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INTEGRAL CHARACTERIZATIONS FOR TIMELIKE AND SPACELIKE CURVES ON THE LORENTZIAN SPHERE S_1^3 *

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Abstract – V. Dannon showed that spherical curves in E^4 can be given by Frenet-like equations, and he then gave an integral characterization for spherical curves in E^4. In this paper, Lorentzian spherical timelike and spacelike curves in the space time R_1^4 are shown to be given by Frenet-like equations of timelike and spacelike curves in the Euclidean space E^3 and the Minkowski 3-space R_1^3. Thus, finding an integral characterization for a Lorentzian spherical R_1^4-timelike and spacelike curve is identical to finding it for E^3 curves and R_1^3-timelike and spacelike curves. In the case of E^3 curves, the integral characterization coincides with Dannon's.

Let $\{T,\ N,\ B\}$ be the moving Frenet frame along the curve $\alpha(s)$ in the Minkowski space R_1^3. Let $\alpha(s)$ be a unit speed C^4-timelike (or spacelike) curve in R_1^3 so that $\alpha'(s)=T$. Then, $\alpha(s)$ is a Frenet curve with curvature $\kappa(s)$ and torsion $\tau(s)$ if and only if there are constant vectors a and b so that
(i) $T'(s)=\kappa(s)\left\{a\cos\xi(s)+b\sin\xi(s)+\int_0^s \cos\left[\xi(s)-\xi(\delta)\right]T(\delta)\kappa(\delta)\,d\delta\right\},\ T$ is timelike,
(ii) $T'(s)=\kappa(s)\left\{ae^{\xi}+be^{-\xi}+\int_0^s \cosh\left(\xi(s)-\xi(\delta)\right)T(\delta)\kappa(\delta)\,d\delta\right\},\ N$ is timelike,
where $\xi(s)=\int_0^s \tau(\delta)\,d\delta.$

Keywords – Lorentzian 3-sphere, timelike curve, spacelike curve, curvature

1. INTRODUCTION

The differential equation characterizing a spherical curve is given by

$$\frac{d}{ds}\left\{\frac{1}{\tau(s)}\frac{d\rho}{ds}\right\}+\rho(s)\,\tau(s)=0, \tag{1}$$

where s is the length of arc, $\rho(s)=1/\kappa(s)$ is the radius of curvature and $\tau(s)$ is the torsion of the curve [1].

Breuer and Gottlieb [2] gave an explicit solution of this differential equation:

$$\rho(s)=a\cos\int_0^s \tau(\delta)\,d\delta+b\sin\int_0^s \tau(\delta)\,d\delta,$$

where a and b are arbitrary constants [3].

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Let $\alpha(s)$ be a unit speed C^3 curve in E^3 so that $\alpha'(s)=T$. In [4], V. Dannon showed that α is a Frenet curve with curvature $\kappa(s)$ and torsion $\tau(s)$ if and only if there are constant vectors a and b so that

$$T'(s)=\kappa(s)\left\{a\cos\xi(s)+b\sin\xi(s)-\int_0^s\cos\left[\xi(s)-\xi(\delta)\right]T(\delta)\kappa(\delta)\,d\delta\right\},$$

where $\xi(s)=\int_0^s\tau(\delta)\,d\delta.$

The differential equation characterizing a Lorentzian spherical curve in Minkowski 3-space R_1^3 and explicit solutions of this differential equation are given in [5, 6].

On the other hand, it is also shown in [5, 6] that a unit speed timelike (resp. spacelike) R_1^3 curve lies on a Lorentzian sphere if and only if there is a function $f\in C^1$ so that

$$f\tau=(1/\kappa)', \quad f'+\tau/\kappa=0 \text{ (resp. } f'-\tau/\kappa=0).$$

Set $\rho=1/\kappa$ and consider those equations rewritten in the form

$$\rho'=\tau f, \quad f'=-\tau\rho \text{ (resp. } f'=\tau\rho).$$

Thus, the characteristic of a Lorentzian spherical curve in R_1^n is the Frenet Pattern. So the problem of getting an integral characterization to a Lorentzian spherical curve is nothing else but the integration of Frenet equations. This means that Lorentzian correspondence of the Breuer-Gottlieb characterization is isomorphic to the integral characterization of Frenet equations in the Lorentzian plane. Then, the methods used to obtain characterizations of Lorentzian spherical curves can be extended to include Frenet equations in R_1^3.

2. PRELIMINARIES

Space-time R_1^4 is a Euclidean space R^4 provided with the standard flat metric given by

$$\mathrm{g}=dx_1^2+dx_2^2+dx_3^2-dx_4^2,$$

where (x_1,x_2,x_3,x_4) is a rectangular coordinate system in R_1^4.

Since g is an indefinite metric, an arbitrary vector $v\in R_1^4$ can have one of three causal characters: it can be spacelike if $\mathrm{g}(v,v)>0$ or $v=0$, timelike if $\mathrm{g}(v,v)<0$, and null (lightlike) if $\mathrm{g}(v,v)=0$ and $v\neq 0$. Similarly, an arbitrary curve $\alpha=\alpha(s)$ in R_1^4 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. Also, recall that the norm of a vector v is given by $\|v\|=\sqrt{|\mathrm{g}(v,v)|}$. Therefore, v is a unit vector if $\mathrm{g}(v,v)=\pm 1$. Two vectors v, w in R_1^4 are said to be orthogonal if $\mathrm{g}(v,w)=0$. The velocity of a curve $\alpha(s)$ is given by $\|\alpha'(s)\|$.

The Lorentzian sphere with center $m=(m_1,m_2,m_3,m_3)\in R_1^4$ and radius $r\in R^+$ in the space-time R_1^4 is the hyperquadric

$$S_1^3=\left\{a=(a_1,a_2,a_3,a_4)\in R_1^4 \;\middle|\; \mathrm{g}(a-m,a-m)=r^2\right\}.$$

Let us denote the moving Frenet frame along the spacelike curve $\alpha(s)$ in the space R_1^4 by $\{T_1,T_2,T_3,T_4\}$. Then T_1,T_2,T_3 and T_4 are the tangent, the principal normal, the first binormal and second binormal vector fields, respectively. A timelike or spacelike curve $\alpha(s)$ is said to be parameterized by arc length function s if $\mathrm{g}(\alpha'(s),\alpha'(s))=\mp 1$.

Let $\alpha(s)$ be a curve in the space time R_1^4 parameterized by arc length function s. Then, for the curve $\alpha(s)$ the following Frenet equations are given in [7]:

If T_1 is timelike and the others are spacelike, then the Frenet formulae has the form

$$\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix},$$

if T_2 is timelike and the others are spacelike, then Frenet equations are given by

$$\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix},$$

and finally, if T_3 is timelike and the others are spacelike, then

$$\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \mu \\ 0 & 0 & \mu & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}.$$

3. S_1^3 CURVES AND R_1^3 FRENET CURVES

Similar to the extension (based on Wong's condition [8]) given by Dannon [4], we give an extension of Petrovic-Torgasev and Sucurovic's condition [5, 6] to R_1^4 for expressing the connection between S_1^3 curves and R_1^3 Frenet curves.

In R_1^4, since T_4 is the fourth orthonormal vector to T_1, T_2, T_3, $\mu(s) = g(T_3', T_4)$ measures the change of direction of the space spanned by T_1, T_2, T_3. Then we have the following cases:

Case 1: T_1 is timelike:

Proposition 3.1. Let $\alpha(s)$ be an R_1^4 unit speed C^5 timelike Frenet curve with curvature functions $\kappa(s)$, $\tau(s)$, $\mu(s)$. Then the following are equivalent:

i) $\alpha(s)$ lies on a R_1^4 Lorentzian sphere.

ii) $\kappa(s) \neq 0$ and there are two C^2 functions, $f(s)$ and $g(s)$, so that

$$\rho' = \tau f, \quad f' = -\tau\rho + \mu g \quad (\rho = 1/\kappa), \quad g' = -\mu f. \tag{2}$$

Note that, the equations (2) are the Frenet formulae of an E^3 curve in R_1^4.

Proof: i) $\Rightarrow$ ii) Assume that $\alpha(s)$ lies on an R_1^4 Lorentzian sphere of radius a which we may assume to have center x_0 at the origin 0. By repeated differentiation of $a^2 = g(\alpha, \alpha)$ and using Frenet equations, we obtain $\kappa \neq 0$ and $\boldsymbol{f}' = \boldsymbol{F}\boldsymbol{f}$, where $f_i = g(\alpha, T_i)$, $i = 2, 3, 4$,
$(\rho = f_2, f = f_3$ and $g = f_4)$.

ii) $\Rightarrow$ **i)** Given $\boldsymbol{f}' = \boldsymbol{F}\boldsymbol{f}$, and define the curve $\gamma(s) = \left(\alpha - \sum_{i=2}^{4} f_i T_i\right)(s)$, then $\frac{d}{ds}\left\{\alpha - \sum_{i=2}^{4} f_i T_i\right\} = 0.$ Therefore, $\alpha - \sum_{i=2}^{4} f_i T_i = \text{const.} \equiv x_0$. Then we have $\alpha - x_0 = \sum_{i=2}^{4} f_i T_i$. Using the orthogonality gives $g(\alpha - x_0, \alpha - x_0) = \sum_{i=2}^{4} f_i^2$. Differentiation of this gives $g'(\alpha - x_0, \alpha - x_0) = 0$, so $g(\alpha - x_0, \alpha - x_0) = \text{const.} = a^2$, i.e. α lies on the Lorentzian 3-sphere of radius a about x_0.

Case 2: T_2 is timelike:

Proposition 3.2. Let $\alpha(s)$ be an R_1^4 unit speed C^5 spacelike Frenet curve with curvature functions $\kappa(s), \tau(s), \mu(s)$. Then the following are equivalent:

i) $\alpha(s)$ lies on an R_1^4 Lorentzian sphere.

ii) $\kappa(s) \neq 0$ and there are two C^2 functions $f(s)$ and $g(s)$ so that

$$\rho' = \tau f, \quad f' = \tau\rho + \mu g \quad (\rho = 1/\kappa),\ g' = -\mu f \text{ and } f^2 + g^2 > \rho^2. \tag{3}$$

Note that the equations (3) are the Frenet formulae of an R_1^3 timelike curve.

Proof: The proof is similar to that of Proposition 3.1.

Case 3: T_3 is timelike:

Proposition 3.3. Let $\alpha(s)$ be an R_1^4 unit speed C^5 spacelike Frenet curve with curvature functions $\kappa(s), \tau(s), \mu(s)$. Then the following are equivalent:

i) $\alpha(s)$ lies on an R_1^4 Lorentzian sphere.

ii) $\kappa(s) \neq 0$ and there are two C^2 functions, $f(s)$ and $g(s)$ so that

$$\rho' = \tau f, \quad f' = \tau\rho + \mu g \quad (\rho = 1/\kappa),\ g' = \mu f \text{ and } \rho^2 + g^2 > f^2. \tag{4}$$

Proof: The proof is similar to that of Proposition 3.1.

4. THE INTEGRAL CHARACTERIZATIONS OF R_1^3 FRENET CURVES

In section 3, we see that R_1^4 timelike or spacelike spherical curve equations have the structure of R_1^3 timelike or spacelike Frenet curves. Consequently, finding an integral characterization for an R_1^4 Lorentzian spherical curve is identical to finding it for an R_1^3 curve. Thus, to obtain characterizations of Lorentzian spherical curves, we extend the method given in [4].

Case 1: T_1 is timelike:

The two bottom E^3 Frenet equations are of the form

$$g'(s) = -\lambda f + \mu h,\ \ h'(s) = -\mu g.$$

Assuming that μ is non vanishing (the conclusions are free from this assumption), then we get

$$g'/\mu = -(\lambda f/\mu) + h.$$

Differentiation, then substitution of h' and eventually application of the change of variable $\xi(s) = \int_0^s \mu(\delta)\, d\delta$, reduce this equation to

$$\ddot{g} + g = -\left(\lambda f/\mu\right)^{\boldsymbol{\cdot}},$$

where the variable is ξ. A particular solution for $g(\xi)$ is

$$-\int_0^s \cos\left[\xi(s)-\xi(\delta)\right]\lambda(\delta)\,f(\delta)\,d\delta.$$

Then we have:

Theorem 4.1. Let $\alpha(s)$ be a C^4 curve in E^3 parameterized by its arc length s so that $\alpha'(s)=T.$ Then the following are equivalent:
i) $\alpha(s)$ has a Frenet system $T,\ N,\ B$, curvature $\kappa(s)$ and torsion $\tau(s)$ that satisfy
the Frenet equations

$$T'(s)=\kappa N,\ \ N'(s)=-\kappa T+\tau B,\ \ B'(s)=-\tau N.$$

ii) There are constant vectors a and b so that

$$T'(s)=\kappa(s)\left\{a\cos\xi+b\sin\xi-\int_0^s \cos\left[\xi(s)-\xi(\delta)\right]T(\delta)\kappa(\delta)d\delta\right\},$$

where $\xi(s)=\int_0^s \tau(\delta)\,d\delta.$

Proof : The characterization coincides with Dannon's.

Case 2: T_2 is timelike:
The two bottom R_1^3 timelike Frenet equations are of the form

$$g'(s)=\lambda f+\mu h,\ \ h'(s)=-\mu g.$$

Assuming that μ is non vanishing (the conclusions are free from this assumption), then we get

$$g'/\mu=(\lambda f/\mu)+h.$$

Differentiation, then substitution of h' and eventually application of the change of variable $\xi(s)=\int_0^s \mu(\delta)\,d\delta,$ reduce this equation to

$$\ddot{g}+g=\left(\lambda f/\mu\right)^{\cdot},$$

where the variable is ξ. A particular solution for $g(\xi)$ is

$$\int_0^s \cos\left[\xi(s)-\xi(\delta)\right]\lambda(\delta)\,f(\delta)\,d\delta.$$

Then we have:

Theorem 4.2. Let $\alpha(s)$ be a C^4 timelike curve in R_1^3 parameterized by its arc length s so that $\alpha'(s)=T.$ Then the following are equivalent:
i) $\alpha(s)$ has a Frenet system $T,\ N,\ B$, curvature $\kappa(s)$ and torsion $\tau(s)$ that satisfy
the Frenet equations

$$T'(s)=\kappa N,\ \ N'(s)=\kappa T+\tau B,\ \ B'(s)=-\tau N.$$

ii) There are constant vectors a and b so that

$$T'(s)=\kappa(s)\left\{a\cos\xi+b\sin\xi+\int_0^s\cos\left[\xi(s)-\xi(\delta)\right]T(\delta)\kappa(\delta)d\delta\right\},$$

where $\xi(s)=\int_0^s\tau(\delta)\,d\delta.$

Proof: **ii**) $\Rightarrow$ **i**) Suppose the condition holds. Put

$$N(s)=\int_0^s\cos\left[\xi(s)-\xi(\delta)\right]T(\delta)\kappa(\delta)d\delta+a\cos\xi+b\sin\xi$$

and

$$B(s)=\int_0^s\sin\left[\xi(\delta)-\xi(s)\right]T(\delta)\kappa(\delta)d\delta-a\sin\xi+b\,\cos\xi.$$

Then N and B satisfy the Frenet equations.

i) $\Rightarrow$ **ii**) If the equations hold, the above N and B solve the coupled system

$$N'=\kappa T+\tau B,\ \ B'=-\tau N.$$

The last equation $\kappa N=T'$ is our condition.

Case 3: T_3 is timelike:

The two bottom R_1^3 spacelike Frenet equations are of the form

$$g'(s)=\lambda f+\mu h,\ \ h'(s)=\mu g.$$

Assuming that μ is non vanishing (the conclusions are free from this assumption), then we get

$$g'/\mu=(\lambda f/\mu)+h\,.$$

Differentiation, then substitution of h' and eventually application of the change of variable $\xi(s)=\int_0^s\mu(\delta)\,d\delta,$ reduce this equation to

$$\ddot{g}-g=\left(\lambda f/\mu\right)^{\boldsymbol{\cdot}},$$

where the variable is ξ. A particular solution for $g(\xi)$ is

$$\int_0^s\cosh\left[\xi(s)-\xi(\delta)\right]\lambda(\delta)f(\delta)\,d\delta.$$

Then we have:

Theorem 4.3. Let $\alpha(s)$ be a C^4 spacelike curve in R_1^3 parameterized by its arc length s so that $\alpha'(s)=T.$ Then the following are equivalent:

i) $\alpha(s)$ has a Frenet system $T,\ N,\ B$, curvature $\kappa(s)$ and torsion $\tau(s)$ that satisfy
the Frenet equations

$$T'(s)=\kappa N,\ \ N'(s)=\kappa T+\tau B,\ \ B'(s)=\tau N.$$

ii) There are constant vectors a and b so that

$$T'(s)=\kappa(s)\left\{\ ae^{\xi}+be^{-\xi}+\int_0^s\cosh\left(\xi(s)-\xi(\delta)\right)T(\delta)\kappa(\delta)d\delta\ \right\},$$

where $\xi(s) = \int_0^s \tau(\delta)\, d\delta$.

Proof: **ii)** $\Rightarrow$ **i)** Suppose the condition holds. Put

$$N(s) = \int_0^s \cosh\left[\xi(s) - \xi(\delta)\right] T(\delta)\kappa(\delta) d\delta + a\, e^{\xi} + b\, e^{-\xi}$$

and

$$B(s) = -\int_0^s \sinh\left[\xi(\delta) - \xi(s)\right] T(\delta)\kappa(\delta) d\delta + a\, e^{\xi} - b\, e^{-\xi}.$$

Then N and B satisfy the Frenet equations.

i) $\Rightarrow$ **ii)** If the equations hold, the above N and B solve the coupled system

$$N' = \kappa T + \tau B, \;\; B' = -\tau N.$$

The last equation $\kappa N = T'$ is our condition.

The method extends to R_1^n timelike Frenet curves through successive application of transformations of the form $\int_0^s A(\delta)\, d\delta$ by using the terminology of [9].

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