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DIRAC STRUCTURES*

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Abstract – In this paper we introduce the concept of Dirac structures on (Hermitian) modules and vector bundles and deduce some of their properties. Among other things we prove that there is a one to one correspondence between the set of all Dirac structures on a (Hermitian) module and the group of all automorphisms of the module. This correspondence enables us to represent Dirac structures on (Hermitian) modules and on vector bundles in a very suitable form and define induced Dirac structures in a natural way.

Keywords – Dirac structure, Hermitian module, Hilbert module, vector bundle

1. INTRODUCTION

The theory of Dirac structures, as a generalization of Poisson and presymplectic structures was introduced by Courant and Weinstein in [1] and [2]. Its algebraic counterpart was given by Dorfman in [3]. In [4] we considered Dirac structures on real Hilbert spaces, which can be extended to complex Hilbert spaces by a simple modification as follows:

Let (H, < | >) be a complex Hilbert space. A subspace of the vector space H×H is called a Dirac structure on H if it is maximally isotropic under the pairing

$$((x,y),(u,v)) \mapsto \frac{<x \mid v> + <y \mid u>}{2}$$

on H×H. Notice that using this definition, the results of [4] on Dirac structures on real Hilbert spaces are also true for Dirac structures on complex Hilbert spaces.

In [4] we showed that, to each Dirac structure on a vector bundle there corresponds a unique Dirac structure on some Hilbert space. More precisely, let L be a Dirac structure on the vector bundle η=(E, π, M, F) in the sense of [1]. Let M be compact. Endow M and η with Riemannian metrics. Let H be the Hilbert space of L^2-sections of η, and let Λ be the set of L^2-sections of L. Then, Λ is a Dirac structure on H. Unfortunately, the converse is not true. Moreover, there is no convenient way to formulate the notion of integrability of Dirac structures on tangent bundles in terms of Dirac structures on the associated Hilbert spaces. The search for overcoming these inconveniencies and also our interest in the deformation quantization of Dirac structures on vector bundles have been our motivations for this work. In this paper we give a simple definition of Dirac structures on general and Hermitian modules and study their properties. We also give a definition of the Dirac structure on general and Hermitian vector bundles. Our definition includes the existing ones and the complex structures on vector bundles as special cases. The problem of integrability of Dirac structures will be considered in future work.

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There are many interesting papers on Dirac structures and their applications The algebraic version of these structures as introduced in this paper gives a very simple representation of Dirac structures and can be used to solve interesting problems in differential geometry.

This paper, which gives the foundations, consists of three parts. In the first part, we give the definition of Dirac structures on arbitrary and Hermitian modules and prove some results. The second part of the paper contains the definition of Dirac structures on arbitrary and Hermitian vector bundles and some facts about them. The relation between Dirac structures on (Hermitian) modules and Dirac structures on (Hermitian) vector bundles is presented in the final section.

In forthcoming papers we will use these facts to investigate and solve some interesting problems in differential geometry. Since the problems require integrability, we have to consider them after introducing this notion and giving some of its properties.

Note that in the following all modules over a C^*-algebra, including real and complex vector spaces with a positive definite conjugate-bilinear functional, will be called a Hermitian module. In the same way, the Riemannian and Hermitian vector bundles will be called Hermitian vector bundles.

2. DIRAC STRUCTURES ON MODULES

In the following $\mathbb{k}$ denotes the field of real or complex numbers. If E is a real vector space, its complexification will be denoted by E^c. R is an arbitrary ring with $\frac{1}{2} \in R$ and M is a right R-module which satisfy the following condition:

For each non-zero element $x \in M$, $2x \neq 0$.

The results of the paper are also true for left R-modules. We also consider Hermitian R-modules, where R is a C^*-algebra. Here, by a Hermitian R-module we mean a module M over the $*$-ring R with a map $<|>: M\times M\to R$, which satisfies the following conditions:

For all $r\in R$, and all $u,v\in M$, we have

1) $<u | vr>=<u | v>r$,
2) $<u | v>^*=<v | u>$,
3) $<u | u> \geq 0$
4) $<u | u>=0$ implies $u=0$.

In this case the map $\|.\|: M\to\mathbb{R}^+$ given by $\|u\|=\|<u|u>\|^{\frac{1}{2}}$ defines a norm on the $\mathbb{k}$-vector space M. The Hermitian R-module M is called a Hilbert R-module if the normed space $(M,\|.\|)$ is complete. For more details on the subject see [5], [6].

For any R-module M the isomorphism π: $M\times M\to M\times M$ is defined by $(x,y)\mapsto(y,x)$. The first (resp. second) projection of $M\times M$ onto M will be denoted by p_1 (resp. p_2). Two submodules of a (Hermitian) R-module are called (ortho)complementary, (if they are orthogonal to each other and) if M is equal to their direct sum. A submodule M' of a (Hermitian) module M is called projective if it has a (ortho)complement in M. Note that when M is a projective module, each projective submodule of M is a projective module in the usual sense. The group of all (isometric) automorphisms of a (Hermitian) module M will be denoted by Aut(M).

Let M be an R-module, and let I: $M\to M$ be the identity map. The homomorphism T: $M\to M$ is called a complex structure on M if $T^2=-I$. Note that since $(T-I)(T+I)=T^2-I=-2I$, $T\pm I: M\to M$ are invertible.

Definition 2.1. Let M be a (Hermitian) module. A submodule L of $M\times M$ is called a Dirac structure on M if L and $\pi(L)$ are (ortho)complementary.

Remark 2.2. Let E be a finite-dimensional real vector space. A Dirac structure on E in the sense of [1] is a subspace of $E\times E^*$ which is maximally isotropic under the pairing

$$<(x,y)\,|\,(u,v)>_{+}=\frac{u(y)+v(x)}{2}.$$

After identifying E^* with E by an inner product on E, it is easy to see that this definition is a special case of our definition of Dirac structures on Hermitian modules [4].

Proposition 2.3. Let L be a Dirac structure on a (Hermitian) module M. Then the homomorphisms $p_1\pm p_2$: $L\to M$ are (isometric) isomorphisms.

Conversely, let L be a submodule of $M\times M$ and let $L\cap\pi(L)=0$ (and $\pi(L)$ be orthogonal to L). Then a sufficient condition for L to be a Dirac structure on M is that the homomorphisms $p_1\pm p_2$: $L\to M$ be surjective.

Proof: Let (x,y) be in L and let x+y=0. Then $(x,-x)\in L\cap\pi(L)$. Thus, x=0. Therefore, p_1+p_2 is injective. On the other hand, for each $w\in M,(w,w)\in M\times M$. Therefore, there exist $(x,y)\in L$ and $(y',x')\in\pi(L)$ such that $(w,w)=(x,y)+(y',x')$. Thus, $w=x+y'=x'+y$. Hence, $x-x'=y-y'$. But then, $(x-x',x-x')=(x-x',y-y')=(x,y)-(x',y')\in L\cap\pi(L)$. Thus, $x=x'$, $y=y'$ and $w=x+y$. Therefore, p_1+p_2 is surjective.

Similarly, it can be shown that p_1-p_2 is an isomorphism.

Assume that M is a Hermitian module. Let $(x,y)\in L$. Then, since $\pi(L)$ is orthogonal to L,

$$<x\pm y\,|\,x\pm y>=<x\,|\,x>+<y\,|\,y>\pm<x\,|\,y>\pm<y\,|\,x>=<x\,|\,x>+<y\,|\,y>.$$

Therefore, $p_1\pm p_2$ are isometries.

Now assume that the submodule L of $M\times M$ is such that ($\pi(L)$ is orthogonal to L) $L\cap\pi(L)=\{0\}$, and $p_1\pm p_2$:$L\to M$ are surjective. Let $(x,y)\in M\times M$. Then, there exists $(u,v)\in L$ such that x+y=u+v. Hence, x-u=v-y. Now, there exists $(z,t)\in L$ such that x-u=z-t. Thus, (x-u,y-v)=(z-t,t-z), i.e., $(x,y)=(u,v)+(z,t)-(t,z)\in L\oplus\pi(L)$. Therefore, L is a Dirac structure on the (Hermitian) module M.

Example 2.4. Let H be a Hilbert space and let T: H→H be a globally defined surjective linear operator. Assume that 1 is not in the spectrum of T^2. Let L denote the graph of T. It is easy to see that $L\cap\pi(L)=\{0\}$. Since $p_1\pm p_2$: $L\to H$ are also surjective, by the above proposition, L is a Dirac structure on the vector space H. L is a Dirac structure on the Hilbert space H, if and only if, T is anti-symmetric [4].

Example 2.5. Let M be a module, and Let T: $M\to M$ be a complex structure on M. Let $L=\{(x,Tx)\in M\times M\mid x\in M\}$. Since $T^2=-I$, we have $L\cap\pi(L)=\{0\}$. On the other hand, as we have seen earlier, $T\pm I$:$M\to M$ are invertible. Hence, $p_1\pm p_2$: $L\to M$, are surjective. Therefore, L is a Dirac structure on M.

As a consequence of the first part of the above proposition we have:

Corollary 2.6. Any two Dirac structures on a (Hermitian) module M are (isometrically) isomorphic as (Hermitian) modules

The set of all Dirac structures on M will be denoted by D(M).

Proposition 2.7. Let M be a (Hermitian) module. Then, there is a one-to-one correspondence between D(M) and Aut(M).

Proof: Let L be a Dirac structure on M. Then by the above proposition

$$A_L = (p_1 - p_2)\left((p_1 + p_2)|_L\right)^{-1} \in Aut(M).$$

Conversely, for $A \in$ Aut(M), let L_A={(x+Ax, x-Ax)|x$\in M$}. Assume that (x+Ax, x-Ax)=(y-Ay, y+Ay). Then x-y=-A(x+y) and x-y=A(x+y). Thus, 0=(x-y)=A(x+y). Since $A \in$ Aut(M), x=y=0. Therefore $L_A \cap \pi(L_A)=\{0\}$. On the other hand, $p_1 \pm p_2 : L \to M$ are surjective. Note also that when M is a Hermitian module, L_A and $\pi(L_A)$ are orthogonal to each other. Therefore, L_A is a Dirac structure on M. Finally, assume that B is also in Aut(M) and $L_A=L_B$. Then, for each $x \in M$ there exists $y \in M$ such that

(x+Ax,x-Ax)=(y+By,y-By).

Thus, x=y and Ax=Bx. Since x is an arbitrary element of M, A=B.

Proposition 2.8. Let L be a Dirac structure on a Hermitian (resp. Hilbert) module M. Then, L is a Hermitian (resp. Hilbert) submodule of $M \times M$ which is maximally isotropic under the pairing

$$<(x,y)\,|\,(u,v)>_+ = \frac{<x\,|\,v> + <y\,|\,u>}{2}$$

Proof: By definition, L is the orthocomplement of $\pi(L)$. Therefore, it is a projective Hermitian (resp. Hilbert) submodule of $M \times M$. Clearly, L is isotropic under above pairing. Let $(z,t) \in M \times M$ be such that for each $(u,v) \in L$, $<(z,t)\,|\,(u,v)>_+=0$. Then, $(t,z) \in \pi(L)$. Hence, $(z,t) \in L$. Therefore, L is maximally isotropic.

The converse of the proposition is also true for Hilbert spaces [4]. The following example shows that this is not true for general Hilbert modules.

Example 2.9. Let η be the tangent bundle of the 2-dimensional sphere S^2 and let g be the standard Riemannian metric on it. Let R denote the C^*-algebra of continuous functions on the sphere and let M be the R-module of L^2 vector fields on it. Clearly, M is a Hilbert module under the pairing <X | Y>=g(X,Y). The submodule of M consisting of all vector fields identically zero on the upper (lower) hemisphere is denoted by M_0(M_1). Clearly, $L=M_0 \times M_1 \subset M \times M$ is maximally isotropic under the pairing $<\,|\,>_+$. Since $L \oplus \pi(L) \neq M \times M$, L is not a Dirac structure on M.

Definition 2.10. Dirac structures L and L' are called transversal if $L \cap L' = \{(0,0)\}$.
For example, $L=M \times \{0\}$ and $L'=\{0\} \times M$ are transversal Dirac structures on M.

Lemma 2.11. A necessary and sufficient condition for L and L' to be transversal is that $A_L^{-1} \circ A_{L'} : M \to M$ has no non-zero fixed point.

Proof: Let $x \neq 0$ and $A_L^{-1} \circ A_{L'} x = x$. Then $A_L x = A_{L'} x$. Hence

$$(x + A_L x \quad x - A_L x) = (x + A_{L'} x \quad x - A_{L'} x) \in L \cap L'.$$

Therefore, $x \neq 0$ and $A_L^{-1} \circ A_{L'} x = x$ implies that $L \cap L' \neq \{0\}$. Now, assume that $L \cap L' \neq \{0\}$. Let $0 \neq (x, y) \in L \cap L'$, and let z=x + y. Then

$$A_L z = (p_1 - p_2) \circ (p_1 + p_2)^{-1}(z) = A_{L'} z.$$

Therefore, $L \cap L' \neq \{0\}$ implies $A_L^{-1} \circ A_{L'} z$ =z for some $z \neq 0$.

Corollary 2.12. Any two distinct Dirac structures on a simple module are transversal.

3. AUTOMORPHISM GROUP

Definition 3.1. Let L (resp. L') be a Dirac structure on a (Hermitian) module M(resp. M'). A Dirac homomorphism from M into M' is a homomorphism T:$M \to M'$ such that T×T carries L into L'.

Lemma 3.2. Let L (resp. L\`) be a Dirac structure on an R-module $M$ (resp. $M$\`), and let $T: M \to M'$ be a Dirac morphism. Then $(T\times T)^{-1}(L') = L$

Proof: Let $(x,y)\in (T\times T)^{-1}(L')$. Then we have
1) $(Tx,Ty)\in L'$.
2) $(x,y) = (x_1,y_1)\oplus(y_2,x_2)\in L\oplus\pi(L)$,
From 2 we have $(Tx_1,Ty_1)+(Ty_2,Tx_2)\in L'$. Since (Tx_1,Ty_1) is also in L\` we have $(Ty_2,Tx_2)\in L'$. But $(Ty_2,Tx_2) = T\times T\circ\pi(x_2,y_2) = \pi\circ T\times T(x_2,y_2)\in\pi(L')$. Therefore $(x_2,y_2)=0$ and $(x,y)\in L$.

Lemma 3.3. Let L (resp. L') be a Dirac structure on a (Hermitian) module M(resp. M'). Then
1) A homomorphism T: $M \to M'$, is a Dirac homomorphism if and only if
$T\circ A_L = A_{L'}\circ T$.
2) The composition of two Dirac homomorphisms is a Dirac homomorphism.
3) If the Dirac homomorphism T: $M \to M'$ is invertible, then T^{-1}:$M' \to M$ is a Dirac homomorphism.
4) The identity mapping I: $M \to M$ is a Dirac homomorphism.

Proof: 1) As we have seen earlier, $L = \{(x + A_L x, \quad x - A_L x), x\in M\}$. Assume that $T\circ A_L = A_{L'}\circ T$. Then, for each x$\in M$,

$$T\times T(x + A_L x, \quad x - A_L x) = (Tx + T\circ A_L x, \quad T\,x - T\circ A_L x) = \\ (Tx + A_{L'}(Tx), \quad T\,x - A_{L'}(Tx))\in L'.$$

Therefore, T is a Dirac homomorphism. Conversely, assume that T is a Dirac homomorphism. Then, for each x$\in M$, there exists a y$\in M'$ such that

$$T\times T(x + A_L x, \quad x - A_L x) = (y + A_{L'} y, \quad y - A_{L'} y).$$

Hence

$$(Tx + T\circ A_L x, \quad T\,x - T\circ A_L x) = (y + A_{L'} y, \quad y - A_{L'} y).$$

This equality yields Tx=y, and $T\circ A_L x = A_{L''} y = A_{L'}\circ Tx.$. Since, x is an arbitrary element of $M, T\circ A_L = A_{L'}\circ T$.
The proof of the rest is immediate. As a consequence of the above lemma we have

Proposition 3.3. Let L be a Dirac structure on a (Hermitian) module M. Then, the group of Dirac automorphisms of M is the centralizer of A_L in Aut(M).
The following assertions are trivial.
1) $A_{\pi(L)} = -A_L$.
2) Aut(L)=Aut(π(L)).
3) A necessary and sufficient condition for Aut(L) to be equal to Aut(M) is that A_L be in the centre of Aut(M). In this case L is called a central Dirac structure on M.

4. CONSTRUCTIONS WITH DIRAC STRUCTURES ON MODULES

Let L be a Dirac structure on M, M' a submodule of M and let M^* be the dual of M. Then:

1) Assume that M' is invariant under A_L. Let A′ denote the restriction of A_L to M'. The Dirac structure $L_{A'}$ is called the Dirac structure on M' induced by L. Clearly, $L_{A'}$ is a submodule of L. Equivalently:

Lemma 4.1. Let (M, L) and (M', L') be Dirac (Hermitian) R-modules and let M' be a submodule of M. A necessary and sufficient condition for (M', L') to be a Dirac submodule of (M, L) is that the canonical injection i:$M'\to M$ be a Dirac morphism.

2) Let L^* be the set of all elements of $(M\times M)^* = M^*\times M^*$, which are identically zero on $\pi(L)$. Observe that $\pi(L^*)$ is the set of those elements of $M^*\times M^*$ which are identically zero on L. This follows from $M\times M=L\oplus\pi(L)$, and $L\cap\pi(L)=\{0\}$. Let $q_1 : M\times M = L\oplus\pi(L)\to L$ and $q_2: M\times M\to L\oplus\pi(L)\to L$ be the first and the second canonical projections, and let $(\alpha,\beta)\in M^*\times M^*$. Clearly, $(\alpha,\beta)=(\alpha,\beta)\circ q_1\oplus(\alpha,\beta)\circ q_2$. But $(\alpha,\beta)\circ q_1 \in L^*$, and $(\alpha,\beta)\circ q_2\in\pi(L^*)$. Therefore, L^* is a Dirac structure on $M^*\times M^*$, which is called the Dirac structure on M^* associated with L. Let B be the automorphism associated with L. Then, it is easy to see that L^* is the Dirac structure on M^* associated with B^{*-1}.

3) Let X be a set, S be the set of all functions from X into the R-module M, and L be a Dirac structure on M. Clearly, S is an R-module. Let Q be the submodule of S×S defined by

$$Q=\{(f,g)\in S\times S \mid \mathrm{Im}(f,g)\subset L\},$$

where $\mathrm{Im}(f,g)=\{(f(x),g(x)) \mid x\in M\}$.Then, $Q\cap\pi(Q)=\{0\}$, and $S\times S = Q\oplus\pi(Q)$.
Therefore, Q is a Dirac structure on S. Its associated automorphism is defined by
$A_Q(f)=A_L\circ f$.

4) Let I be a set and let $\{M_i \mid i\in I\}$ be a family of R-modules. For each $i\in I$, let L be a Dirac structure on M_i. Let $M=\oplus M_i$, $L=\{(\oplus_{i\in I} x_i, \oplus_{i\in I} y_i) \mid (x_i,y_i)\in L_i\}$. It is easy to see that L is a Dirac structure on M, and its associated automorphism is $\oplus_{i\in I} A_{L_i}$.

The proof of the following assertions are immediate.

5) Let L be a Dirac structure on an (Hermitian) R-module M. Let φ: R′→R be an injective (∗-) homomorphism. Then L is a Dirac structure on M considered as an (Hermitian) R′-module.

6) Let L be a Dirac structure on a Hilbert R-module M. Assume that R^+ is the unitization of R. Then, L is a Dirac structure on the Hilbert R^+-module M.

7) Let L be a Dirac structure on a real vector space E. Then, L^c is a Dirac structure on the complex vector space E^c. Also, when L is a Dirac structure on a real Hilbert space H, L^c is a Dirac structure on the complex Hilbert space H^c.

Lemma 4.2. Let (M, L) and (M', L') be Dirac R-modules. Assume that $\varphi: M\to M'$ is a Dirac morphism. Then, ($Ker\,\varphi$, $Ker(\varphi\times\varphi)\cap L$) is a Dirac submodule of (M, L).

Proof: Let $(x,y)\in Ker(\varphi\times\varphi)\cap L$. Then, $\varphi(x)=\varphi(y)=0$. So, $x,y\in Ker(\varphi)$. Therefore, $p_1\pm p_2$: $Ker(\varphi\times\varphi)\cap L\to Ker\,\varphi$ is injective. Now assume that $z\in Ker\,\varphi\subset M$. There exists $(x,y)\in L$ such that $x+y=(p_1+p_2)(x,y)=z$. Since, $(p_1+p_2)(\varphi\times\varphi)(x,y)=\varphi\circ(p_1+p_2)(x,y)=\varphi(z)=0$, and $p_1+p_2 : L\to M$ is an isomorphism, $(\varphi\times\varphi)(x,y)=0$. Therefore, $(x,y)\in Ker(\varphi\times\varphi)\cap L$. In the same way we see that p_1-p_2:$Ker(\varphi\times\varphi)\cap L\to Ker\,\varphi$, is an isomophism. Clearly, $(Ker(\varphi\times\varphi)\cap L)\cap\pi(Ker(\varphi\times\varphi)\cap L)=\{0\}$. By Proposition 2.3 ($Ker(\varphi)$, $Ker(\varphi\times\varphi)\cap L$) is a Dirac submodule of (M,L).

Lemma 4.3. Let (M,L) and (M',L') be Dirac R-modules and let $\varphi: M\to M'$ be a Dirac morphism. Then, $(\mathrm{Im}(\varphi), \mathrm{Im}(\varphi\times\varphi)\cap L')$ is a Dirac submodule of (M',L').

Proof: Let $z\in M$. Then, there exists (x,y) $\in$ L such that (p_1+p_2)(x,y)=x+y=z. Since φ is a Dirac morphism

$$\varphi(z)=\varphi(x)+\varphi(y)=(p_1'+p_2')(\varphi(x),\varphi(y))=(p_1'+p_2')(\varphi\times\varphi)(x, y).$$

Therefore, $p_1'+p_2':(\varphi\times\varphi)(L)\to\varphi(M)$ is surjective. In the same way we see that $p_1'-p_2':(\varphi+\varphi)(L)\to\varphi(M)$ is also surjective. Since φ is a Dirac morphism, by Lemma 3.2 $\varphi\times\varphi(L)=\mathrm{Im}(\varphi\times\varphi)\cap L'$ and $\varphi\times\varphi(\pi(L))=\mathrm{Im}(\varphi\times\varphi)\cap\pi(L')$. So, $(\mathrm{Im}\,\varphi, \mathrm{Im}(\varphi\times\varphi)\cap L')$ is a Dirac submodule of (M',L').

Corollary 4.4. With the above notations and conventions $\varphi: (M,L)\to(M',L')$ is a Dirac morphism if and only if $(\varphi(M), \varphi\times\varphi(L))$ is a Dirac submodule of (M', L').

Lemma 4.5. Let (M', L') be a Dirac submodule of the Dirac module (M, L).Then, $\left(\left(\frac{M}{M'}\right),\left(\frac{L}{L'}\right)\right)$ is a Dirac module.

Proof: Since (M, L) and (M', L') are Dirac modules, $M\times M=L\oplus\pi(L)$ and $M'\times M'=L'\oplus\pi(L')$.
On the other hand,

$$\frac{M}{M'}\times\frac{M}{M'}=\frac{M\times M}{M'\times M'}=\frac{L}{L''}\oplus\frac{\pi(L)}{\pi(L')}=\frac{L}{L'}\oplus\pi\left(\frac{L}{L'}\right).$$

The proof is complete.

Definition 4.6. Let R be a commutative ring and for 1≤i≤n, let L_i be a Dirac structure on M_i, and let $A_i=A_{L_i}$. Clearly, $A=\otimes_{i=1}^{n}A_i$ is an automorphism of $M=\otimes_{i=1}^{n}M_i$. The Dirac structure L_A on M is called the tensor product of L_i.

Definition 4.7. Let R be as above and let L be a Dirac Structure on an R-module M. Then, $A=\wedge^n A_L$ is an automorphism of $\wedge^n M$. So, L_A is a Dirac structure on $\wedge^n M$ called the *n-th exterior product* of L.

5. PROJECTIVE DIRAC STRUCTURES

Definition 5.1. Let L be a Dirac structure on a (Hermitian) module M. Let $L_1=Ker(p_1)\cap L$ and let $M_1=p_2(L_1)$. L is called a *projective Dirac structure* if there exists a submodule M_0 (ortho)complement to M_1 in M which contains $p_1(L)$. The submodule $(p_1+p_2)|_L^{-1}(M_0)$ of L will be denoted by L_0. Clearly, $L=L_0\oplus L_1$, and $p_2(L_0)\subset M_0$.

Remark 5.2. When L is a Dirac structure on the Hermitian module M, it is clear that the submodule M_0, the orthocomplement of M_1, if it exists, is unique, and M_1 is orthogonal to $p_1(L)$. Thus, $p_1(L)$ is automatically contained in M_0. Moreover, since $p_1+p_2: L\to M$ is an isometry, L_0 and L_1 are orthocomplementary.

Proposition 5.3. Let *L* be a projective Dirac structure on *M*. Then, L_0 is a Dirac structure on M_0. Conversely, let M_0 and M_1 be (ortho)complementary in *M*. Let L_0 be a Dirac structure on M_0, and let $L_1 = \{(0,y) \mid y \in M_1\}$. Then, $L = L_0 \oplus L_1$ is a Dirac structure on *M*.

Proof: Clearly, $L_0 \cap \pi(L_0) = \{0\}$, and when *M* is a Hermitian module, L_0 and $\pi(L_0)$ are orthogonal to each other. As explained above, $p_1 \pm p_2 : L_0 \to M_0$ are bijective. Therefore, L_0 is a Dirac structure on M_0.

To prove the last part of the proposition, assume that (x,y) and (z,t) are in *L*. Then, there exist y_1 and y_2 in L_1 such that $(x, y-y_1)$ and $(z\ ,\ t-y_2)$ are in L_0. Assume that (x,y)=(t,z). Then, x=t and y=z. Moreover, from $x+y-y_1$ and $y+x-y_2$ in M_0 we have $y_1 = y_2$, and from $(x, y-y_1)-(y, x-y_1) \in L_0$ we have x=y. Since $(x, y-y_1) \in L_0$,
$y_1 \in M_0 \cap M_1 = 0$. Finally, from $(x,x) \in L_0$ we have x=0. Thus, $L \cap \pi(L) = \{0\}$. Let $m \in M$. Then, there exist $x \in M_0, y \in M_1$ such that m=x+y. Now, there exists $(u,v) \in L_0$ such that x=u+v. Thus, $(u,v+y) \in L$ and u+v+y=m. Therefore, $p_1 + p_2 : L \to M$ is surjective. In the same way one can see that $p_1 + p_2 : L \to M$ is surjective. Note that when *M* is a Hermitian module, *L* and π(*L*) are orthogonal to each other. Therefore, *L* is a Dirac structure on *M*.

Proposition 5.4. Let M_0 and M_1 be submodules of the (Hermitian) module *M*. A necessary and sufficient condition for $L = M_0 \times M_1$ to be a Dirac structure on *M* is that they are (ortho) complementary.
The proof is immediate.

Remark 5.5. Let *L* be a projective Dirac structure on *M*. With the notations of proposition 5.3 it is clear that (M_0, L_0) and (M_1, L_1) are Dirac submodules of (*M, L*).

Let *L* be a projective Dirac structure on *M*. As we have seen earlier, L_0 is a Dirac structure on M_0. *L*et $T : p_1(L) \to M$ be defined by $Tx = y \Leftrightarrow (x,y) \in L_0$. Clearly, *T* is a well-defined linear operator which will be called the homomorphism of M_0 induced by *L*. Note that, in general, *T* is not defined on all of M_0.

Proposition 5.6. Let *M* be a Hermitian R-module. Then:
1) If *L* is a projective Dirac structure on *M*, the associated homomorphism $T : M_0 \to M_0$ is anti-self-adjoint.
2) Conversely, assume that *M* is a Hilbert module. Let *T*:*M*→*M* be an anti-self-adjoint linear operator. Then *L*, the graph of *T,* is a Dirac structure on *M*.

Proof: 1) Let x and y be in $p_1(L) = \mathrm{Dom}(T)$. Then, $(x,Tx) \in L$ and $(Ty,y) \in \pi(L)$. Therefore, they are orthogonal and their inner product is zero, i.e., $\langle Ty|x\rangle + \langle y|Tx\rangle = 0$. Thus, *T* is anti-symmetric. Now, assume that there exist $u,v \in M_0$ such that for each $x \in \mathrm{Dom}(T)$, $\langle x|v\rangle = \langle Tx|u\rangle$. Then, for each $x \in \mathrm{Dom}(T)$

$$\langle (x,Tx)|(-v,u)\rangle = \langle x|-v\rangle + \langle Tx|u\rangle = -\langle x|v\rangle + \langle Tx|u\rangle = 0.$$

Thus, (-v,u) is orthogonal to L_0. But (-v,u) is also orthogonal to $Ker(p_1) \cap L$. Hence,
$(u,-v) \in L$. Thus, $u \in \mathrm{Dom}(T)$ and v=-*T*u. Therefore, *T* is anti-self-adjoint.
2) Since *T* is anti-self-adjoint, it is an anti-self-adjoint element of the C*-algebra $\mathbf{L}_M$ [7]. Therefore, 1 and -1 are not in the spectrum of T. Hence, $p_1 \pm p_2 : L \to M$ are surjective. On the other hand, π(*L*) is orthogonal to *L*. Therefore, by Proposition 1.2, *L* is a Dirac structure on *M* [4, *L*emma 2.18].

Remark 5.7. Let L be a projective Dirac structure on the (Hermitian) module M and let A and $T : M_0 \to M_0$ be its associated isomorphism and its induced homomorphism. Clearly, M_0 and M_1 are invariant under A and the restriction of A to M_1 is the identity. Let A_0 denote the restriction of A to M_0. Then, since $I \pm T : p_1(L) \to M_0$ are isomorphisms, B=(I-T)(I+T)$^{-1}$: $M_0 \to M_0$ is an isomorphism. Let $x \in M_0$. Then, there exists $z \in p_1(L)$ such that x=z+Tz. Hence, Bx=z-Tz. Thus, (x+Bx,x-Bx)=2(z,Tz)$\in L_0$. Since B$\in$Aut(M_0), L'={(x+Bx,x-Bx)|x$\in M_0$} is a Dirac structure on M_0. But $L' \subset L_0$. Hence, $L' = L_0$. Therefore, by Proposition 1.4, A_0=B=(I-T)(I+T)$^{-1}$. This equality implies that $T = (I - A_0)(I + A_0)^{-1}$. Therefore, A_0 and T are related to each other by the so-called Cayley transform.

6. THE RELATION BETWEEN DIRAC STRUCTURES ON HILBERT MODULES AND ON HILBERT SPACES

Let M be a Hilbert R-module and let α be a state of R. Let H be the associated Hilbert space. The canonical mapping M→H will be denoted by q. Let L be a Dirac structure on M. The closure of the image of L under q×q in H×H will be denoted by Λ. Let (x,y) and (u,v) be in L. Then α(<x,v>)+α(<y,u>)=α(<x,v>+<y,u>)=0, since, (x,y) and (v,u) are orthogonal to each other. Therefore, (q×q)(L) and π((q×q)(L)) are orthogonal to each other. Since (q×q)(L) is dense in Λ, Λ and π(Λ) are orthocomplementary. Therefore, Λ is a Dirac structure on H.

7. DIRAC STRUCTURES ON VECTOR BUNDLES

Let M be a smooth manifold and let η=(E,π,M,F) be a (Hermitian) vector bundle over M. The group of its strong bundle (isometric) automorphisms will be denoted by Aut(η). The first(resp. second) projection of η×η onto η will be denoted by P_1(resp. P_2). Let P:η×η→η×η be the the strong bundle isomorphism given as follows:
For each u,v$\in$E, P(u,v)=(v,u).

Definition 7.1. Let M be a smooth manifold and let η be a (Hermitian) smooth real or complex vector bundle over M. A subbundle λ of η×η is called a Dirac structure on η if λ and P(λ) are (ortho)complement.

We mention that a Dirac structure on η in the sense of [1] is a subbundle of the Whitney sum of η and its dual, such that for each x$\in M$, the fiber over x of λ is a Dirac structure on the fiber over x of η, in the sense of [1]. Since we can identify η with its dual by a Riemannian metric, this definition is a special case of our definition of Dirac structures on Hermitian vector bundles.

Proposition 7.2. Let λ be a Dirac structure on a (Hermitian) vector bundle (E, π, M, F). Then, $P_1 \pm P_2 : \lambda \to \eta$ are strong bundle (isometric) isomorphisms.

Conversely, let λ be a subbundle of the vector bundle η×η, λ∩P(λ)=0 (and P(λ) be orthogonal to λ) λ∩P(λ)={0}. Then a sufficient condition for λ to be a Dirac structure on η is that the strong bundle maps $P_1 \pm P_2$ be surjective.

The proof is the same as the proof of Proposition 2.3.

Example 7.3. Let J be a complex structure on a vector bundle η. It can be easily proved that the graph of J is a Dirac structure on η. Compare Example 1.4.
As a consequence of the first part of the above proposition we have:

Corollary 7.4. Any two Dirac structures on a (Hermitian) vector bundle are (isometrically) isomorphic.

The set of all Dirac structures on a (Hermitian) vector bundle η will be denoted by D(η).

Proposition 7.5. Let η be a (Hermitian) vector bundle. Then, there is a one-to-one correspondence between D(η) and Aut(η).
The proof is the same as the proof of Proposition 1.6.

Proposition 7.6. Let η=(E,π,*M*,F,< | >) be a Hermitian vector bundle, and let λ be a subbundle of η×η. A necessary and sufficient condition for λ to be a Dirac structure on η is that λ be maximally isotropic with respect to the pairing

$$< (x,y) | (u,v) >_+ = \frac{< x | v > + < y | u >}{2}.$$

Proof: Since, by definition, λ is a Dirac structure on the Hermitian vector bundle η, if and only if λ and P(λ) are orthocomplement, the proof is clear.

8. COMPLEMENTS OF DIRAC STRUCTURES ON VECTOR BUNDLES

Definition 8.1. Let for $i = 1,2, \lambda_i$ be Dirac structures on (Hermitian) vector bundles $\eta_i = (E_i, \pi_i, M_i, F_i)$. A bundle map $(T, f): \eta_1 \to \eta_2$ is said to be Dirac if the restriction of the bundle map $(T \times T, f): \eta_1 \times \eta_1 \to \eta_2 \times \eta_2$, to λ_1, carries λ_1 into λ_2.

Lemma 8.2. Let $\lambda_i, i = 1,2,$ be Dirac structures on (Hermitian) vector bundles $\eta_i = (E_i, \pi_i, M_i, F_i)$. Then
1) A bundle map $(T, f): \eta_1 \to \eta_2$, , is Dirac, if and only if $T \circ A_{\lambda_1} = A_{\lambda_2} \circ T$.
2) The composition of two Dirac bundle maps is a Dirac bundle map.
3) If the Dirac bundle map (T,f) is invertible, then (T^{-1},f^{-1}) is also a Dirac bundle map.
4) The identity bundle map is Dirac.
The proof is the same as that of Lemma 3.2.
From the above lemma we have the following.

Proposition 8.3. Let λ be a Dirac structure on a (Hermitian) vector bundle η. Then the Dirac gauge group of λ is the centralizer of A_λ in Aut(η).

The proof of the following assertions is straightforward.
1) Let λ be a Dirac structure on a (Hermitian) vector bundle η. Assume that the subbundle η′ is invariant under A_λ. Then, λ′=λ∩(η′×η′) is a Dirac structure on the (Hermitian) vector bundle η′. Moreover, $A_{\lambda'} = A_\lambda |_{\eta'}$.
2) Let *M* and *M*′ be finite-dimensional manifolds and let f: *M*′→*M* be a smooth map. Assume that λ is a Dirac structure on a (Hermitian) vector bundle η over *M*. Then, f* (λ) is a Dirac structure on the (hermitian) vector bundle f* (η) over *M*′.
3) Let λ be a Dirac structure on a (Riemannian) real vector bundle η. Then, λ^c is a Dirac structure on the (Hermitian) vector bundle η^c }.
4) Let λ be a Dirac structure on a vector bundle η, with dual η*. Let λ* be the subbundle of η*×η* which annihilates P(λ). Then, λ* is a Dirac structure on η*, and $A_{\lambda^*} = \left(A_\lambda^*\right)^{-1}$.
5) Let $(\eta_i)_{i \in I}$ be a family of (Hermitian) vector bundles over a manifold *M*, and let η be their (orthogonal) Whitney sum. For each i∈I, let λ_i be a Dirac structure on η_i , and let $p_{ik} : \eta_i \times \eta_i \to \eta_i, k$ =1, 2 be the

first and the second projections. Let λ be the Whitney sum of λ_i's. Then, the image of λ under the bundle map

$$\left(\oplus_{i\in I} P_{iI}, \oplus_{i\in I} P_{i2}\right) : \lambda \to \eta \times \eta$$

is a Dirac structure on η.

9. THE RELATION BETWEEN DIRAC STRUCTURES ON MODULES AND DIRAC STRUCTURES ON VECTOR BUNDLES

Let *M* be a smooth manifold, and let η=(E, π, *M*, F) be a vector bundle over *M*. For simplicity assume that *M* is compact. Let R denote the C^*-algebra of continuous functions on *M*, and let H denote the R-module of sections of η. With the aid of a Hermitian inner product we identify η with its dual. Then, H becomes a Hermitian R-module.

Let *L* be the total space of a Dirac structure λ on the vector bundle η, and let Λ denote the set of all sections of λ. Clearly, Λ is a submodule of H×H. It is clear that Λ∩π(Λ)={0}. On the other hand, since $P_1 \pm P_2 : L \to E$ are surjective, $P_1 \pm P_2 : \wedge \to H$ are surjective. Therefore, Λ is a Dirac structure on H.

Now, let H be a finitely generated projective R-module, and let Λ be a Dirac structure on H. Clearly, Λ is a projective R-module. Therefore, there exists a vector bundle η over *M*, the module of sections of which is H, and there exists a subbundle λ of η×η with total space *L* admitting Λ as its module of sections. Since Λ∩π(Λ)={0}, we have *L*∩P(*L*)={0}. Assume that for $m \in M, (u,v) \in E_m \times E_m$. Let (x,y) be a section of η×η such that x(m)=u, and y(m)=v. Since (x,y)∈H×H, there exist $(x_1, y_1) \in \wedge$ and $(y_2, x_2) \in \pi(\wedge)$ such that $(x, y) = (x_1, y_1) + (y_2, x_2)$. Hence, $(u,v) = (x_1(m), y_1(m)) + (y_2(m), x_2(m)) \in E_m \times E_m$. Therefore, λ is a Dirac structure on η.

Let *M* be a compact manifold. As is well-known, there is a one-to-one correspondence between Hermitian $C(M)$-module H and Hermitian vector bundles η over *M* [8]. If two finitely generated submodules of H are orthogonal to each other, their corresponding subbundles of η are also orthogonal, and vice versa.

From the above considerations we have the following.

Proposition 9.1. Let *M* be a compact smooth manifold. Assume that η is a (Hermitian) vector bundle over *M*, and λ is a subbundle of η. Let H (resp. Λ) be the C(*M*)-module of sections of η (resp. λ). Then a necessary and sufficient condition for λ to be a Dirac structure on the (Hermitian) vector bundle η is that Λ be a Dirac structure on the (Hermitian) $C(M)$-module H.

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