CONFORMAL VECTOR FIELDS ON TANGENT BUNDLE
WITH A SPECIAL LIFT FINSLER METRIC

E. PEYGHAN\textsuperscript{1**, A. RAZAVI\textsuperscript{2 AND A. HEYDARI\textsuperscript{3}}}

\textsuperscript{1}Department of Mathematics, Faculty of Science, University of Arak, Arak, I. R. of Iran
\textsuperscript{2}Department of Mathematics and Computer Science, Amirkabir University, Tehran, I. R. of Iran
\textsuperscript{3}Faculty of Science, Tarbiatmodares University, Tehran, I. R. of Iran
Emails: e-peyghan@araku.ac.ir, arazavi@aut.ac.ir, aheydari@modares.ac.ir

Abstract – On a Finsler manifold, we define conformal vector fields and their complete lifts and prove that in certain conditions they are homothetic.

Keywords – Conformal vector field, complete lift, finsler manifold, lift metric

1. PRELIMINARIES

Let \((M, g)\) be a Riemannian manifold, a vector field \(V\) on \(M\) is called a conformal vector field if its local 1-parameter group of transformations is a local conformal transformation. It is well known that \(V\) is a conformal vector field on \(M\) if and only if there is a scalar function \(\lambda\) on \(M\) such that \(L_{V} g = 2\lambda g\). When \(\lambda\) is a constant, \(V\) is called homothetic, especially when \(\lambda = 0\), \(V\) is a killing vector field or an infinitesimal isometry \([1]\).

On a Finsler manifold \((M, F)\), let \(V\) be a vector field with the complete lift \(V^c\), then \(V\) is called conformal vector field if there is a scalar function \(\rho\) on \(TM\) such that \(L_{V^c} g = 2\rho g\), where \(g = (g_{ij})\) is the corresponding fundamental Finsler tensor defined by \(g_{ij}(x,y) = \frac{1}{2}F^2_{ij}(x,y)\).

Let \(TM\) be the tangent space with a canonical coordinate system \((x^{i}, y^{i})\), then the vertical tangent bundle of \(TM_0 = TM \setminus \{0\}\) is defined by

\[ VTM = \text{span}\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\}. \]

A non-linear connection on \(TM_0\) is a complementary distribution \(HTM\) defined by

\[ HTM = \text{span}\{\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n}\}, \]

where \(\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}\), and \(N_i^j\) are the connection coefficients. \(HTM\) is a vector bundle completely determined by the smooth functions \(N_i^j(x,y)\) on \(TM\) \([2, 3]\). Moreover, we have

\[ TTM_0 = VTM \oplus HTM \]
Let $\nabla$ be a linear connection on $VTM$, then $(HTM, \nabla)$ is called a Finsler connection on $M$. Indeed, a Finsler connection is a triad $(N, F, C)$ where $N(N_{ij}^k)$ is a nonlinear, $F(F_{ij}^k)$ is the horizontal part and $C(C_{ij}^k)$ is the vertical part of this connection. Now let $(M, F)$ be Finsler manifold then a Finsler connection is called a metric Finsler connection if $g$ is parallel with respect to $\nabla$. According to the Miron framework this means $g$ is both horizontally and vertically a metric [4, 5, 6]. The Cartan connection is a metric Finsler connection for which the deflection, horizontal, and vertical torsion tensor fields vanish.

The curvature tensor of a metric Finsler connection is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

where $X, Y \in \mathcal{X}(TM)$.

They are called horizontal or vertical according to the choice of $X$ and $Y$ in $HTM$ or $VTM$. Then we have [5]

$$R^h_{kji} = \delta_i F^h_{kj} - \delta_j F^h_{ki} + F^m_{kij} F^h_{mi} - F^m_{kij} F^h_{mi} + C^h_{km} R^m_{ji},$$

$$R^h_{ij} = \delta_j N^h_i - \delta_i N^h_j,$$

where we have put $\partial_i = \partial / \partial x_i$, $\partial_j = \partial / \partial y_j$, $\partial_i = \partial - N^m_i \partial_m$. When $\nabla$ is a Cartan connection then $N^h_i = y^m F^h_{mi}$.

**Proposition 1.** [4] Let $M$ be an $n$-dimensional Finsler manifold with a Cartan connection, then we have the following equations:

1. $F^h_{ij} = \frac{1}{2} g^{hu} (\delta_i g_{mj} + \delta_j g_{im} - \delta_m g_{ij});$

2. $C^h_{ik} = \frac{1}{2} \partial_k g_{ij}$ where $C^h_{ik} = C^m_{ik} g_{mj};$

3. $y^m C_{mj} = 0;$

4. $R^h_{ij} = y^m R^h_{mij}.$

The Cartan horizontal and vertical covariant derivative of a tensor field of type $(1,2)$ are locally as follows:

$$\nabla^h T^h_{kij} := T^h_{kij} + F^h_{kij} T^m_{ji} - F^m_{kij} T^h_{ji} - F^m_{kij} T^h_{ji};$$

$$\nabla^h T^h_{kij} := T^h_{kij} + C^h_{mij} T^m_{ji} - C^m_{mij} T^h_{ji} - C^m_{mij} T^h_{ji}.$$ 

2. LIFT METRICS AND CONFORMAL VECTOR FIELDS

a) Complete Lift Vector Fields and Lie Derivative

Let $V = v^i \partial_i$ be a vector field on $M$. Then $V$ induces an infinitesimal point transformation on $M$. This is naturally extended to a point transformation of the tangent bundle $TM$ which is called extended point transformation. Let $V$ be a vector field on $M$ and $\{\varphi_t\}$ the local 1-parameter group of $M$ generated by $V$. Let $\tilde{\varphi}_t$ be the extended point transformation of $\varphi_t$, then $\{\tilde{\varphi}_t\}$ induces a vector field $V^C$ on $TM$ which is called the complete lift of $V$ [7, 8].

It can be shown that the extended point transformation is a transformation induced by the complete lift vector field of $V$, $V^C = v^i \partial_i + y^i \nabla_j v^i \partial_j$ with respect to the decomposition (1), where $\nabla$ is a linear connection.
The Lie derivation of an arbitrary tensor, $T_i^k$, is given locally by [9]:

$$L_\nu T_i^k = v^a \nabla_a T_i^k + v^a \nabla_a y^b \nabla_b T_i^k - T_i^a \nabla_a y^k + T_a^k \nabla_v y^a$$

or equivalently,

$$L_\nu T_i^k = v^a \partial_a T_i^k + y^a \partial_\nu y^b \partial_{\nu} T_i^k - T_i^a \partial_\nu y^k + T_a^k \partial_\nu y^a.$$

So we have

$$L_\nu y^i = v^a \partial_a y^i + y^a \partial_\nu y^b \partial_{\nu} y^i - y^a \partial_\nu y^i = y^a \partial_a y^i - y^a \partial_\nu y^i = 0,$$

(3)

$$L_\nu g_{ij} = v^a \partial_a g_{ij} + y^a \partial_\nu g_{ij} \partial_{\nu} g_{ij} + g_{ai} \partial_\nu y^a + g_{ai} \partial_\nu y^a.$$

(4)

where $\nabla$ is a linear connection.

In Finsler geometry, $L_\nu$ is replaced by $L_{\nu}$, where $\nu^i$ is the lift of $V$. We also have this interchanging formula between Cartan covariant derivatives and Lie derivatives.

$$\nabla_k L_{\nu} g_{ij} - L_{\nu} \nabla_k g_{ij} = g_{ai} L_{\nu} F_{i,k}^a + g_{ai} L_{\nu} F_{j,k}^a.$$

(5)

### b) A Lift Metric on Tangent Bundle

V. Oproiu introduced a family of Riemannian metrics on the tangent space of Riemannian manifolds and considered locally symmetric, Kählerian and anti-Hermitian conditions with these metrics [10-12]. Then Abbasi-Sarih proved in [13] that the Oproiu metrics form a particular subclass of the so-called $g$-natural metrics on the tangent space [14, 15]. Also in [16], Boeckx-Vanhecke obtained an almost contact metric on the unit tangent space.

In this section we consider a new Riemannian metric on the tangent space, and in the next section obtain some conditions which reduce the conformal vector fields to be homothetic.

Let $(M, F)$ be a Finsler manifold, define a tensor field $G$ on $TM$ by

$$G(x, y) = \alpha h_{ij}(x, y) dx^i dx^j + 2\beta h_{ij}(x, y) dx^i \delta y^j + \gamma h_{ij}(x, y) \delta y^i \delta y^j$$

where $\alpha, \beta$ and $\gamma$ are real numbers and $h_{ij}(x, y)$ are components of a generalized Lagrange metric [6, 17]. It is clear that $G$ is nonsingular if $\alpha \gamma - \beta^2 \neq 0$ and positive definite if $\alpha \gamma - \beta^2 > 0$, defining, respectively, a pseudo-Riemannian or Riemannian lift metrics on $T(M)$.

We are going to consider the metric $G$ with $h_{ij}(x, y)$ of the following special deformation of $g_{ij}(x)$

$$h_{ij}(x, y) = a(F^2) g_{ij}(x, y),$$

where $y^i = g_{ij}(x, y) y^j$ and $a : \text{Im}(F^2) \subseteq R_+ \rightarrow R_+$ with $a > 0$. For shortness we set $g_1 = h_{ij} dx^i dx^j$, $g_2 = 2h_{ij} dx^i \delta y^j$ and $g_3 = h_{ij} \delta y^i \delta y^j$, therefore $G = \alpha g_1 + \beta g_2 + \gamma g_3$.

### 3. MAIN RESULTS

Analogous to the Riemannian geometry, by straightforward calculation we have the following results in Finsler geometry [18, 19].

**Lemma 1.** Let $(M, F)$ be a Finsler manifold with Cartan connection, then we have

1. $[\delta_i, \delta_j] = R_{ij}^{\ k} \partial_k$;
2. $[\delta_i, \delta_j] = \partial_j N_{ij} \partial_k$;

Winter 2008  
Lemma 2. Let \( (M,F) \) be a Finsler manifold with Cartan connection, then we have
\[
\begin{align*}
(1) & \quad L_{v^i} \partial_{v^j} = -\partial_y v^h \delta_y^i - L_{v^i} \partial_y N^h \delta_y^j; \\
(2) & \quad L_{v^i} \partial_v = -\partial_y v^h \partial_v; \\
(3) & \quad L_{v^i} dx^h = \partial_m v^h dx^m; \\
(4) & \quad L_{v^i} \delta y^h = L_{v^i} N^h \partial_m + \partial_m v^h \delta y^m.
\end{align*}
\]

**Proof:** First we give the proof of part (2). By a simple calculation, we have:
\[
L_{v^i} \partial_v = [v^c, \partial_v] = [v^b \delta_y^h + y^m v^h \mid_m \partial_v, \partial_v] = v^b \partial_y (v^h \delta_y^h) - y^m v^h \mid_m \partial_y + y^m v^h \mid_m \partial_v - \partial_y (y^m v^h \mid_m \partial_v) = \partial_y (y^m v^h \mid_m \partial_v) = \partial_v v^h \partial_v.
\]
The proof of part (1) is similar to (2).

Since \( (dx^h, \delta y^h) \) is the dual basis of \( (\delta_y, \partial_v) \), if we put
\[
L_{v^i} \delta y^h = \alpha^h_m \partial_m + \beta^h_m \delta y^m,
\]
then we have
\[
0 = L_{v^i} (\delta y^h (\delta_y^j)) = (L_{v^i} \delta y^h) \delta_y^j + \delta y^h (L_{v^i} \delta_y^j) = \alpha^h_m - L_{v^i} N^h_i,
\]
and
\[
0 = L_{v^i} (\delta y^h (\delta_v)) = (L_{v^i} \delta y^h) \delta_v + \delta y^h (L_{v^i} \delta_v) = \beta^h_i - \delta_v v^h.
\]
Thus we get (4). In the same way as the proof of part (4), we can prove (3).

Lemma 3. Let \( (M,g) \) be a Finsler manifold with Cartan connection, then we have
\[
\begin{align*}
(1) & \quad L_{v^i} g_{ij} = a(F^2)(2\varphi g_{ij} + L_{v^i} g_{ij})dx^i dx^j; \\
(2) & \quad L_{v^i} g_{ij} = 2a(F^2)g_{ij} + 2a(F^2)(2\varphi g_{ij} + L_{v^i} g_{ij})\delta y^i \delta y^j; \\
(3) & \quad L_{v^i} g_{ij} = 2a(F^2)g_{ij} + (2\varphi g_{ij} + L_{v^i} g_{ij})\delta y^i \delta y^j.
\end{align*}
\]
where \( \varphi = y^m v^h \mid_m \frac{a(F^2)}{a(F^2)}\).

**Proof:** From the above lemma, we get
\[
L_{v^i} g_{ij} = L_{v^i} (h_{ij} \delta y^i \delta y^j) = V^c (a(F^2)g_{ij})dx^i dx^j + 2a(F^2)g_{ij} (L_{v^i} dx^i) dx^j = (v^b \delta_y^h \mid_m \partial_y) a(F^2)g_{ij} + (v^b \delta_y^h \mid_m \partial_v) a(F^2)
\]
\[
+ 2a(F^2)g_{ij} (\partial_y \delta y^i \partial y^j) dx^j = 2a(F^2) \varphi g_{ij} dx^i dx^j + a(F^2) L_{v^i} g_{ij} dx^i dx^j.
\]
Thus we have (1), (2) and (3) are easily proof in the same way as the proof of (1).

**Definition 1.** Let $X$ be a conformal vector field on $TM$ with the associated function $\rho$. $X$ is called quasi-inessential vector field if $\rho - \varphi$ is a function of $(x^i)$, namely there exists a function $\Omega$ of $(x^i)$ such that $\rho = \Omega + \varphi$. If $\Omega$ is constant, then $X$ is called quasi-homothetic vector field. Moreover, if $\Omega = 0$ then $X$ is called quasi-isometry vector field on $TM$.

**Remark:** These classes of vector fields contain the classes of inessential, homothetic and isometry vector fields as special cases, respectively (for $\varphi = 0$). Hence, the forthcoming results hold for inessential, homothetic and isometry vector fields.

**Theorem 1.** Let $(M,F)$ be a $C^\infty$ connected Finsler manifold, $TM$ its tangent bundle and $G$ the Riemannian (or pseudo-Riemannian) metric on $TM$ derived from $g$. Then every complete lift conformal vector field on $TM$ is quasi-homothetic.

**Proof:** Let $V$ be a vector field on $M$, $V^e$ the complete lift vector field of $V$ which is conformal, and let $G$ be a pseudo-Riemannian metric on $TM$ derived from $g$. We have by definition $L_v G = 2\rho G$. The Lie derivative of $G$ gives

$$L_v G = a a (F^2) (2\varphi g_{ij} + L_v g_{ij}) dx^i dx^j + 2\beta a (F^2) (2\varphi g_{ij} + L_v g_{ij}) dx^i \delta y^j$$

$$+ 2\beta a (F^2) g_{ij} L_v N^a_{ij} dx^i dx^j + \gamma a (F^2) (2\varphi g_{ij} + L_v g_{ij}) \delta y^i \delta y^j$$

$$+ \gamma a (F^2) g_{ij} L_v N^a_{ij} \delta y^i \delta y^j = 0.$$  \hspace{1cm} (6)

So we have

$$L_v G = a (F^2) [\alpha (2\varphi g_{ij} + L_v g_{ij}) + 2\beta g_{ij} (L_v N^a_{ij})] dx^i dx^j$$

$$+ \alpha (F^2) [2\beta (2\varphi g_{ij} + L_v g_{ij}) + 2\gamma g_{ij} (L_v N^a_{ij})] dx^i \delta y^j$$

$$+ \gamma a (F^2) (2\varphi g_{ij} + L_v g_{ij}) \delta y^i \delta y^j = 2\rho G.$$  \hspace{1cm} (7)

Comparing with the definition of $G$, we find

$$\alpha L_v g_{ij} + \beta (g_{ij} L_v N^a_{ij} + g_{ij} L_v N^a_{ij}) = 2\alpha \Omega g_{ij};$$ \hspace{1cm} (7)

$$\beta L_v g_{ij} + \gamma g_{ij} L_v N^a_{ij} = 2\beta \Omega g_{ij};$$ \hspace{1cm} (8)

$$\gamma L_v g_{ij} = 2\gamma \Omega g_{ij}.$$ \hspace{1cm} (9)

Where $\Omega = \rho - \varphi$.

I) If $\gamma \neq 0$, then from (9) we have

$$L_v g_{ij} = 2\Omega g_{ij}$$

and from (8) we have

$$L_v N^a_{ij} = 0.$$  \hspace{1cm} (10)

Using this and $N^a_{ij} = y^m F^h_{mi}$, we get
\[0 = L_{\nu} N_i^h = L_{\nu} \left( y^m F_m^i \right) = y^m L_{\nu} F_m^i, \quad (10)\]

where the last equality follows from equation (3).

II) If \( \gamma = 0 \), since \( \alpha \gamma - \beta^2 \neq 0 \) we have \( \beta \neq 0 \). From (8) we get

\[L_{\nu} g_{ij} = 2\Omega g_{ij}\]

and from (7) we have

\[g_{ai} L_{\nu} N_i^a + g_{aj} L_{\nu} N_i^a = 0.\]

Using this, equation (3) and \( N_i^h = y^m F_m^i \), we have

\[y^m (g_{ai} L_{\nu} F_m^i + g_{aj} L_{\nu} F_m^a) = 0. \quad (11)\]

In each case I and II we have

\[L_{\nu} g_{ij} = 2\Omega g_{ij} \quad (12)\]

or from equation (4)

\[v^a \partial_a g_{ij} + g_{ai} \partial_j v^a + g_{aj} \partial_i v^a + y^b \partial_j v^b \partial_i \Omega = 2\Omega g_{ij}.\]

Applying \( \partial_\kappa \) to both sides of the above equation, we find that

\[2v^a \partial_\kappa C_{ijk} + 2C_{aqk} \partial_j v^a + 2C_{iak} \partial_j v^a + 2y^a \partial_i v^b \partial_\kappa C_{ijk} = 2g_{ij} \partial_\kappa \Omega + 4\Omega C_{ijk}.\]

By using \( y^i C_{ijk} = 0 \), we obtain \( \partial_\kappa \Omega = 0 \). Therefore \( \Omega \) is a function of \( x \) alone. From (5) we have

\[y^k \left( \nabla_k L_{\nu} g_{ij} - L_{\nu} \nabla_k g_{ij} \right) = y^k \left( g_{ai} L_{\nu} F_i^a + g_{aj} L_{\nu} F_j^a \right).\]

By using (10), (11) and (12) in each case I and II we find that

\[y^k \nabla_k \Omega = 0.\]

Since \( \Omega \) is a function of \( x \) alone, we obtain \( \partial_\nu \Omega = 0 \). This, together with the connectedness of \( M \), shows that \( \Omega \) is constant.

**Note:** In a special case when \( a'(F^2) = 0 \) e.g. \( a(t) = (t - F^2)^2 + 1 \) follows from lemma 3, that \( \phi = 0 \) and hence \( L_{\nu} G = 2\rho G \), where \( \rho \) depends on \( x \) only. Therefore we have:

**Corollary 1.** Let \((M,F)\) be a \(C^\infty\) connected Finsler manifold, \(TM\) its tangent bundle and \(G\) the Riemannian (or pseudo-Riemannian) metric on \(TM\) derived from \(g\) with \(a'(F^2) = 0\). Then every complete lift conformal vector field on \(TM\) is homothetic.

**REFERENCES**


