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CAPACITY ON FINSLER SPACES*

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Abstract – Here, the concept of electric capacity on Finsler spaces is introduced and the fundamental conformal invariant property is proved, i.e. the capacity of a compact set on a connected non-compact Finsler manifold is conformal invariant. This work enables mathematicians and theoretical physicists to become more familiar with the global Finsler geometry and one of its new applications.

Keywords – Capacity, conformal invariant, Finsler space

1. INTRODUCTION

Finsler space is the most natural and advanced generalization of Euclidean space, which has many applications in theoretical physics. The physical notion of capacity is the electrical capacity of a 2-dimensional conducting surface, which is defined as the ratio of a given positive charge on the conductor to the value of the potential on its surface.

The capacity of a set as a mathematical concept was introduced first by N. Wiener in 1924 and was subsequently developed by O. Forstman [1], C. J. de La Vallee Poussin, and several other physicists and mathematicians in connection with the potential theory.

The concept of conformal capacity was introduced by Loewner [2] and has been extensively developed for $\mathbb{R}^n$ [3-6]. In particular, it was used by G.D. Mostow to prove his famous theorem on the rigidity of hyperbolic spaces [5]. The concept of capacity on Riemannian geometry was introduced by J. Ferrand [7] and developed in the joint work’s of M. Vuorinan and G.J. Martin [8] and [9].

Here, we introduce the concept of capacity for Finsler spaces and prove that, it depends only on the conformal structure of (M,g), more precisely:

Theorem: Let (M,g) be a connected non-compact Finsler manifold, then the capacity of a compact set on M is a conformal invariant.

1. PRELIMINARIES

1.1. Finsler metric

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Let M be an n-dimensional C^∞ manifold. For a point $x \in M$, denote by T_xM the tangent space of M at x. The tangent bundle TM on M is the union of tangent spaces T_xM. We will denote the elements of TM by (x, y) where $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) := x$. Throughout this paper we use the *Einstein summation convention* for the expressions with repeated indices. That is, wherever an index appears twice, once as a subscript, and once as a superscript, then that term is summed over all values of that index.

A *Finsler structure* on a manifold M is a function $F : TM_0 \to [0,\infty)$ with the following properties: (i) F is C^∞ on TM_0. (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e. $\forall \lambda > 0 \quad F(x,\lambda y) = \lambda F(x, y)$. (iii) The Hessian of F^2 with elements $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^i y^j}$ is positive definite on TM_0. We recall that, g_{ij} is a homogeneous tensor of degree zero in y and $g_{ij}(x, y)y^i y^j = g(y, y)$, where $g(\ ,\ )$ is the local scalar product on any point of TM_0. Then the pair (M, g) is called a *Finsler manifold.* The Finsler structure F is Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

1.2. Notations on conformal geometry of Finsler manifolds

Let's consider two n-dimensional Finsler manifolds (M, g) and (M', g') with Finsler structures F and F' and with line elements (x, y) and (x', y') respectively. Throughout this paper we shall assume that coordinate systems on (M, g) and (M', g') have been chosen so that $x'^i = x^i$ and $y'^i = y^i$ holds for all i, unless a contrary assumption is explicitly made. Using this assumption these manifolds can be denoted simply by M and M', respectively. Let u and v be two tangent vectors at a point x of a Finsler manifold (M, g). The *angle* θ *of* v *with respect to* u is defined by

$$\cos\theta = \frac{g_{ij}(x,u)u^i v^j}{\sqrt{g_{ij}(x,u)u^i u^j}\sqrt{g_{ij}(x,u)v^i v^j}}.$$

Clearly this notion of angle is not symmetric. A diffeomorphism $f : M \to M'$ between two Finsler manifolds is called *conformal* if for each $p \in M$, $(f_*)_p$ preserves the angles of any tangent vector, with respect to any y in M. In this case the two Finsler manifolds are called *conformal equivalent* or simply *conformal*. If $M = M'$ then f is called a *conformal transformation* or *conformal automorphism*. It can be easily checked that a diffeomorphism is conformal if and only if $f^* g' = e^{2\sigma} g$ for some function $\sigma : M \to IR$ (this result is due to Knebelman [10]. In fact, the sufficient condition implies that the function $\sigma(x, y)$ be independent of direction y, or equivalently $\partial\sigma/\partial y^i = 0$). The diffeomorphism f is called an *isometry* if $f^* g' = g$. Two Finsler structures F and F' are called *conformal* if $F'(x, y) = e^\sigma F(x, y)$ or equivalently, $g' = e^{2\sigma(x)} g$. Locally we have $g'_{ij}(x, y) = e^{2\sigma(x)} g_{ij}(x, y)$, and $g'^{ij}(x, y) = e^{-2\sigma(x)} g^{ij}(x, y)$.

1.3. Some vector bundles and their properties

Let $\pi : TM \longrightarrow M$ be the natural projection from TM to M. The *pull-back tangent space* π^*TM is defined by $\pi^*TM := \{(x, y, v) \mid y \in T_xM_0, v \in T_xM\}$. The *pull-back cotangent space* π^*T^*M is the dual of π^*TM. Both π^*TM and π^*T^*M are n-dimensional vector spaces over TM_0 [11, 12]. We denote by S_xM the set consisting of all rays $[y] := \{\lambda y \mid \lambda > 0\}$, where $y \in T_xM_0$. Let $SM = \bigcup_{x\in M} S_xM$, then SM has a natural $(2n-1)$ dimensional manifold structure and the total space of a fiber bundle, called *Sphere bundle* over M. We denote the elements of SM by $(x,[y])$ where $y \in T_xM_0$. Let $p : SM \longrightarrow M$ denote the natural projection from SM to M. The *pull-back tangent space* p^*TM is defined by $p^*TM := \{(x,[y], v) \mid y \in T_xM_0, v \in T_xM\}$. The *pull-back cotangent space*

p^*T^*M is the dual of p^*TM. Both p^*TM and p^*T^*M are total spaces of vector bundles over SM. We use the following Lemma for replacing the C^∞ functions on TM_0 by those on SM.

Lemma 1.1. [13] Let η be the function $\eta : TM_0 \longrightarrow SM$, where $\eta(x, y) = (x,[y])$ and $f \in C^\infty(TM_0)$. Then there exists a function $g \in C^\infty(SM)$ satisfying $\eta^* g = f$ if and only if $f(x, y) = f(x, \lambda y)$, where $y \in T_x M_0, \lambda > 0$ and η^* is the pull-back of η.

Let $f \in C^\infty(M)$, the vertical lift of f denoted by $f^V \in C^\infty(TM_0)$, be defined by $f^V : TM \longrightarrow IR$, where $f^V(x, y) := f \circ \pi(x, y) = f(x)$. f^V is independent of y and from Lemma 1.1 there is a function g on $C^\infty(SM)$ related to f^V by means of $\eta^* g = f^V$. In the sequel g is denoted by f^V for simplicity. It is well known that, if the differentiable manifold M is compact then the Sphere bundle SM is compact, and also it is orientable whether M is orientable or not [14, 15]).

1.4. Nonlinear connections

1.4.1. Nonlinear connection on the tangent bundle TM

Consider $\pi_* : TTM \longrightarrow TM$ and put $ker\pi_*^v = \{z \in TTM \mid \pi_*^v(z) = 0\}, \forall v \in TM$, then the vertical vector bundle on M is defined by $VTM = \bigcup_{v\in TM} ker\pi_*^v$. A *non-linear connection* or a *horizontal distribution* on TM is a complementary distribution HTM for VTM on TTM. These functions are called coefficients of the non-linear connection and will be noted in the sequel by N_i^j. It is clear that HTM is a vector sub-bundle of TTM called horizontal vector bundle. Therefore we have the decomposition $TTM = VTM \oplus HTM$.

Using the induced coordinates (x^i, y^i) on TM, where x^i and y^i are called, respectively, *position* and *direction* of a point on TM, we have the local field of frames $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$ on TTM. Let $\{dx^i, dy^i\}$ be the dual of $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$. It is well known that we can choose a local field of frames $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y_i}\}$ adapted to the above decomposition, i.e. $\frac{\delta}{\delta x^i} \in \chi(HTM)$ and $\frac{\partial}{\partial y_i} \in \chi(VTM)$. They are sections of horizontal and vertical bundles, HTM and VTM, defined by $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x_i} - N_i^j \frac{\partial}{\partial y_j}$, where $N_i^j(x, y)$ are the coefficients of non linear $\gamma^i{}_{jk} := \frac{1}{2} g^{is}(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j})$ and $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$.

1.4.2. Nonlinear connections on the sphere bundle SM

Using the coefficients of non linear connection on TM, one can define a non linear connection on SM by using the objects which are invariant under positive re-scaling $y \mapsto \lambda y$. Our preference for remaining on SM forces us to work with $\frac{N^i{}_j}{F} := \gamma^i{}_{jk} l^k - C^i{}_{jk} \gamma^k{}_{rs} l^r l^s$, where $l^i = \frac{y^i}{F}$. We also prefer to work with the local field of frames $\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^j}\}$ and $\{dx^i, \frac{\delta y^j}{F}\}$, which are invariant under the positive re-scaling of y, and therefore, live over SM. They can also be used as a local field of frames over tangent bundle p^*TM and cotangent bundle p^*T^*M respectively.

1.5. A Riemannian metric on SM

It turns out that the manifold TM_0 has a natural Riemannian metric, known in the literature as *Sasaki metric* [12, 16]); $\tilde{g} = g_{ij}(x, y)dx^i \otimes dx^j + g_{ij}(x, y)\frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}$, where $g_{ij}(x, y)$ is the Hessian of Finsler structure F^2. They are functions on TM_0 and invariant under positive re-scaling of y, therefore they can be considered as functions on SM. With respect to this metric, the *horizontal subspace* spanned by $\frac{\delta}{\delta x^j}$ is orthogonal to the *vertical subspace* spanned by $F\frac{\partial}{\partial y^i}$. The metric $\tilde{g}$ is invariant under the positive re-scaling of y and can be considered as a Riemannian metric on SM.

1.6. Hilbert form

Consider the pull-back vector bundle p^*TM over SM. The pull-back tangent bundle p^*TM has a canonical section l defined by $l_{(x,[y])} = (x,[y], \frac{y}{F(x,y)})$. We use the local coordinate system (x^i, y^i) for

SM , where y^i are homogeneous coordinates up to a positive factor. Let $\{\partial_i\}$ be a natural local field of frames for p^*TM , where $\partial_i := (x,[y],\frac{\partial}{\partial x^i})$. The natural dual co-frame for p^*T^*M is noted by $\{dx^i\}$. The Finsler structure $F(x,y)$ induces a canonical 1-form on SM defined by $\omega := l_i dx^i$, where $l_i = g_{ij} l^j$ and ω is called the *Hilbert form* of F . Using $g_{ij} = FF_{y^i y^j} + F_{y^i} F_{y^j}$ and $\frac{\delta F}{\delta x^i} = 0$, with a straightforward calculation we get

$$d\omega = -(g_{ij} - l_i l_j) dx^i \wedge \frac{\delta y^j}{F}. \tag{1}$$

1.7. Gradient vector field

For a Riemannian manifold $(SM, \tilde{g})$, the gradient vector field of a function $f \in C^\infty(SM)$ is given by $\tilde{g}(\nabla f, \tilde{X}) = df(\tilde{X}), \forall \tilde{X} \in \chi(SM)$. Using the local coordinate system $(x^i,[y^i])$ for SM , the vector field $\tilde{X} \in \chi(SM)$ is given by $\tilde{X} = X^i(x,y)\frac{\delta}{\delta x^i} + Y^i(x,y) F \frac{\partial}{\partial y^j}$ where $X^i(x,y)$ and $Y^i(x,y)$ are C^∞ functions on SM . A simple calculation shows that locally

$$\nabla f = g^{ij} \frac{\delta f}{\delta x^i} \frac{\delta}{\delta x^j} + F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial}{\partial y^j}.$$

The norm of ∇f with respect to the Riemannian metric $\tilde{g}$ is given by

$$|\nabla f|^2 = \tilde{g}(\nabla f, \nabla f) = g^{ij} \frac{\delta f}{\delta x^i} \frac{\delta f}{\delta x^j} + F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}. \tag{2}$$

2. EXTENSION OF SOME DEFINITIONS TO FINSLER MANIFOLDS

In what follows, (M,g) denotes a connected Finsler manifold of class C^1 with dimension $n \geq 2$. Let $(SM, \tilde{g})$ be its Riemannian Sphere bundle.
We consider the volume element $\eta(g)$ on SM defined as follows:

$$\eta(g) := \frac{(-1)^N}{(n-1)!} \omega \wedge (d\omega)^{n-1}, \tag{3}$$

where $N = \frac{n(n-1)}{2}$ and ω is the Hilbert form of F (This volume element was used for the first time in Finsler geometry by Akbar-Zadeh in his thesis [11] and [17]). Let $C(M)$ be the linear space of continuous real valued functions on M , $u \in C(M)$ and u^V its vertical lift on SM . For M , compact or not, we denote by $H(M)$ the set of all functions in $C(M)$, admitting a generalized L^n-integrable gradient ∇u^V satisfying

$$I(u,M) = \int_{SM} |\nabla u^V|^n \eta(g) < \infty.$$

If M is non-compact let us denote by $H_0(M)$ the subspace of functions $u \in H(M)$ for which the vertical lift u^V has a compact support in SM . A *relatively compact* subset is a subset whose closure is compact. A function $u \in C(M)$ will be called *monotone* if for any relatively compact domain D of M

$$\sup_{x\in\partial D} u(x) = \sup_{x\in D} u(x); \qquad \inf_{x\in\partial D} u(x) = \inf_{x\in D} u(x).$$

We denote by $H^*(M)$ the set of monotone functions $u \in H(M)$.
We define notion of capacity as follows:

Definition 2.1. Capacity of a compact subset C of a non-compact Finsler manifold M is defined by

$$Cap_{M}(C) := \inf_{u} I(u, M),$$

where the infimum is taken over the functions $u \in H_0(M)$ with $u = 1$ on C and $0 \le u(x) \le 1$ for all x, these functions are said to be admissible for C.

The non-compactness condition of M is a necessary condition. In fact, if M is compact, then by putting $u = 1$ in H_0M we have $I(u, M) = 0$, therefore the capacity of all subsets is zero and there is nothing to say.

A *relative continuum* is a closed subset C of M such that $C \cup \{\infty\}$ is connected in Alexandrov's compactification $\overline{M} = M \cup \{\infty\}$. To avoid ambiguities, the connected closed sets of M that are not reduced to one point will be called *continua*. In what follows we want to associate conformal invariant function, which is determined entirely by the conformal structure of manifold M, at every double point of M.

Definition 2.2. Let (M, g) be a Finsler manifold. For all (x_1, x_2) in $M^2 := M \times M$ we set

$$\mu_{M}(x_1, x_2) = \inf_{C \in \alpha(x_1, x_2)} Cap_{M}(C),$$

where $\alpha(x_1, x_2)$ is the set of all compact continua subsets of M containing x_1 and x_2.

3. CONFORMAL PROPERTY OF CAPACITY

Lemma 3.1. Let (M, g) and (M', g') be two conformal related Finsler manifolds, then there exist an orientation preserving diffeomorphism between their sphere bundles.

Proof: Let $f : (M, g) \longrightarrow (M', g')$ be a diffeomorphism between two Finsler manifolds. We define a mapping h between their sphere bundles as follows $h : SM \longrightarrow SM'$, where $h(x,[y]) = (f(x),[f_*(y)])$, and f_* is the differential map of f. Since f_* is a linear map, h is well defined. If f is conformal then $f^* g' = \lambda g$, where λ is a positive real valued function on M and for components of Finsler metrics g and g' defined on TM and TM' we have $\lambda g = f^* g' = f^*(g'_{ij} dx'^i dx'^j)$, by definition $(f_*)^* g'_{ij}(f^* dx'^i)(f^* dx'^j) = (f_*)^* g'_{ij} dx^i dx^j$, and therefore $(f_*)^* g'_{ij} = \lambda g_{ij}$ or equivalently, $h^* g'_{ij} = \lambda g_{ij}$. Let ω' be the Hilbert form related to the Finsler metric g'. By definition

$$\omega' = g'_{ij} \frac{y'^j}{F} dx'^i = g'_{ij} \frac{y'^j}{\sqrt{g'_{mn} y'^m y'^n}} dx'^i.$$

Therefore,

$$h^* \omega' = h^*(g'_{ij}) \frac{h^*(y'^j)}{\sqrt{h^*(g'_{mn} y'^m y'^n)}} h^*(dx'^i) = \sqrt{\lambda}\omega. \tag{4}$$

By applying h^* to (1) we get by straight forward calculation

$$h^* d\omega' = \sqrt{\lambda} d\omega. \tag{5}$$

So if $\eta(g)$ and $\eta(g')$ denote the volume elements of SM and SM' respectively, then from (3), (4) and (5) we get

$$h^*(\eta(g')) = (\sqrt{\lambda})^n \eta(g). \tag{6}$$

Therefore h is an orientation preserving diffeomorphism.

Lemma 3.2. Let f be a diffeomorphism between Finsler manifolds (M, g) and (M', g'), and h a mapping between their sphere bundles with Sasaki metrics, $(SM, \tilde{g})$ and $(SM', \tilde{g}')$. If $u \in H_0(M')$ then we have

1. $|\nabla u^V|^n = (g'^{ij} \frac{\delta u^V}{\delta x'^i} \frac{\delta u^V}{\delta x'^j})^{\frac{n}{2}}$,
2. $(u \circ f)^V = u^V \circ h$,
3. $h^* \frac{\delta u^V}{\delta x'^i} = \frac{\delta (u \circ f)^V}{\delta x^i}$.

Therefore, the following diagram is commutative:

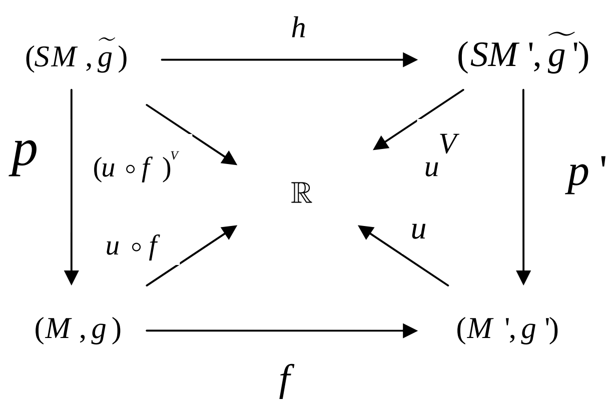

Diagram 1.

Proof:

1. Since the vertical lift of $u \in H_0(M')$ is a function of position alone, $\frac{\partial u^V}{\partial y^i} = 0$. Therefore the first assertion follows from (2).
2. Let's consider the projections $p : SM \to M$ and $p' : SM' \to M'$. The vertical lifts of u and $u \circ f$, are by definition, $u^V(x', [y']) = u \circ p'(x', [y']) = u(x')$ and

$$(u \circ f)^V(x, [y]) = (u \circ f) \circ p(x, [y]) = (u \circ f)(x).$$

From which we have

$$(u \circ f)^V (x, [y]) = (u \circ f)(x) =$$
$$u^V (f(x), [f_*(x)]) = u^V (h(x, [y])) = u^V \circ h(x, [y]).$$

This proves the assertion (2).

3. By definition of h^* we have $h^*(\frac{\delta}{\delta x'^i}.u^V) = h^*(\frac{\delta}{\delta x'^i}).h^* u^V = \frac{\delta}{\delta x^i}.(u^V \circ h)$, and from (2) we get assertion (3).

Now we are in a position to prove the following theorem:

Theorem 3.3. Let (M, g) be a connected non-compact Finsler manifold, then the capacity of a compact set on M is a conformal invariant.

Proof: We show that the notion of capacity depends only on the conformal structure of M, or equivalently, for any conformal map f from Finsler manifold (M,g) onto another Finsler manifold (M',g'), we have

$$Cap_{M}(C) = Cap_{M'}(f(C)).$$

Since SM and SM' are two smooth, orientable manifolds with boundary, then for a smooth, orientation preserving diffeomorphism function $h: SM \longrightarrow SM'$ defined in Lemma 3.1, clearly (by a classical result in differential Geometry, [18]) we have

$$\int_{SM'} \omega = \int_{SM} h^{*}\omega, \qquad \omega \in \Omega^{2n-1} SM'.$$

So we get,

$$I(u,M') = \int_{S(M')} |\nabla u^{V}|^{n} \eta(g') = \int_{SM} h^{*}(|\nabla u^{V}|^{n} \eta(g')). \tag{7}$$

Using Lemma 3.2, a straightforward calculation shows that

$$h^{*} |\nabla u^{V}|^{n} = (\sqrt{\lambda})^{-n} |\nabla (u \circ f)^{V}|^{n}. \tag{8}$$

Using (6) in Lemma 3.1, and relations (7) and (8) we get

$$I(u,M') = \int_{SM} |\nabla (u \circ f)^{V}|^{n} \eta(g) = I(u \circ f, M). \tag{9}$$

Let C be a compact set in M, then we have

$$Cap_{M}(C) = \inf_{v \in H_0 M,\, v|_C = 1} I(v,M), Cap_{M'}(f(C)) = \inf_{u \in H_0 M',\, u|_{f(C)} = 1} I(u,M').$$

Put

$$A = \{I(v,M) \mid v \in H_0 M, v|_{C} = 1\},$$

$$B = \{I(u,M') \mid u \in H_0 M', u|_{f(C)} = 1\}.$$

We first show that $B \subseteq A$. For all $I(u,M') \in B$, we easily have the following assertions.

- Since $support(u^{V})$ is compact in SM', $h^{-1}(support(u^{V})) = support(u \circ f)^{V}$ is compact in SM and by definition $u \circ f \in H_0(M)$.
- $(u \circ f)|_{C} = 1$ since $u|_{f(C)} = 1$.
- From (9) we have $I(u \circ f, M) = I(u,M')$.

Therefore, $I(u \circ f, M) \in A$ and $B \subseteq A$. By the same argument we have $A \subseteq B$. Hence, $Cap_{M}(C) = Cap_{M'}(f(C))$.

Theorem 3.3, implies that the function μ_{M} is invariant under any conformal mapping. More precisely, if f is a conformal mapping between Finsler manifolds (M,g) and (M',g'), then for all $x_1, x_2 \in M$ we have

$$\mu_{M}(x_1, x_2) = \mu_{M'}(f(x_1), f(x_2)),$$

In the Riemannian geometry this function is of general interest in the study of global conformal geometry, which can be the subject of further studies in Finsler geometry.

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