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$\alpha-$ SEPARABLE AND O-TOPOLOGICAL GROUP*

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Abstract – We introduce some new concepts of topological spaces which say $\alpha - separable$ topological space and O-topological group, $\alpha -$ first axiom, $\alpha -$ second axiom, and we find some relations between them with some applications in normed spaces.

Keywords – Topological group, O-topological group, $\alpha -$ separable space, $\alpha -$ first axiom and $\alpha -$ second axiom

1. INTRODUCTION AND DEFINITIONS

In this paper, we introduce the new concepts as $\alpha -$ separable space, $\alpha -$ basis, and other concepts which are extensions of the concepts of separable space and first countablity axiom and others, and we make a new structure for a topological group called O-topological group. For these new concepts, we obtain some useful results that have previously been studied for first countablity basis axiom, second countablity basis axiom and separable space. In this paper, for topological space (X,τ), if $A \subseteq X$, then clA means the closure of A. We have some new definitions and notions which are as follows:

Let (X,τ) be a topological space and α be a cardinal number. Then, we say that X is $\alpha -$ separable space if there exists a subset $A \subset X$ such that $cardA = \alpha$, $clA = X$, and moreover, for each $\gamma < \alpha$ we are not able to find any $B \subset X$ such that $cardB = \gamma$, $clB = X$. In other words, α is the smallest cardinal number that for some $A \subset X$ we have $cardA = \alpha$ and $clA = X$.

Without loss, we generally let every $\alpha -$ separable space be separable space whenever $\alpha \leq card\aleph$, in which $\aleph$ is the set of natural numbers.

For example, the space $L^{\infty}([0,1])$ by topology induced by metric $d(f,g) = \sup_{x\in[0,1]} \left| f(x) - g(x) \right|$ is $c -$ separable space whenever $c = card\Re$. Of course, we know that $L^{\infty}([0,1])$ is not separable space. By Theorem 10 from this paper, we will generally introduce the same examples.

In this paper we use some of the following notions and terminology:

1. The notion $\leq$ refers to $<$ or $=$.

2. For a topological group space $(X,\tau,*)$, the notion $\sum_{t=1}^{m} x_t$ means $x_1 * x_2 * ..... * x_m$, and if the sequence $(\sum_{t=1}^{m} x_t)_{m\in N}$ is unbounded, then we set $\sum_{t\in N} x_t = \infty$.

Suppose that J is a net and $I_1 \underset{\neq}{\subset} I_2 \underset{\neq}{\subset} I_3 ...$ are finite subsets of J. Then we write $\sum_{t} x_t = +\infty$ if the sequence $(\sum_{t\in I_k} x_t)_{k=1}^{\infty}$ is unbounded.

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3. For every group X, the element e is the identity element of X and we define $X^{-}=\{x\in X : x<e\}$ and $X^{+}=\{x\in X : x>e\}$.

4. For a topological group space $(X,\tau,*)$, we say that $\sum_{t=1}^{\infty} x_t = x$ if the sequence $(\sum_{t=1}^{m} x_t)_{m=1}^{\infty}$ converges to x.

Let $(X,*)$ be a group and the relation $\leq$ be partially ordered on the set X. Also, let $Y\subseteq X$. We say that $\sup Y = e$ if and only if $x\leq e$ for all $x\in Y$, and if $x\leq u$ for all $x\in Y$, this implies that $e\leq u$.

More generally, $\sup Y = y_0$ if and only if $\sup\{y_0^{-1}y : y\in Y\}=e$, and in a similar way, $\inf Y = y_0$ if and only if $\sup\{y_0 y^{-1} : y\in Y\}=e$ [1,2].

Let $(X,*,\leq)$ be a commutative group and partially ordered. A set $Y\subseteq X$ is said to be directed to e if for each $y,z\in Y$ there exists $u\in Y$ such that $y\leq u$ and $z\leq u$, and moreover, $\sup Y = e$. A net $(x_\alpha)_{\alpha\in I}$ of X is said to be O-convergence to e if and only if there exists a nonempty set $Y\subset X$ which is directed to e, and for each $y\in Y$ there exists $\beta\in J$ such that $y^{-1}\leq x_\alpha\leq y$ for all $\alpha\geq\beta$. If a net $(x_\alpha)_{\alpha\in I}$ is O-converges to e, we will write $\lim_t x_t = e$.

A partially ordered set, X, is called an O-space if convergence with respect to the topology is equivalent to O-convergence [3].

Let $(X,\tau,*,\leq)$ be a topological group and O-space in which $x\leq y$ iff $e\leq x^{-1}*y$, and the space (X,τ) satisfies the least–upper bounded property. Then, $(X,\tau,*,\leq)$ is called O-topological group space.

For example, the real number $(\Re,+,\leq)$ by standard topology, and $(L^{\infty}([0,1]),+,\leq)$ by topology induced by metric $d(f,g)=\sup_{x\in[0,1]}|f(x)-g(x)|$ and ordinary operator, $+$ and the relation $\leq$ are an O-topological group.

Now, we define some new concepts in topological spaces which are extensions of the concepts of first countable basis and second countable basis, and as a consequence, we make some relations between them and $\alpha-$separable space.

Let (X,τ) be a topological space and α be a cardinal number. We say that the element $x\in X$ has $\alpha-basis$ if there exist a family $(B_i)_{i\in I}$ which is the basis at x such that $cardI=\alpha$ and for each $\gamma<\alpha$ we are not able to find any basis at x such as $(C_j)_{j\in J}$ with $cardJ=\gamma$. In other words, α is the smallest cardinal number such that the collection $(B_i)_{i\in I}$ is the basis at x with $cardI=\alpha$. If each point of X has $\alpha-basis$, then it is said that X satisfies the $\alpha-$first axiom.

For example, if $\Re$ is real numbers and $\aleph$ is natural numbers, then the space $\Re^{\aleph}$ with product topology and box topology satisfies the first countable basis and $c-$first axiom, respectively, whenever $c=card\Re$ [4].

Now, let (X,τ) be a topological space and α be a cardinal number. We say that X has $\alpha-$second axiom if there is a family $(B_i)_{i\in I}$ of basis for τ such that $cardI=\alpha$ and for all $\gamma<\alpha$ we are not able to find any basis for τ as $(C_j)_{j\in J}$ in which $cardJ=\gamma$. In other words, α is the smallest cardinal number with $cardI=\alpha$ such that the collection $(B_i)_{i\in I}$ is the basis for X.

As a natural example, let J be a set with $cardJ=\alpha$. Then the space $\Re^{J}$ with product topology and box topology satisfies the $\alpha-$second axiom and $2^{\alpha}-$second axiom, respectively.

Generally, for the above concepts, we can introduce a simple example as follows.

Let X be a set with discrete topology and $cardX=\alpha$. Then, the space X is $\alpha-$separable space and satisfies the $\alpha-$first axiom and $\alpha-$second axiom. Of course, we know that if $\alpha>card\aleph$, then the space X is not a separable space or does not satisfy the first countable basis or second countable basis.

2. THE SOME RELATIONS BETWEEN THE NEW CONCEPTS

Theorem 1. If the topological space (X,τ) satisfies the $\alpha-$second axiom, then (X,τ) is $\alpha-$separable space. The converse holds when X satisfies the $\gamma-$first axiom whenever $\gamma<\alpha$.

Proof: Let the topological space (X,τ) satisfy the $\alpha-$second axiom. Then there is the smallest basis $(B_i)_{i\in I}$ for X such that $cardI=\alpha$. Without loss of generality, we usually suppose that $B_i \cap B_j=\phi$ whenever $i\neq j$. We choose an arbitrary element $x_i \in B_i$ for all $i\in I$, and set $A=\{x_i : x_i \in B_i\}$. It is clear that A is dense in X and it is the smallest subset of X with respect to cardinal number such that $cardA=\alpha$. Consequently, X is $\alpha-$separable space.

Conversely, since X is $\alpha-$separable space, there is the smallest subset $A\subset X$ with respect to the cardinal number such that $cardA=\alpha$ and $clA=X$. We set $A=\{x_i : i\in I\}$ where $cardI=\alpha$. Since X is $\gamma-basis$, for each $i\in I$, x_i has $\gamma-basis\,(\gamma<\alpha)$ as $(B_{i\,j})_{j\in J}$. Then, the collection $(B_{i\,j})_{i\in I, j\in J}$ is the smallest basis for topological space (X,τ) with respect to the cardinal number such that $card(B_{i\,j})_{i\in I, j\in J}=\alpha$. Consequently, the space (X,τ) satisfies the $\alpha-$second axiom and the proof is complete.

In general, if the topological space (X,τ) satisfies the $\gamma-$first axiom and $\alpha-$separable space, then (X,τ) is $Max\{\alpha,\gamma\}-$second axiom.

Theorem 2. Let $(X,\tau,*.\leq)$ be an O-topological group, the relation $\leq$ be a total order on X, and $\{x_t\}_{t\in J}$ be a net in X^+ such that $\lim_t x_t = e$, $\sum_t x_t=+\infty$. If $cardJ=\alpha$, then X^+ is $\gamma-separable$ space whenever $\gamma\leq\alpha$.

Proof: Without loss of generality, we get $x_t \neq x_s$ for all $t\neq s,(t,s\in J)$. Let x be an arbitrary element of X^+. We define $X_1=\{x_t | x_t \leq x, t\in J\}$. Since the net $\{x_t\}_{t\in J}$ which is O- converges to e, then the set X_1 is not empty and has supremum, which is taken by x_{t_1}. We define $X_2, X_3,\cdots,X_{K-1}$ as $X_K=\{x_t \mid x_t < x_{t_j}, 1\leq j\leq k-1, x_t\leq x*\left(\sum_{j=1}^{k-1} x_{t_j}^{-1}\right)\}$ and we set $SupX_k = x_{t_k}$. We claim that $x=\sum_{j=1}^{k} x_{t_j}$ or $x=\sum_{j=1}^{\infty} x_{t_j}$.

If we take $\sum_{j=1}^{\infty} x_{t_j} < x$, then $\sum_{j=1}^{\infty} x_{t_j} * x_\beta < x$ holds for some $\beta\neq t_j\,(j\in N)$. So, there is $k\in N$ such that $\sum_{j=1}^{k} x_{t_j} * x_\beta < x$. We set $B=\left\{x_{t_j} \middle| x_\beta < x_{t_j}, j\in N\right\}$ in which $B\neq\phi$, and also put $MinB = x_{t_d}$. Therefore, we have $x_{t_{d+1}} < x_\beta < x_{t_d}$, and by noticing the definition of X_{d+1} we have $x_\beta \in X_{d+1}$. So we conclude that $x_\beta \leq x_{t_{d+1}}$, which is a contradiction.

Therefore, the set $\{\sum_{j=1}^{m} x_{t_j} : m\in N, t\in J\}$ has a cardinal number similar to α, and is dense in X^+. Now, we find the smallest subset $A\subset X$ with respect to the cardinal number such that $clA=X$ and $cardA\leq\alpha$.

Corollary 3. Let $(X,\tau,*,\leq)$ be an O-topological group, the relation $\leq$ be totally ordered on X, and $\{x_t\}_{t\in J}$ be a net in X^+ such that $\lim_t x_t = e$, $\sum_t x_t = +\infty$ whenever $cardJ=\alpha$, then the space X is $\gamma-separable$ space where $\gamma\leq\alpha$.

Proof: It is similar that X^- is $\gamma-separable$ space, and we know that $X=X^-\bigcup X^+\bigcup\{e\}$. Hence, the space X is also $\gamma-separable$ space whenever $\gamma\leq\alpha$.

Corollary 4. Let X be an O-topological group, the relation $\leq$ be totally ordered on X, and $\{x_n\}_n \subseteq X^+$ where $\sum_{n=1}^{\infty} x_n = +\infty$, $\lim_{n\to\infty} x_n = e$. Then, X^+ is separable space.

Corollary 5. Let X be an O-topological group, the relation $\leq$ be totally ordered on X, and $\{x_n\}_{n\in N} \subseteq X^+$ such that $\sum_{n=1}^{\infty} x_n = +\infty$, $\lim_{n\to\infty} x_n = e$. Then, for each x of X^+, there exists a subsequence $\{x_{n_j}\}_j$ such that $x = \sum_j x_{n_j}$.

Example 6. Suppose that $\{x_n\}_{n=1}^{\infty}$ is an arbitrary sequence of R^+ (positive real number) such that $\sum_{n=1}^{\infty} x_n = +\infty$ and $\lim_{n\to\infty} x_n = 0$, then the set $\{(-1)^m \sum_{j=1}^{m} x_{n_j} : m \in N\}$ is dense in R.

In other words, if $\{x_n\}_n$ is an arbitrary sequence in real numbers R^+ with $\lim_{n\to\infty} x_n = 0$ and $\sum_{n=1}^{\infty} x_n = +\infty$, then for each x belonging to R^+ there exists a subsequence as $\{x_{n_j}\}_j$ such that $x = \sum_j x_{n_j}$. For special example see [5].

3. SOME DISCUSSION ON THE $\alpha - separable$

In this part, we study some applications of $\alpha -$separable space in normed spaces and will prove some theorems that have been studied in separable space. We suppose that X is a normed space and X^* is a dual space of X or conjugate space of X, that is, the normed space $L(X,R)$ of all bounded linear functionals on X with the operator norm. In this section, each topology is induced by a norm on a vector space.

A subset D of a normed vector space X is said to be fundamental if it generates a dense subspace of X, that is, if, for every $x \in X$ and every $\varepsilon > 0$ there is a finite subset $\{x_1, x_2, ...x_n\}$ of D and scalars $\lambda_1, \lambda_2, ...\lambda_n \in R$ such that $\left\| x - \sum_{i=1}^{n} \lambda_j x_j \right\| < \varepsilon$. For details, see [6].

Let X be a normed space and let A and B be subsets of X and X^*, respectively. Define $A^\perp$ and $^\perp B$ by the formula as

$$A^\perp = \{x^* : x^* x = 0 \; for\ each\ \ x \in A\} \quad \text{and} \quad {}^\perp B = \{x \;\; : x^* x = 0 \;\; for\ each\ \ x^* \in B\}$$

$A^\perp$ is recalled as the annihilator of A in X^* and $^\perp B$ is recalled as the annihilator of B in X.

Let (X,τ) be a vector topological space on field real number and $A \subset X$. Then we define the set $\langle A \rangle$, the smallest subspace of X which includes A, and $[A]$, the smallest closed subspace of X which includes A [7].

Theorem 7. A normed space X over R is $\alpha - separable$ space if and only if it contains a subset A with $cardA = \alpha$ such that A is a fundamental family of vectors for X.

Proof: Let A be a fundamental family of vectors, a subset of X with $cardA = \alpha$. Let B be a set of linear combinations of the elements of A with coefficients on the field Q (rational numbers). Then B is dense in X because its closure, generated by A, is X. On the other hand, B has cardinal number α, since it is the image of the $\bigcup_{n=1}^{\infty} Q^n \times A^n$ under map f defined by $f(\lambda_1, \lambda_2, ...\lambda_n, x_1, x_2, ...x_n) = \sum_{j=1}^{n} \lambda_j x_j$.

Theorem 8. A normed space X is $\alpha - separable$ space if and only if it has the smallest topology basis B in which $cardB = \alpha$.

Proof: Let B be the smallest topology basis for X with $cardB = \alpha$. We set $B = (B_t)_{t\in J}$ where $cardJ = \alpha$. Then we set the net $(x_t)_{t\in J}$ where $x_t \in B_t$. Thus, the set $\{x_t : t \in J\}$ is a fundamental for X. Consequently, by preceding Theorem 7, X is $\alpha - separable$ space.

Conversely, let $A = (x_t)_{t\in J}$ be the smallest net with respect to the cardinal number with $cardJ = \alpha$ and $cl(A) = X$. We define the set as $B_t = \{B(x_t, \frac{1}{n}) : n \in N\}$ for all $t \in J$ whenever $B(x_t, \frac{1}{n}) = \{x \in X : \|x_t - x\| < \frac{1}{n}\}$. If we put $B = \bigcup_{t\in J} B_t$, then B is a topological basis for X with cardinal number α.

Theorem 9. Let (X, τ) be a vector topological space on field real number, and the infinite set $A \subset X$ be an $\alpha - separable$ space, then $\langle A\rangle$ and $[A]$ are both $\alpha - separable$ space.

Proof: Since A is $\alpha - separable$ space, there exist subset $B \subset A$ such that $cardB = \alpha$ and $cl(B) = A$. Let S be a set consisting of all linear combinations of the elements of B formed using only the scalar coefficient from Q.

Then $S = \langle B\rangle$, so $S \cong \bigcup_{n=0}^{\infty} Q^n \times B$. We conclude $cardS = \alpha$, and so $cl(S) = \langle A\rangle$, which implies $\langle A\rangle$ is $\alpha - separable$. Since a dense subset of $\langle A\rangle$ is also a dense subset of $[A]$, the set $[A]$ is also $\alpha - separable$.

Theorem 10. Let $1 \le p < \infty$ and $L^p(X)$ be all functions f such that $\int_X |f|^p d\mu < \infty$. If X is $\alpha - separable$ space, then $L^p(X)$ is $\alpha - separable$ space and for $p = \infty$, the space $L^\infty(X)$ is also $2^\alpha - separable$.

Proof: Suppose that $(X, \sum, \mu)$ is a positive measure space in which the collection $\sum$ is Borel sets. By Theorem 8 we have $card\sum = \alpha$ and set $\sum = \{A_t : t \in J\}$, where $cardJ = \alpha$. We choose the collection as $\Psi(x) = \sum_{n=1}^{m} c_{t_n} \chi_{A_{t_n}}(x)$ whenever $t_n \in J$ and $c_{t_n} \in Q$, which is dense in $L^p(X)$ [8]. Consequently, $L^p(X)$ is $\alpha - separable$ space.

Now, let A be a subset of X with $cardA = \alpha$ and $cl(A) = X$. We define the set as $A = \{x_t : t \in J\}$ whenever $cardJ = \alpha$ and set $S = \{\chi_B : B \in A\}$. Obviously $cardS = 2^\alpha$ and by Theorem 9, we have $card\langle S\rangle = 2^\alpha$. It is clear $cl\langle S\rangle = L^\infty(X)$, so the space $L^\infty(X)$ is also $2^\alpha - separable$ and the proof is complete.

Theorem 11. Let X be a normed space and let A and B be the subset of X and X^* respectively. Then, we have

a) The set $A^\perp$ and ${}^\perp B$ are the closed subspace of X^* and X respectively.

b) ${}^\perp(A^\perp) = [A]$ ($[A]$ is the smallest closed subspace that includes A)

c) If A is a subspace of X, then ${}^\perp(A^\perp) = cl(A)$.

Proof: See the proposition of (1.10.14) from [7].

Theorem 12. Let X be a normed space. If X^* is $\alpha - separable$ space, then X is $\alpha - separable$ space.

Proof: Let J be an arbitrary index set with $cardJ = \alpha$, and the set $\{x_t^* : t \in J\}$ be dense in X^*. For each $t \in J$, let x_t be an element of $\{x \in X : \|x\| < 1\}$ such that $\|x_t^*(x_t)\| \geq \frac{1}{2}\|x_t^*\|$.

If $x^* \in X^*$ and $x^* \neq 0$, then there is an $t \in J$ such that $\|x^* - x_t^*\| < \frac{1}{4}\|x^*\|$, from which it follows that $\|x_t^*\| \geq \|x^*\| - \|x^* - x_t^*\| > \frac{3}{4}\|x^*\| > 0$ and that

$$|x^* x_t| \geq |x_t^* x_t| - |(x^* - x_t^*)(x_t)| \geq \|x_t^* x_t\| - |x^* - x_t^*| > \frac{1}{2}\|x_t^*\| - \frac{1}{4}\|x^*\| > \frac{1}{2}\|x_t^*\| - \frac{3}{4}\|x_t^*\| > 0$$

Therefore, the annihilator of $\{x_t : t \in J\}$ contains only the zero element of X^*, so by Theorem11 we have $[\{x_t : t \in J\}] = {}^{\perp}(\{x_t : t \in J\}^{\perp}) = {}^{\perp}\{0\} = X$.

By Theorem 9, we know that the set $[\{x_t : t \in J\}]$ is $\alpha - separable$ space, then X is $\alpha - separable$ space.

Corollary 13. Let X be a reflexive $(X^{**} = X)$ normed space. Then X is $\alpha - separable$ space if and only if X^* is $\alpha - separable$ space.

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