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A GENERALIZED SUMMABILITY FACTOR THEOREM FOR ABSOLUTE SUMMABILITY AND QUASI β – POWER INCREASING SEQUENCES*

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Abstract – The object of this paper is to establish a summability factor theorem for summability $|A,\delta|_k, k \geq 1$ where A is the lower triangular matrix with non-negative entries satisfying certain conditions.

Keywords – Generalized absolute summability, weighted mean matrix, Cesaro matrix, summability factor

1. INTRODUCTION

A positive sequence $\{b_n\}$ is said to be almost increasing if there exists an increasing sequence $\{c_n\}$ and positive constans A and B such that $Ac_n \leq b_n \leq Bc_n$.

A sequence $\{\lambda_n\}$ is said to be of bounded variation (bv) if $\sum_n |\Delta\lambda_n| < \infty$. Let $bv_0 = bv \cap c_0$, where c_0 denotes the set of all null sequences.

Let A be a lower triangular matrix, $\{ s_n \}$ a sequence. Then

$$A_n := \sum_{v=0}^{n} a_{nv} s_v .$$

A series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty , \tag{1}$$

and it is said to be summable $|A,\delta|_k, k \geq 1$ and $\delta \geq 0$ if ([1])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |A_n - A_{n-1}|^k < \infty. \tag{2}$$

We may associate with A two lower triangular matrices, $\overline{A}$ and $\hat{A}$, defined as follows:

$$\overline{a}_{nv} = \sum_{r=v}^{n} a_{nr} \quad n,v = 0,1,2 ,...,$$

and

$$\hat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \quad n = 1,2,3,... .$$

We may write

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$$T_n = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=i}^{n} a_{nv} = \sum_{i=0}^{n} \overline{a}_{ni} a_i \lambda_i .$$

Thus

$$\begin{aligned} T_n - T_{n-1} &= \sum_{i=0}^{n} \overline{a}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \overline{a}_{n-1,i} a_i \lambda_i \\ &= \sum_{i=0}^{n} \overline{a}_{ni} a_i \lambda_i - \sum_{i=0}^{n} \overline{a}_{n-1,i} a_i \lambda_i \\ &= \sum_{i=0}^{n} (\overline{a}_{ni} - \overline{a}_{n-1,i}) a_i \lambda_i \\ &= \sum_{i=0}^{n} \widehat{a}_{ni} a_i \lambda_i = \sum_{i=1}^{n} \widehat{a}_{ni} \lambda_i (s_i - s_{i-1}) \\ &= \sum_{i=1}^{n} \widehat{a}_{ni} \lambda_i s_i - \sum_{i=1}^{n} \widehat{a}_{ni} \lambda_i s_{i-1} \\ &= \sum_{i=1}^{n-1} \widehat{a}_{ni} \lambda_i s_i + a_{nn} \lambda_n s_n - \sum_{i=1}^{n} \widehat{a}_{ni} \lambda_i s_{i-1} \\ &= \sum_{i=1}^{n-1} \widehat{a}_{ni} \lambda_i s_i + a_{nn} \lambda_n s_n - \sum_{i=0}^{n-1} \widehat{a}_{n,i+1} \lambda_{i+1} s_i \\ &= \sum_{i=1}^{n-1} (\widehat{a}_{ni} \lambda_i - \widehat{a}_{n,i+1} \lambda_{i+1}) s_i + a_{nn} \lambda_n s_n . \end{aligned}$$

We may write

$$\begin{aligned} (\widehat{a}_{ni} \lambda_i - \widehat{a}_{n,i+1} \lambda_{i+1}) &= \widehat{a}_{ni} \lambda_i - \widehat{a}_{n,i+1} \lambda_{i+1} - \widehat{a}_{n,i+1} \lambda_i + \widehat{a}_{n,i+1} \lambda_i \\ &= (\widehat{a}_{ni} - \widehat{a}_{n,i+1}) \lambda_i + \widehat{a}_{n,i+1} (\lambda_i - \lambda_{i+1}) \\ &= \lambda_i \Delta_i \widehat{a}_{ni} + \widehat{a}_{n,i+1} \Delta \lambda_i . \end{aligned}$$

Therefore,

$$\begin{aligned} T_n - T_{n-1} &= \sum_{i=1}^{n-1} \Delta_i \widehat{a}_{ni} \lambda_i s_i + \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} \Delta \lambda_i s_i + a_{nn} \lambda_n s_n . \\ &= T_{n1} + T_{n2} + T_{n3}, \text{ say.} \end{aligned}$$

A triangle is a lower triangular matrix with all nonzero main diagonal entries.

A positive sequence $\{\gamma_n\}$ is said to be a quasi $\beta-$ power increasing sequence if there exists a constant $K = K(\beta,\gamma) \geq 1$ such that $Kn^{\beta}\gamma_n \geq m^{\beta}\gamma_m$ holds for all $n \geq m \geq 1$, (see, [2]).

It should be noted that every almost increasing sequence is a quasi $\beta-$ power increasing sequence for any non-negative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$.

2. MAIN RESULT

Theorem 1. Let A be a lower triangular matrix with non-negative entries satisfying

(i) $\bar{a}_{n0} = 1,\ n = 0,1,...,$

(ii) $a_{n-1,v} \geq a_{nv}$ for $n \geq v+1,$

(iii) $na_{nn} = O(1).$

(iv) $\sum_{n=i+1}^{m+1} n^{\delta k} \left|\Delta_i \hat{a}_{ni}\right| = O(i^{\delta k} a_{ii})$ and

(v) $\sum_{n=i+1}^{m+1} n^{\delta k} \hat{a}_{n,i+1} = O(i^{\delta k})$.

Let $(\lambda_n) \in bv_0$ and let $\{X_n\}$ be a quasi $\beta-$ power increasing sequence for some $0<\beta<1$ and the sequences $\{\beta_n\}$ and $\{\lambda_n\}$ are such that

(vi) $\left|\Delta\lambda_n\right| \leq \beta_n,$

(vii) $lim\, \beta_n = 0,$

(viii) $\sum_{n=1}^{\infty} n\left|\Delta\beta_n\right| X_n < \infty,$ and

(ix) $\left|\lambda_n\right| X_n = O(1)$

are satisfied. If

(x) $\sum_{n=1}^{m} n^{\delta k-1} \left|s_n\right|^k = O(X_m),$

then the series $\sum a_n \lambda_n$ is summable $\left|A,\delta\right|_k, k \geq 1,\ \delta \geq 0.$

Remark: It may be noted that if we take $\delta = 0$ in the above theorem, then we get a result of Savas [3] on $\left|A_k\right|$-summability factors. Furthermore, we must note that in the theorem of [3] the condition $(\lambda_n) \in bv_0$ should be added.

We need the following lemma for the proof of our theorem.

Lemma ([3]). Under the conditions on $\{X_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ as taken in the statement of the theorem, the following conditions hold when (viii) is satisfied:

(1) $n\beta_n X_n = O(1)$ and

(2) $\sum_{n=1}^{\infty} \beta_n X_n < \infty$.

Proof of Theorem 1. To complete the proof it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \left|T_{nr}\right|^k < \infty, \text{ for } r = 1, 2, 3.$$

As in [3], it is easy to see that $\hat{a}_{n,i+1} \geq 0$.

From (ix), it follows that $\lambda_n = O(1)$, using Hölder's inequality and (iii)

$$I_1 := \sum_{n=1}^{m+1} n^{\delta k+k-1} \left|T_{n1}\right|^k \leq \sum_{n=1}^{m+1} n^{\delta k+k-1} \Big(\sum_{i=1}^{n-1} \left|\Delta_i \hat{a}_{ni}\right| \left|\lambda_i\right| \left|s_i\right|\Big)^k$$

$$\leq \sum_{n=1}^{m+1} n^{\delta k+k-1} (\sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}| \, |\lambda_i|^k |s_i|^k) \mathrm{x} (\sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}|)^{k-1}.$$

Using (ix) and (x) and the condition (2) of Lemma.

$$I_1 := O(1) \sum_{n=1}^{m+1} n^{\delta k} (na_{nn})^{k-1} \sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}| |\lambda_i|^k |s_i|^k$$

$$:= O(1) \sum_{n=1}^{m+1} n^{\delta k} (na_{nn})^{k-1} \sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}| |\lambda_i| |\lambda_i|^{k-1} |s_i|^k$$

$$:= O(1) \sum_{i=1}^{m} |\lambda_i| |s_i|^k \sum_{n=i+1}^{m+1} n^{\delta k} (na_{nn})^{k-1} |\Delta_i \hat{a}_{ni}|$$

$$:= O(1) \sum_{i=1}^{m} |\lambda_i| |s_i|^k i^{\delta k} a_{ii}$$

$$:= O(1) \left[\sum_{i=1}^{m-1} \beta_i X_i + |\lambda_m| X_m \right]$$

$$:= O(1).$$

Using Hölder’s inequality, (iii) and (v)

$$I_2 = \sum_{n=1}^{m+1} n^{\delta k+k-1} |T_{n2}|^k \leq \sum_{n=1}^{m+1} n^{\delta k+k-1} \left| \sum_{i=1}^{n-1} \hat{a}_{n,i+1} s_i \Delta\lambda_i \right|^k$$

$$\leq \sum_{n=1}^{m+1} n^{\delta k+k-1} (\sum_{i=1}^{n-1} \hat{a}_{n,i+1} |s_i| |\Delta\lambda_i|)^k$$

$$\leq \sum_{n=1}^{m+1} n^{\delta k+k-1} (\sum_{i=1}^{n-1} \hat{a}_{n,i+1} |s_i| \beta_{\mathrm{i}})^k$$

$$\leq \sum_{n=1}^{m+1} n^{\delta k+k-1} (\sum_{i=1}^{n-1} \hat{a}_{n,i+1} |s_i|^{\mathrm{k}} \beta_{\mathrm{i}}) \mathrm{x} (\sum_{i=1}^{n-1} \hat{a}_{n,i+1} \beta_{\mathrm{i}})^{k-1}.$$

$$:= O(1) \sum_{n=1}^{m+1} n^{\delta k} (na_{nn})^{k-1} \sum_{i=0}^{n-1} \hat{a}_{n,i+1} |s_i|^k \beta_i$$

$$:= O(1) \sum_{i=1}^{m} \beta_i |s_i|^k \sum_{n=i+1}^{m+1} n^{\delta k} (na_{nn})^{k-1} \hat{a}_{n,i+1}$$

$$:= O(1) \sum_{i=1}^{m} \beta_i |s_i|^k \sum_{n=i+1}^{m+1} \hat{a}_{n,i+1}.$$

Using (viii) and (x)

$$I_2 := O(1) \sum_{i=1}^{m} i^{\delta k} \beta_i |s_i|^k = O(1) \sum_{i=1}^{m} i^{\delta k} i \beta_i \frac{1}{i} |s_i|^k$$

$$: = O(1)\sum_{i=1}^{m-1}\left|\Delta(i\beta_i)\right|X_i + O(1)m\beta_m X_m$$

$$:= O(1)\sum_{i=1}^{m-1} i\left|\Delta(\beta_i)\right|X_i + O(1)\sum_{i=1}^{m-1}\beta_{i+1}X_{i+1} + O(1)m\beta_m X_m$$

$$:= O(1),$$

again using the conditions of Lemma.
Using (iii) and (ix),

$$\sum_{n=1}^{m+1} n^{\delta k+k-1}\left|T_{n3}\right|^k \le \sum_{n=1}^{m+1} n^{\delta k+k-1}\left|a_{nn}\lambda_n s_n\right|^k$$

$$: = O(1)\sum_{n=1}^{m} n^{\delta k}(na_{nn})^{k-1}a_{nn}\left|\lambda_n\right|^k\left|s_n\right|^k$$

$$:= O(1)\sum_{n=1}^{m} n^{\delta k}\, a_{nn}\left|\lambda_n\right|^{k-1}\left|\lambda_n\right|\left|s_n\right|^k$$

$$: = O(1),$$

as in the proof of I_1.
Setting $\delta = 0$, in the theorem yields the following corollary.

Corollary 1. ([3]). Let A be a triangle satisfying conditions (i)-(iii) of Theorem 1.
Let $(\lambda_n) \in bv_0$ and let $\{X_n\}$ be a quasi $\beta-$ power increasing sequence for some 0< β <1 and the sequences $\{\beta_n\}$ and $\{\lambda_n\}$ are such that conditions (vi) – (ix) of Theorem 1 are satisfied. If

$$\text{(x)}\ \sum_{n=1}^{m}\frac{1}{n}\left|s_n\right|^k = O(X_m),$$

then the series $\sum a_n\lambda_n$ is summable $\left|A\right|_k, k \ge 1$.

Corollary 2. Let $\{p_n\}$ be a positive sequence such that $P_n = \sum_{k=0}^{n} p_k \to \infty$, and satisfies
(i) $np_n = O(P_n)$,
(ii) $\sum_{n=v+1}^{m+1} n^{\delta k}\left|\frac{p_n}{P_n P_{n-1}}\right| = O(\frac{v^{\delta k}}{P_v})$.

Let $(\lambda_n) \in bv_0$ and let $\{X_n\}$ be a quasi $\beta-$ power increasing sequence for some 0< β <1 and the sequences $\{\beta_n\}$ and $\{\lambda_n\}$ are such that conditions (vi) – (ix) of Theorem 1 are satisfied. If

$$\text{(x)}\ \sum_{n=1}^{m} n^{\delta k-1}\left|s_k\right|^k = O(X_m),$$

then the series $\sum a_n\lambda_n$ is summable $\left|N, p, \delta\right|_k, k \ge 1,\ \delta \ge 0$.

Proof: Conditions (vi)-(ix) and (x) of Corollary 2 are, respectively, conditions (vi)-(ix) and (x) of Theorem 1. Conditions (i) and (ii) of Theorem 1 are automotically satisfied for any weighted mean method. Condition (iii) of Theorem 1 becomes condition (i) of Corollary 1 and conditions (iv) and (v) of Theorem 1 become condition (ii) of Corollary 2.

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