,\};$$
$$(c_1, c_2)(c_1', c_2') = (c_1 c_1', c_2 c_2');$$
$$(c_1, c_2)^* = (\overline{c_1}, \overline{c_2}).$$

The group of integers, $\mathbb{Z}$, is a unital inverse semigroup. With $A = \mathbb{C}^2$ and $S = \mathbb{Z}$, define

$$E_0 = A, E_1 = \{(0, a): a \in A\}, E_{-1} = \{(a, 0): a \in A\},$$

and $E_n = \{\,(0,0)\,\}$ for all n, except $n = -1, 0, 1$. Let α_0 be the identity map on A, $\alpha_1\,((a, 0)) = (0, a)$ be the forward shift and $\alpha_n = (\alpha_1)^n$ for all $n \neq 0$. Obviously, (α_n, E_{-n}, E_n) is a partial automorphism of A.

Definition 1.2. Let S be an inverse semigroup with identity e, and A be a C^*-algebra. By an *action* of S on A, we mean a *semigroup homomorphism*

$$s \longmapsto (\alpha_s, E_{s^*}, E_s): S \longrightarrow PAut(A),$$

with $E_e = A$.

Proposition 1.3. Let A be a C^*-algebra, S be a unital inverse semigroup with unit element e and α be an action of S on A. Then we have,

(i) $\alpha_{s^*} = \alpha_s{}^{-1}$ for all s in S, α_e is the identity map on A and if s is an idempotent element of S, then α_s is the identity map on $E_s = E_{s^*}$.

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(ii) $\alpha_t(E_{t^*}E_s) = E_{ts}$ for all s, t in S.

Proof: (i) We know that α is a homomorphism. Therefore, $\alpha_s = \alpha(s) = \alpha(ss^*s) = \alpha(s)\alpha(s^*)\alpha(s) = \alpha_s\alpha_{s^*}\alpha_s$ and $\alpha_s = \alpha_s\alpha_s{}^{-1}\alpha_s$. Uniqueness of inverses in inverse semigroups (Exel, 1998), implies that $\alpha_{s^*} = \alpha_s{}^{-1}$. Moreover,

$$\alpha_e\alpha_s = \alpha_{es} = \alpha_s = \alpha_{se} = \alpha_s\,\alpha_e,$$

that is, $\alpha_e = i_A$. For an idempotent element s we have $sss = s^2 = s$ and $ss^*s = s$. By uniqueness of inverse of s we have $s = s^*$. Consequently $\alpha_s = \alpha_{s^*}$ and $E_s = E_{s^*}$. Since $\alpha_s = \alpha_{s^2} = \alpha_s\,\alpha_s = \alpha_s\alpha_{s^*} = i_{E_s}$, we observe that α_s is the identity map on $E_s = E_{s^*}$.
(ii) Since E_s and E_{t^*} are closed ideals in the C^*-algebra A we have $E_sE_{t^*} = E_s \cap E_{t^*}$. Therefore,

$$\begin{aligned}\alpha_t(E_sE_{t^*}) &= \alpha_t(E_s \cap E_{t^*}) = ran\,(\alpha_t\alpha_s) \\ &= ran\,(\alpha(t)\,\alpha(s)) \\ &= ran\,(\alpha(ts)) \\ &= ran\,(\alpha_{ts}) = E_{ts}.\end{aligned}$$

A triple (A, S, α) in which A is a C^*-algebra, S is a unital inverse semigroup and α is an action of S on A is called a *semipartial dynamical system*.

Definition 1.4. Given a semipartial dynamical system, (A, S, α), by a covariant representation of (A, S, α) we mean a triple (π, v, H) where $\pi : A \longrightarrow B(H)$ is a non-degenerate $*-$representation of A on a Hilbert space H and $v : S \longrightarrow B(H)$ is a multiplicative map such that
(i) $v_s\pi(a)\,v_{s^*} = \pi\,(\alpha_s(a))$ for all $a \in E_{s^*}$;
(ii) v_s is a partial isometry with initial space $\pi(E_{s^*})H$ and final space $\pi(E_s)H$.

It should be noted that $v_{s^*} = (v_s)^*$ and $v_e = 1_H$.
Let (A, S, α) be a semipartial dynamical system and $L_A = \{\, x \in l^1(S, A) : x(s) \in E_s \,\}$ be a closed subspace of $l^1(S, A)$. Define a multiplication and involution on L_A by

$$(x * y)(s) = \sum_{rt=s} \alpha_r\big[\alpha_{r^*}\big(x(r)\big)y(t)\big]$$

and

$$x^*(s) = \alpha_s[x(s^*)^*],$$

for $x, y \in L_A$ and $r, s, t \in S$. By Proposition 1.3. we see that $(x * y)(s) \in E_s$ for every $s \in S$. Therefore $x * y \in L_A$. Also, $x(s^*) \in E_s$ for every x in L_A, E_{s^*} is an ideal of A, $(x(s^*))^* \in E_{s^*}$ and $\alpha_s((x(s^*))^*) \in E_s$. That is, $x^* \in L_A$. Obviously, $\|x * y\| \le \|x\|\|y\|$ and $\|x^*\| = \|x\|$ where $\|.\|$ denotes the norm of L_A inherited from $l^1(S, A)$. As a result, L_A is a Banach $*$-algebra [(Sieben, 1997), prop. 4.1], and if (π, v, H) is a covariant representation of (A, S, α) then $\pi \times v$ where $\pi \times v : L_A \longrightarrow B(H)$ by $(\pi \times v)(x) = \sum_{s \in S} \pi\big(x(s)\big)v_s$ is a non-degenerate representation of L_A [(Sieben, 1997), prop. 4.3].

We close this section with the following crucial definition.

Definition 1.5. Let (A, S, α) be a semipartial dynamical system. Define a seminorm $\|.\|_c$ on L_A by

$$\|x\|_c = sup\{\|\pi \times v(x)\| : (\pi, v, H) \text{ is a covariant representation of } (A, S, \alpha)\}.$$

Let $I = \{x \in L_A : \|x\|_c = 0\}$. The *crossed product* $A \times_\alpha S$ is the C^*-algebra obtained by completing the quotient $\frac{L_A}{I}$ with respect to $\|x\|_c$.

2. On semipartial dynamical system $(C_0(X), S, \alpha)$

In this section we will mostly be concerned with $(C_0(X), S, \alpha)$ where X is a locally compact Hausdorff space and α is that action of S on $C_0(X)$ which arises from partial homeomorphisms of X, that is, for every $s \in S$ there is an open subset U_s of X and a homeomorphism $\theta_s : U_{s^*} \longrightarrow U_s$ such that $U_e = X$ and θ_e is the identity map on X. The action α of S on $C_0(X)$ *corresponding to the partial homeomorphism* θ is given by

$$\alpha_s(f)(x) = f(\theta_{s^*}(x))$$

for $s \in S$ and $f \in C_0(U_{s^*})$.

Given a unital inverse semigroup S and a locally compact Hausdorff space X, by a *topological action* of S on X we mean a pair $\theta = (\{U_s\}_{s\in S}, \{\theta_s\}_{s\in S})$, where for each s in S, U_s is an open subset of X, $\theta_s : U_{s^*} \longrightarrow U_s$ is a homeomorphism, $U_e = X$ and θ_e is the identity map on X. Let $\theta = (\{U_s\}_{s\in S}, \{\theta_s\}_{s\in S})$, be a topological action of S on X as above. Then $E_s = C_0(U_s)$ will be identified, in the usual way, with the ideal of functions in $C_0(X)$ vanishing off U_s.

The major new results of this section are theorems 2.2, 2.3 and 2.6.

Definition 2.1. The topological action θ of S on X is *topologically free* if for every $s \in S - \{e\}$ the set

$$F_s := \{x \in U_{s^*} : \theta_s(x) = x\}$$

has empty interior.

Although F_s need not be closed in X, we will show that it is closed in U_{s^*}. For this, let x be a limit point of F_s and $x \in U_{s^*}$. There exists a net $\{x_i\}$ of elements of F_s such that $x_i \longrightarrow x$. Since θ_s is a homeomorphism we have $\theta_s(x_i) \longrightarrow \theta_s(x)$. From $\theta_s(x_i) = x_i$ we see that $x_i \longrightarrow \theta_s(x)$. Uniqueness of the limit of a net shows that $\theta_s(x) = x$, that is, $x \in F_s$. This shows that F_s is closed in the domain of θ_s.

Important facts about nowhere dense sets can be found in (Goffman and Pedrick, 1991).

Theorem 2.2. The topological action θ of a unital inverse semigroup S on X is topologically free if and only if for every $s \in S - \{e\}$, the set F_s is nowhere dense.

Proof: The "if " part is trivial. For the " only if " let θ be topologically free. We know that F_s is closed relative to U_{s^*}. As a consequence $F_s = C \cap U_{s^*}$ in which C is a closed subset of X. If V is open and $V \subset \overline{F_s}$, then

$$V \cap U_{s^*} \subset \overline{F_s} \cap U_{s^*} = \overline{(C \cap U_{s^*})} \cap U_{s^*} \subseteq \bar{C} \cap U_{s^*} = C \cap U_{s^*} = F_s.$$

Since F_s has empty interior and $V \cap U_{s^*}$ is open we see that $V \cap U_{s^*} = \phi$. So the open sets U_{s^*} and V are separated. Now, since

$$V \subset \overline{F_s} = \overline{C \cap U_{s^*}} \subseteq C \cap \overline{U_{s^*}} \subset \overline{U_{s^*}}$$

we see that $V = \phi$. That is F_s is nowhere dense.

In the remainder of this work we denote by δ_s $(s \in S)$ the function in L_A which takes the value 1 at s and zero at every other element of S.

Theorem 2.3. Let $s \in S - \{e\}, f \in E_s = C_0(U_s)$, and $x_0 \notin F_s$. For every $\varepsilon > 0$ there exists $h \in C_0(X)$ such that:
(i) $h(x_0) = 1$;
(ii) $\|h(f\,\delta_s)h\| \leq \varepsilon$, and
(iii) $0 \leq h \leq 1$.

Proof: Since $x_0 \notin F_s$ let us separate the proof into two cases according to x_0 being in the domain U_s of θ_{s^*} or not. Let $x_0 \notin U_s$. From $f \in E_s$ we see that the set $K := \{x \in U_s : |f(x)| \geq \varepsilon\}$ is a closed subset of U_s and $x_0 \notin K$. So by the Urysohn's lemma there exists h in $C_0(X)$ such that $0 \leq h \leq 1$, $h(K) = 0$ and $h(x_0) = 1$.

Now since the restriction of the function h to the set U_s implies that $hf \in E_s$ by the definition of δ_s, we conclude that $(hf)\delta_s \in L_A \subseteq l^1(S, A) \subseteq C_0(X) \times_\alpha S$. So that

$$\|((hf)\delta_s)(h\delta_e)\| \leq \|hf\| = sup\{|h(x)f(x)| : x \in U_s\} = sup(\{|h(x)f(x)| : x \in K\} \cup \{|h(x)|\ |f(x)| : x \in U_s - K\}) \leq \varepsilon.$$

This shows that (ii) holds.

If $x_0 \in U_s$ then $\theta_{s^*}(x_0) \neq x_0$, since X is Hausdorff, there are disjoint open sets V_1 and V_2 such that $x_0 \in V_1 \subset U_s$ and $\theta_{s^*}(x_0) \in V_2 \subset U_{s^*}$. If $V := \theta_s(V_2) \cap V_1$, then $x_0 \in V_1$ and $\theta_{s^*}(V) \subset V_2$. Since $V_1 \cap V_2 = \phi$ we have $\theta_{s^*}(V) \cap V = \phi$. Now there exists h in $C_0(X)$ such that $0 \leq h \leq 1, h(x_0) = 1$ and $h(X - V) = 0$. Obviously,
(i) and (iii) hold. To show (ii) holds, we know that $hf\delta_s h = ((hf)\delta_s)(h\delta_e) = \alpha_s(\alpha_{s^*}(hf)h)\delta_{se} = 0$, simply because the support of $\alpha_{s^*}(hf)$ is contained in $\theta_{s^*}(V)$, the support of h is in V and $\theta_{s^*}(V) \cap V = \phi$.

Here we need to introduce the important notion of conditional expectation.

Definition 2.4. Let B be a C^* −subalgebras of a C^* −algebras A. By a conditional expectation from A to B we mean a completely positive contraction $\theta: A \longrightarrow B$ such that $\theta(b) = b, \theta(bx) = b\theta(x)$, and $\theta(xb) = \theta(x)b$ for all $x \in A,\ b \in B$.

It should be noted that the conditional expectation θ is a positive map on A. Therefore $\theta(a^*) = (\theta(a))^*$ for all $a \in A$, and it is not hard to see that the conditional expectation property from right multiplication by elements of B is a consequence of that for left multiplication and conversely (Rieffel, 1974).

Using [(Rajarama Bhat, 2000), 6.2.1] we can consider $C_0(X)$ as a C^* −subalgebra of the partial crossed product $C_0(X) \times_\alpha S$. That is the conditional expectation from $C_0(X) \times_\alpha S$ onto $C_0(X)$ is well defined, and is denoted by E. In general, conditional expectations onto subalgebras are not unique, but there are situations where conditional expectations with additional natural properties are unique [(Blackadar, 2006), II.6.10.4].

Definition 2.5. A semipartial dynamical systm (A, S, α) is said to be *topologically free* if the set of fixed points for the partial homeomorphism associated to each non-trivial semigroup element has empty interior.

Since the conditional expection $E: C_0(X) \times_\alpha S \longrightarrow C_0(X)$ is contractive we can state and prove the following theorem.

Theorem 2.6. If $(C_0(X), S, \alpha)$ is a topologically free semipartial dynamical system, then for every $c \in C_0(X) \times_\alpha S$ and every $\varepsilon > 0$ there exists $h \in C_0(X)$ such that:
(i) $\|hE(c)h\| \geq \|E(c)\| - \varepsilon$,
(ii) $\|hE(c)h - hch\| \leq \varepsilon$,
(iii) $0 \leq h \leq 1$.

Proof: Let c be a finite linear combination of the form $\sum_{t \in T} a_t\delta_t$, where T denotes a finite subset of S. Define $E(c) = a_e$ if $e \in T$ and $E(c) = 0$ if $e \notin T$. Since

$$\|a_e\| = sup\{|a_e(x)| : x \in X\},$$

for given $\varepsilon > 0$, the set $V = \{x \in X : |a_e(x)| \geq ||a_e|| - \varepsilon\}$ is a non-empty open set.

Since the topological action α is topologically free, then there exists $x_0 \in V$ such that $x_0 \notin F_t$ for every $t \in T$. Take $f_t = a_t\delta_t \in D_t$, for $\frac{\varepsilon}{|T|}$ by Theorem 2.3 there exist functions h_t such that

$$h_t(x_0) = 1,\ \|h_t(a_t\,\delta_t)\,h_t\| \leq \frac{\varepsilon}{|T|} \text{ and } 0 \leq h_t \leq 1.$$

Let $h = \prod_{t\in T-\{e\}} h_t$. Obviously $0 \leq h_t \leq 1$, that is, (iii) holds. Also (i) holds, simply because $x_0 \in V$ and

$$\begin{aligned}\|ha_eh\| &= sup\{\ h(x)a_e(x)\,h(x)\ : x \in X\}\\ &\geq |h(x_0)\,a_e(x_0)\,h(x_0)|\\ &= |\,a_e(x_0)| > \|a_e\| - \varepsilon.\end{aligned}$$

In order to prove (ii), we have

$$\begin{aligned}\|ha_eh - hch\| &= \left\| ha_eh - \sum_{t\in T} ha_t\delta_t h \right\|\\ &= \left\| \sum_{t\in T-\{e\}} ha_t\delta_t h \right\|\\ &\leq \sum_{t\in T-\{e\}} \|ha_t\delta_t h\| < |T|\frac{\varepsilon}{|T|} = \varepsilon.\end{aligned}$$

For arbitrary element c, since c is the limit of a net in $C_0(X) \times_\alpha S$ and E is contractive, a standard approaximation argument finishes the proof.

3. Invariant and quotient ideals

As before, X is a locally compact Hausdorff space, S is a unital inverse semigroup, θ is a topological action of S on X and α is the action of S on $C_0(X)$ which is corresponding to θ. Also, an ideal I in $C_0(X)$ is called *invariant* under the corresponding action α on $C_0(X)$ or simply *α-invariant* if $\alpha_s(I \cap E_{s^*}) \subseteq I$ for every s in S.

The major new results of this section are Lemma 3.2, Corollary 3.3, Theorem 3.4 and Conjecture 3.5.

Lemma 3.1. If α is an action of S on the C^* −algebra $A = C_0(X)$ and I is an α −invariant ideal of A then

$$\alpha_t\,(E_{t^*}\ \cap\ I) = E_t\ \cap\ I.$$

Proof: Obviously, $\alpha_t\,(E_{t^*}\ \cap\ I) \subseteq E_t \cap I$. Now let $y \in \alpha_t\,(E_t\ \cap\ I)$. Since $y \in E_t$, there exists x in E_{t^*} such that $y = \alpha_t(x)$. We claim that $x \in I$ and as a consequence $y = \alpha_t(x) \in \alpha_t(E_{t^*}\ \cap\ I)$. If $x \notin I$ then $x \notin (E_{t^*}\ \cap\ I)$ and $y = \alpha_t(x) \notin \alpha_t(E_{t^*}\ \cap\ I) \subset I$. That is, $y \notin I$ and it contradicts to the hypothesis.

Let α be an action of S on $A = C_0(X)$. For each invariant ideal I of A there is a restriction of α to an action of S on I. That is, if $\alpha = \{(\alpha_t,\ E_{t^*}, E_t)\}_{t\in S}$ is an action of S on A and $\alpha_t : E_{t^*} \longrightarrow E_t$ is a partial automorphism of A, then $\Theta = \{(\theta_t,\ E_{t^*} \cap I, E_t \cap I)\}_{t\in S}$ in which $\theta_t = \alpha_t|_I$ and $E_t \cap I = \theta_t\,(E_{t^*} \cap\ I)$ is an action of S on I, by Lemma 3.1. Also, $\dot{\alpha} = \{(\dot{\alpha}_t,\ \dot{E_{t^*}}, \dot{E_t})\}_{t\in S}$ in which $\dot{E_{t^*}} = \{a + I \in A/I : a \in E_{t^*}\}$ and $\dot{\alpha_t} : \dot{E_{t^*}} \longrightarrow \dot{E_t} = \alpha_t(E_{t^*}) + I$ defined by $\dot{\alpha_t}(a + I) = \alpha_t(a) + I$ is a *quotient action modulo I* of S on A/I.

Now we make an attempt to investigate the relation between the quotient of the crossed product $A \times_\alpha S$ modulo the ideal generated by I and the crossed product of $\frac{A}{I}$ by the quotient action modulo I. That is, the relation between $\frac{A \times_\alpha S}{\langle I\rangle}$ and $\frac{A}{I \times_\alpha S}$.

Lemma 3.2. Let α be an action of S on a C^* −algebra A and I be an α −invariant ideal of S, then the map from $l^1(S,I)$ to $l^1(S,A)$ induces an injection from $I \times_\alpha S$ to $A \times_\alpha S$.

Proof: Let $L_A = \{x \in l^1(S,A) : x(s) \in E_s\}$ and $L_I = \{x \in l^1(S,I) : x(s) \in E_s\}$ where in L_A the ideal E_s is an ideal of A but in L_I, the ideal E_s is an ideal of I. As we showed in (Tabatabaie Shourijeh, 2006), L_A and L_I are closed subalgebra of $l^1(S,A)$. The inclusion map from $l^1(S,I)$ into $l^1(S,A)$ maps L_I into L_A simply because if $b \in l^1(S,I)$, i.e., $b = \sum_{s\in S} a_s\,\delta_s$ where each $a_s \in E_s$, then $i(b) = b \in l^1(S,I)$. Note that we used the fact that, ideals of ideals of a C^* −algebra are, themselves, ideals of that algebra. Thus the inclusion map induces inclusion map i from $I \times_\alpha S$ to $A \times_\alpha S$. In order to prove that i is injective it is enough to show that every covariant representation of $(I,\ S,\ \alpha)$ extends to a covariant representation of $(A\ S,\ \alpha)$. Therefore, let (π, v, H) be an arbitrary covariant representation of $(I,\ S,\ \alpha)$. Since (π, H) is a representation of I without loss of generality we can assume that $\pi : I \longrightarrow B(H)$ is non-degenerate. By using [(Dixmier, 1977), Prop. 2.10.4] there exists a unique extension π' of π to a representation of A on H and we have

$$v_s\pi'(a)\,v_{s^*} = \pi'(\alpha_s(a))$$

for all $a \in E_{s^*}$. That is, (π', v, H) is a covariant representation of (A, S, α).

Corollary 3.3. If I is an α −invariant closed two-sided ideal of A then $I \times_\alpha S$ is a closed proper two-sided ideal of $A \times_\alpha S$.

Theorem 3.4. Suppose α is an action of S on A and assume I is an α −invariant ideal of A. Then the map $a\delta_s \in I \times_\alpha S \longrightarrow a\,\delta_s \in A \times_\alpha S$ extends to an injection of $I \times_\alpha S$ onto the ideal $< I >$ generated by I in $A \times_\alpha S$, and $< I > \cap A = I$.

Proof: Obviously, Lemma 3.2 and Corollary 3.3 show that $I \times_\alpha S$ injects as an ideal in $A \times_\alpha S$. Therefore, we can identify $I \times_\alpha S$ with

$$\overline{span}\{a\delta_s : a \in E_s \cap I, s \in S\}.$$

Also, we can identity I with its canonical image $I\delta_e$ in $A \times_\alpha S$. Since $< I >$ is the smallest ideal containing I we have $< I > \subseteq I \times_\alpha S$. In order to prove the reverse inclusion it suffices to show that $a\delta_s \in < I >$ for every $a \in E_s \cap I$ and $s \in S$. Therefore, let $a \in E_s \cap I$ and let b_λ be an approximate unit for the ideal E_s. Since $a\,b_\lambda\,\delta_s = (a\delta_e)(b_\lambda\,\delta_s) \in < I >$ and $a\,\delta_s = \lim_{\lambda\to\infty} a\,b_\lambda\delta_s \in < I >$ we have $I \times_\alpha S \subseteq < I >$. That is, $I \times_\alpha S = < I >$ and as a consequence $I = < I > \cap A$.

Since the map $a\delta_s \longrightarrow (a+I)\delta_s$ induces a $*-$homomorphism from $l^1(S,A)$ onto $l^1(S,A/I)$ we have the following conjecture.

Conjecture 3.5. Under the assumptions of Theorem 3.4 we have the following exact sequence.

$$0 \longrightarrow I \times_\alpha S \longrightarrow A \times_\alpha S \longrightarrow (A/I) \times_{\dot{\alpha}} S \longrightarrow 0.$$

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