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WEIGHTED STATISTICAL CONVERGENCE[*]

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Abstract – In this paper, the notion of $(\overline{N}, p_n)$- summability to generalize the concept of statistical convergence is used. We call this new method weighted statistically convergence. We also establish its relationship with statistical convergence, $(C,1)$-summability and strong $(\overline{N}, p_n)$-summability.

Keywords – Norlund-type means, weighted statistical convergence, sequence spaces, Cesaro summability

1. INTRODUCTION

The idea of statistical convergence which is closely related to the concept of natural density or asymptotic density of a subset of the set of natural numbers N, was first introduced by Fast [1]. The concept of statistical convergence plays an important role in the summability theory and functional analysis. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg [2]. Afterwards, the statistical convergence has been studied as a summability method by many researchers such as Fridy [3], Freedman et al. [4], Kolk [5, 6], Fridy and Miller [7], Fridy and Orhan [8, 9], Mursaleen [10] and Savaş [11]. Also, some topological properties of statistical convergence sequence spaces have been studied by Salat [12]. Besides in [13, 14], Connor showed the relations between statistical convergence and functional analysis. Quite recently, Mursaleen et al. [15] have proved some inequalities on statistical summability $(C,1)$. Recent developments concerning this area can be found in [16-20].

In general, statistical convergence of weighted means is studied as a class of regular matrix transformations. In this work, we introduce and study the concept of weighted statistical convergence. The relations among strong $(\overline{N}, p_n)$-summability, $(C,1)$-summability and statistical convergence with respect to this novel method are also investigated.

Let $K \subseteq \mathbb{N}$ and $K_n = \{k \in K : k \le n\}$. Then the natural density of K is defined by $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in K : |x_n - L| \ge \varepsilon\}$ has natural density zero; in this case we write $S - \lim x = L$. The symbol S denotes the set of all statistically convergent sequences.

Let (p_n) be a sequence of the positive real constant such that $P_n = p_0 + p_1 + ... + p_n$ and $p_n \ne 0, p_0 > 0$. We have

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$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k x_k \ . \tag{1.1}$$

It is well known that (1.1) is a transformation from a sequence space into another sequence space. Also, this transformation is regular if $P_n \to \infty$. The pair $(\overline{N}, p_n)$ denotes the set of all sequences (t_n) of Nörlund-type transformations. The sequence (t_n) is the mean of the sequence (x_k) generated by the coefficients of the sequence (p_n). The sequence (x_k) is said to be $(\overline{N}, p_n)$-summable to L if $t_n \to L$ as $n \to \infty$, and we write $x_k \to L(\overline{N}, p_n)$. If $p_n = 1$ for all n in (1.1), we have

$$t_n = \frac{1}{n}\sum_{k=1}^{n} x_k \ .$$

This is denoted by $(C,1)$ and called Cesaro summability. Hardy [21] showed that the sequence $(-1)^n$ is $(C,1)$-summable but it is not $(\overline{N}, 2^n)$-summable. Therefore, the inclusion $(\overline{N}, p_n) \subset (C,1)$ is proper.

In addition, if $\lim_{n\to\infty} \frac{1}{P_n}\sum_{k=0}^{n} p_k |x_k - L| = 0$, the sequence $x = (x_k)$ is said to be strongly $(\overline{N}, p_n)$-summable to L and it is denoted by

$$\left|\overline{N}, p_n\right| = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{P_n}\sum_{k=0}^{n} p_k |x_k - L| = 0 \ \textit{for some } L \right\} \tag{1.2}$$

Besides, if $p_n = 1$ for all n in (1.2), this is denoted by

$$|C,1| = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_k - L| = 0 \ \textit{for some } L \right\}$$

and called the space of sequences of strongly Cesaro summable to L, i.e., $x_k \to L(|C,1|)$.
The matrix $A = (a_{nk})$ in $(\overline{N}, p_n)$-summability is given by

$$a_{nk} = \begin{cases} \frac{p_k}{P_n} & \textit{if } \ k \le n, \\ 0 & \textit{if } \ k > n. \end{cases}$$

Also, the weighted means are equivalent to $(C,1)$ summability over c (see, [22]), the space of convergent sequences.
Before we state the main results of this work, let us give the definition of a new statistical method.

Definition 1. A sequence $x = (x_k)$ is said to be weighted statistical convergent if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n} \left| \{ k \le n : p_k |x_k - L| \ge \varepsilon \} \right| = 0$$

The set of weighted statistical convergence sequence is denoted by $S_{\overline{N}}$ as follows:

$$S_{\overline{N}} = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{P_n} \left| \{ k \le n : p_k |x_k - L| \ge \varepsilon \} \right| = 0, \ \textit{for some } L \right\}.$$

If the sequence $x = (x_k)$ is $S_{\overline{N}}$-convergence, then we also use the notation $x_k \to L(S_{\overline{N}})$.

2. MAIN RESULTS

In this section, we find the relationships of $S_{\overline{N}}$ with $\left|\overline{N}, p_n\right|$ and $(C,1)$. Firstly, let us begin the following theorem.

Theorem 1. If the sequence (x_k) is $\left|\overline{N}, p_n\right|$-summable to *L*, then the sequence (x_k) is $S_{\overline{N}}$-convergent and the inclusion $\left|\overline{N}, p_n\right| \subset S_{\overline{N}}$ is proper.

Proof: Let the sequence x_k be $\left|\overline{N}, p_n\right|$-summable to *L* and $K_\varepsilon = \{k \le n : p_k |x_k - L| \ge \varepsilon\}$. Then, for a given $\varepsilon > 0$, we have

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k |x_k - L| = \frac{1}{P_n}\sum_{k \in K_\varepsilon} p_k |x_k - L| + \frac{1}{P_n}\sum_{k \notin K_\varepsilon} p_k |x_k - L|$$
$$\ge \frac{1}{P_n}\left|\{k \le n : p_k |x_k - L| \ge \varepsilon\}\right|.$$

Hence, we obtain that the sequence x_k is $S_{\overline{N}}$-convergent to *L*. In the following example, it is shown that the inclusion is proper. Let us define the sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} \sqrt{k} & if \;\; k = n^2, \\ 0 & if \;\; k \ne n^2. \end{cases}$$

Let $p_n = 1,2,3,...$. Then we have

$$\frac{1}{P_n}\left|\{k \le n : p_k |x_k - 0| \ge \varepsilon\}\right| = \frac{\sqrt{n}}{P_n} \to 0 \;\; as \;\; n \to \infty.$$

On the other hand,

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k |x_k - 0| = \frac{1}{P_n}\sum_{k=1}^{n} p_{k^2} x_{k^2} = \frac{3n^4 + 6n^3 + 3n^2}{12P_n} \to \infty \;\; as \;\; n \to \infty.$$

Hence it can be seen that the inclusion $\left|\overline{N}, p_n\right| \subset S_{\overline{N}}$ is proper.

Theorem 2. Let $P_n \to \infty$ and $p_k |x_k - L| \le M$ for all k$\in$IN. If $x_k \to L(S_{\overline{N}})$, then $x_k \to L\left|\overline{N}, p_n\right|$ and hence $x_k \to L(C,1)$.

Proof: Let $x_k \to L(S_{\overline{N}})$ and $K_\varepsilon = \{k \le n : p_k |x_k - L| \ge \varepsilon\}$. Since $P_n \to \infty$ and $p_k |x_k - L| \le M$ for all k$\in$IN, then for a given $\varepsilon > 0$, we have

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k |x_k - L| = \frac{1}{P_n}\sum_{k \in K_\varepsilon} p_k |x_k - L| + \frac{1}{P_n}\sum_{k \notin K_\varepsilon} p_k |x_k - L|$$
$$\le \frac{M}{P_n}\left|\{k \le n : p_k |x_k - 0| \ge \varepsilon\}\right| + \varepsilon.$$

Since ε is arbitrary, we have $x_k \to L\left|\overline{N}, p_n\right|$. Also, under conditons given in [21], we give the inclusions $(\overline{N}, p_n) \subset (C,1)$ and $\left|\overline{N}, p_n\right| \subset (\overline{N}, p_n)$, that is;

$$\frac{1}{n}\sum_{k=1}^{n}(x_k - L) \le \frac{1}{P_n}\sum_{k=0}^{n} p_k (x_k - L) \le \frac{1}{P_n}\sum_{k=0}^{n} p_k |x_k - L|.$$

So we obtain that $x_k \to L(C,1)$. This completes the proof.

In this section, we establish the relationships between S and $S_{\overline{N}}$ methods.

Theorem 3. Let $\left(\frac{P_n}{n}\right) > 1$ for all $n \in \mathbb{N}$. If $x_k \to L(S)$, then $x_k \to L(S_{\overline{N}})$ and the inclusion is proper.

Proof: For $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{n}\left|\{k \le n : |x_k - L| \ge \varepsilon\}\right| &= \frac{1}{n}\left|\{k \le n : p_k |x_k - L| \ge \varepsilon\}\right| \\ &= \left(\frac{P_n}{n}\right)\frac{1}{P_n}\left|\{k \le n : p_k |x_k - L| \ge \varepsilon\}\right| \\ &\ge \frac{1}{P_n}\left|\{k \le n : p_k |x_k - L| \ge \varepsilon\}\right|.\end{aligned}$$

This completes the proof.

For the inclusion relation, we take $p_k = \frac{1}{k+1}$ and $x_k = (-1)^{k+1}$. Then we easily see that $x_k \in S_{\overline{N}}$. On the other hand, $(x_k) \in (C,1)$ but $(x_k) \notin S$. This is the desired result.

Theorem 4. If the sequence (P_n) is a bounded sequence such that $\limsup\left\{\frac{P_n}{n}\right\} < \infty$, then $S_{\overline{N}}$ is equivalent to S.

Proof: For a given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{n}\left|\{k \le n : |x_k - L| \ge \varepsilon\}\right| &= \frac{1}{n}\left|\{k \le n : p_k |x_k - L| \ge \varepsilon\}\right| \\ &\le \left(\frac{P_n}{n}\right)\frac{1}{P_n}\left|\{k \le n : p_k |x_k - L| \ge \varepsilon\}\right| \\ &\le \frac{1}{P_n}\left|\{k \le n : p_k |x_k - L| \ge \varepsilon\}\right|.\end{aligned}$$

Since $\limsup\left\{\frac{P_n}{n}\right\} < \infty$, we get $x_k \to L(S_{\overline{N}}) \Rightarrow x_k \to L(S)$, i.e $S_{\overline{N}} \subset S$. Hence by Theorem 3, the result follows.

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