

THE INVOLUTE-EVOLUTE OFFSETS OF RULED SURFACES*

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Abstract – In this study, a generalization of the theory of involute-evolute curves is presented for ruled surfaces based on line geometry. Using lines instead of points, two ruled surfaces which are offset in the sense of involute-evolute are defined. Moreover, the found results are clarified using computer-aided examples.

Keywords – Ruled Surface, involute-evolute, differential geometry

1. INTRODUCTION

A surface is said to be "ruled" if it is generated by moving a straight line continuously in Euclidean space E^3 . Ruled surfaces are one of the simplest objects in geometric modeling.

One important fact about ruled surfaces is that they can be generated by straight lines. One would never know this from looking at the surface or its usual equation in terms of x, y and z coordinates, but ruled surfaces can all be rewritten to highlight the generating lines. A practical application of ruled surfaces is that they are used in civil engineering. Since building materials such as wood are straight, they can be thought of as straight lines. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since straight lines exist everywhere on the surface.

Since the discovery of preconstraint concrete in 1930, architectural construction in the shape of ruled surfaces have been innumerable, including water-towers, chimney-pieces, roofs, and spiral staircases. Eero Saarinen (1910-1961) used ruled surfaces in his buildings at Yale and M.I.T., but the man who has used ruled surfaces more than anyone else is designer, architect and builder Félix Candela who makes extensive use of cylinders and the most familiar ruled surfaces.

In other cases, such as the Chapel at Lomas de Cuernavaca, Morelos, Mexico, the entire building is a gigantic hyperbolic paraboloid. A restaurant at Xochimilco, D. F., is made of four intersecting hyperbolic paraboloids giving the impression of an immense scallop-edged shell.

There are recent works about ruled surfaces: Aslaner studied the time-like hyperruled surfaces in the Minkowski 4-space [1]. Tosun and Gungor described a time-like complementary ruled surface in the Minkowski n -space. Also, they investigated relations connected with an asymptotic and tangential bundle of the time-like complementary ruled surface [2]. Karadağ and Keleş studied the integral invariants of the ruled surfaces in the line space corresponding to the closed spherical curves by the transference principle of E. Study; for this they used the area vector of the closed dual spherical curve [3].

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Huyghens first introduced the concepts of the involute and the evolute in 1973 [4]. A pair of curves are said to be involute-evolute mates in Euclidean space E^3 if there exists a one-to-one correspondence between their points such that one's tangent and the other's principal normal are linear dependent at their corresponding points. Methods for the generation of parallel offsets for a certain class of surfaces have been developed by Farouki [5-6]. Using the same techniques, the theory of Bertrand curves has been developed for the ruled and developable surfaces by Ravani and Ku [7].

In this paper, involute-evolute offsets of ruled surfaces are considered. Using line geometry, it is shown that a theory similar to that of involute-evolute curves can be developed for a ruled surface. The condition for two ruled surfaces to be involute-evolute mates is developed and the results are clarified using computer-aided examples.

2. PRELIMINARIES

A ruled surface φ in 3-dimensional Euclidean space E^3 is a surface swept out by a straight line parallel to \mathbf{e} along a curve α and has the parametric representation

$$\varphi(s, v) = \alpha(s) + v\mathbf{e}(s), \quad \|\mathbf{e}\| = 1.$$

The curve $\alpha = \alpha(s)$ is called the *base curve* and the various positions of the generating lines $\mathbf{e}(s)$ are called the *rulings* of the surface φ . The curve, which is drawn by $\mathbf{e}(s)$ on the unit sphere S^2 is called the *spherical indicatrix curve* and \mathbf{e} is also called the *spherical indicatrix vector* of φ . The unit normal of φ along a general generator $\mathbf{l} = \varphi(s_0, v)$ of the ruled surface approaches a limiting direction as v infinitely decreases. This direction is called the *asymptotic normal direction* and is defined as

$$\mathbf{g}(s) = \frac{\mathbf{e} \wedge \mathbf{e}_s}{\|\mathbf{e}_s\|}, \quad \mathbf{e}_s = \frac{d\mathbf{e}}{ds}.$$

The point at which the unit normal of φ is perpendicular to \mathbf{g} is called the *striction point* (or central point) on \mathbf{l} and the curve drawn by these points is called the *striction curve* of φ . For the striction curve of φ , we have

$$\mathbf{c}(s) = \alpha(s) - \frac{\langle \alpha_s, \mathbf{e}_s \rangle}{\langle \mathbf{e}_s, \mathbf{e}_s \rangle} \mathbf{e}(s).$$

The direction of the unit normal at a striction point is called the *central normal* of φ and is given by

$$\mathbf{t} = \frac{\mathbf{e}_s}{\|\mathbf{e}_s\|}.$$

Thus, we have the orthonormal system $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$. This system is called the *geodesic Frenet trihedron* of φ . For the geodesic *Frenet vectors* \mathbf{e}, \mathbf{t} and \mathbf{g} , we can write

$$\begin{aligned} \mathbf{e}_q &= \mathbf{t} \\ \mathbf{t}_q &= \gamma \mathbf{g} - \mathbf{e}, \\ \mathbf{g}_q &= -\gamma \mathbf{t} \end{aligned} \quad (1)$$

where q and γ are the *arc-parameter* of the spherical indicatrix curve (\mathbf{e}) and the *geodesic curvature* of (\mathbf{e}) with respect to the unit sphere S^2 , respectively, [7].

In this paper, the striction curve of the ruled surface φ will be taken as the base curve. In this case, *Iranian Journal of Science & Technology, Trans. A, Volume 33, Number A2* *Spring 2009*

for the parametrization of φ , we can write

$$\varphi(s, v) = c(s) + ve(s).$$

3. INVOLUTE-EVOLUTE OFFSETS OF RULED SURFACES IN E^3

Let φ and φ^* be two ruled surfaces in E^3 . φ is said to be an *involute offset* of φ^* (or φ^* is said to be an *evolute offset* of φ), if there exists a one-to-one correspondence between their rulings such that the central normal of φ and the spherical indicatrix vector of φ^* are linearly dependent at the striction points of their corresponding rulings.

The base ruled surface, $\varphi(s, v)$, can be expressed as

$$\varphi(s, v) = c(s) + ve(s),$$

where c is its striction curve and s is the arc length along c . The equation of the offset surface φ^* , in terms of its base surface φ , can be written as

$$\varphi^*(s, v) = c^*(s) + ve^*(s) = [c(s) + R(s)t(s)] + vt(s), \tag{2}$$

where R is the distance between the corresponding striction points of φ and φ^* .

Moreover, since the striction curve of φ^* is its base curve, we have

$$R(s) = -\frac{\langle c_s, t_s \rangle}{\langle t_s, t_s \rangle}.$$

If e, t and g are the geodesic Frenet vectors of φ , then the geodesic Frenet vectors of the evolute offset φ^* of φ are given by

$$\begin{bmatrix} e^* \\ t^* \\ g^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & \sin\theta \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix}, \tag{3}$$

where θ is the angle between e and g^* .

Now, we can give the following theorem for θ :

Theorem 1. Let φ^* be the evolute offset of φ . If γ is constant, then θ is constant and also for the converse, if $\theta \neq 0$ then the converse is true.

Proof: From the definition of t^* , we get

$$t^* = \frac{t_q}{\|t_q\|}.$$

Because of equation (1), we obtain

$$t^* = -\frac{1}{\sqrt{1+\gamma^2}}e + \frac{\gamma}{\sqrt{1+\gamma^2}}g.$$

The last equation implies that

$$\sin \theta = \frac{1}{\sqrt{1+\gamma^2}} \text{ and } \cos \theta = \frac{\gamma}{\sqrt{1+\gamma^2}} .$$

This proves our claim.

Let φ^* be the evolute offset of φ . For the distribution parameter P_{e^*} of φ^* , we can write

$$P_{e^*} = \frac{\det(\mathbf{c}_s^*, \mathbf{e}_s^*, \mathbf{e}_s^*)}{\|\mathbf{e}_s^*\|^2} .$$

From Eq. (1), it is easy to see that

$$P_{e^*} = \frac{1}{1+\gamma^2} [P_e + \gamma \langle \mathbf{e}, \mathbf{c}_s \rangle] , \tag{4}$$

where P_e is the distribution parameter of φ .

If φ is developable, then the spherical indicatrix vector \mathbf{e} of φ is tangent to its striction curve. So, from eq. (4), we can give the following theorem without proof:

Theorem 2. Let φ^* be the evolute offset of a developable ruled surface φ . φ^* is developable if and only if the spherical indicatrix curve (e) of φ is a geodesic curve.

Let φ^* be an evolute offset of φ . If φ is closed, then there exists a positive integer P such that $\varphi(s+P, v) = \varphi(s, v)$. So, from eq. (2), we can give the following theorem without proof:

Theorem 3. Let φ^* be an evolute offset of the closed ruled surface φ with period P . φ^* is closed if and only if $R = R(s)$ is a function with a period P .

Let φ_t and φ_{t^*} be the ruled surfaces which are swept out by the central normals at the corresponding striction points of φ and φ^* where φ^* is the evolute offset of φ . Then, it is easily seen that φ_{t^*} is an evolute offset of φ_t . Furthermore, if we choose asymptotic normals instead of the central normals, then φ_{g^*} is not an evolute offset of φ_g .

Examples:

1) Let us consider the closed ruled surface $\varphi(s, v) = (\sin s + v \frac{4}{5} \cos s, -\frac{4}{5} \cos s + v \sin s, v \frac{3}{5} \cos s)$. The closed evolute offset of φ is $\varphi^*(s, v) = (\frac{9}{25} \sin s - v \frac{4}{5} \sin s, v \cos s, -\frac{12}{25} \sin s - v \frac{3}{5} \sin s)$, (see Fig. 1). Furthermore, for the ruled surfaces which are swept out by the central normals at corresponding striction points of φ and φ^* , we obtain $\varphi_t(s, v) = (\sin s - v \frac{4}{5} \sin s, -\frac{4}{5} \cos s + v \cos s, -v \frac{3}{5} \sin s)$ and $\varphi_{t^*}(s, v) = (\frac{9}{25} \sin s - v \frac{4}{5} \cos s, -v \sin s, -\frac{12}{25} \sin s - v \frac{3}{5} \cos s)$, (see Fig. 2).

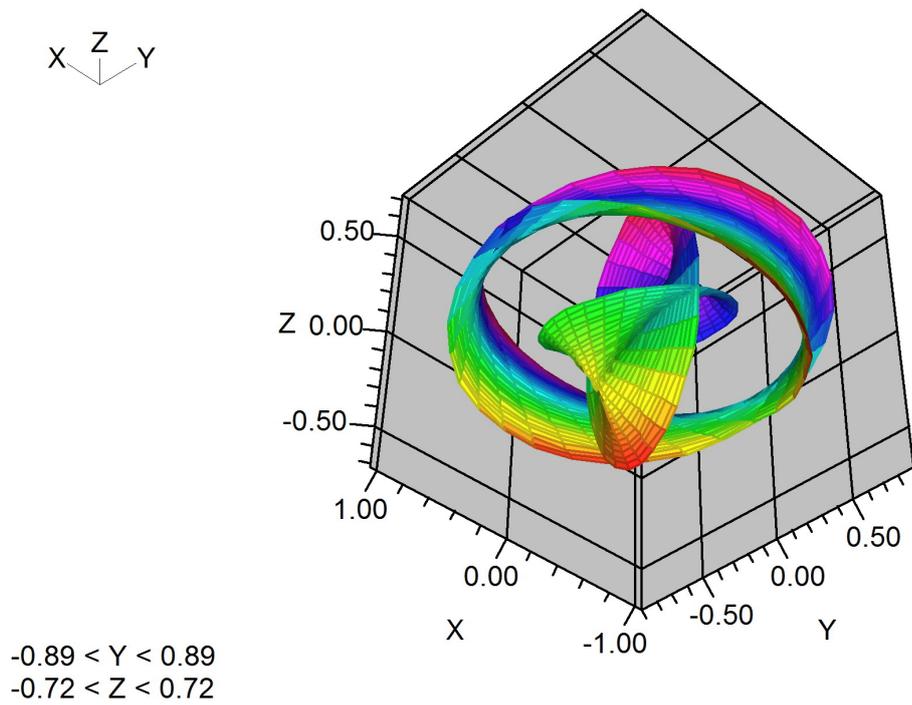


Fig. 1. Closed ruled surface φ and its closed evolute offset

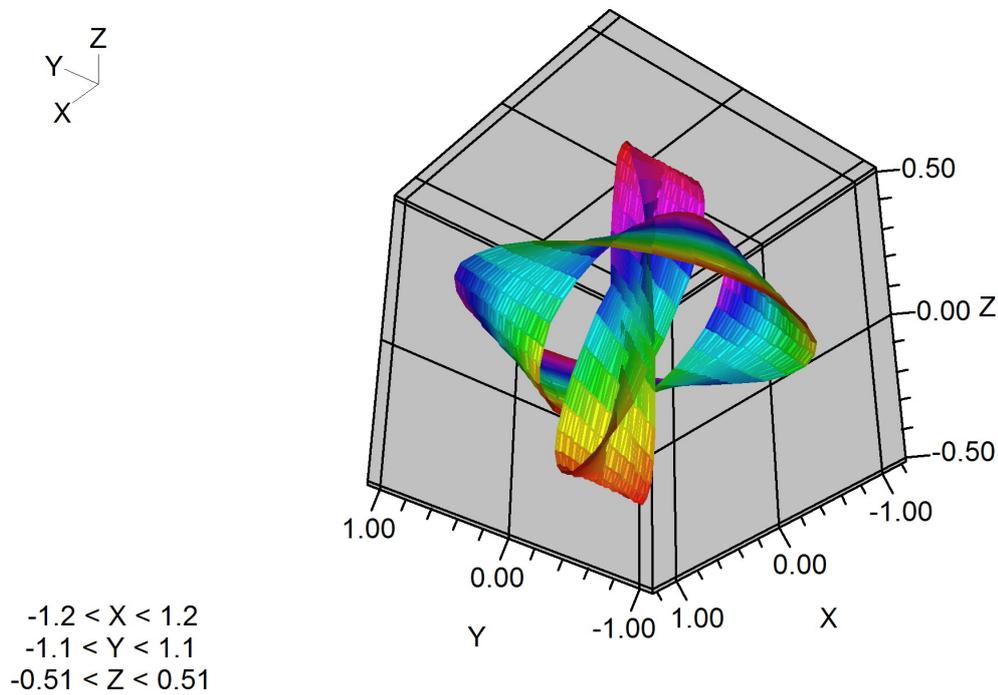


Fig. 2. Ruled surface φ_t and its evolute offset

2) Let $\varphi_1(s, v) = (\cos(\frac{\sqrt{2}}{2}s) - v \frac{\sqrt{2}}{2} \sin(\frac{\sqrt{2}}{2}s), \sin(\frac{\sqrt{2}}{2}s) + v \frac{\sqrt{2}}{2} \cos(\frac{\sqrt{2}}{2}s), \frac{\sqrt{2}}{2}s + v \frac{\sqrt{2}}{2})$ be a developable ruled surface. The developable evolute offset of φ_1 is $\varphi_1^*(s, v) = (-v \cos(\frac{\sqrt{2}}{2}s), -v \sin(\frac{\sqrt{2}}{2}s), 0)$ (see Fig. 3).

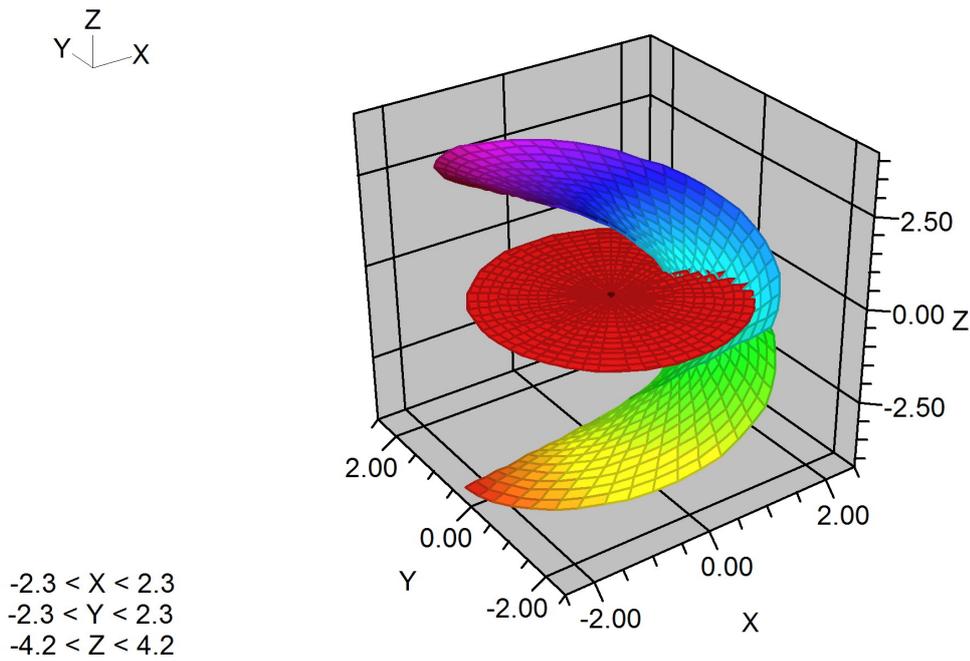


Fig. 3. Developable ruled surface φ_1 and its developable evolute offset

3) Let $\varphi_2(s,v) = (\frac{1}{\sqrt{2}}s^2 + v\frac{s}{\sqrt{2s^2+1}}, \frac{1}{\sqrt{2}}s^2 + v\frac{s}{\sqrt{2s^2+1}}, \sqrt{2}s + v\frac{1}{\sqrt{2s^2+1}})$ be a ruled surface. The evolute offset of φ_2 is $\varphi_2^*(s,v) = (\frac{1}{\sqrt{2}}s^2 + \sqrt{2}(2s^2+1)^2 + v\frac{1}{\sqrt{4s^2+2}}, \frac{1}{\sqrt{2}}s^2 + \sqrt{2}(2s^2+1)^2 + v\frac{1}{\sqrt{4s^2+2}}, -4\sqrt{2}(s^5+s^3) - v\frac{2s}{\sqrt{4s^2+2}})$ (see Fig. 4).

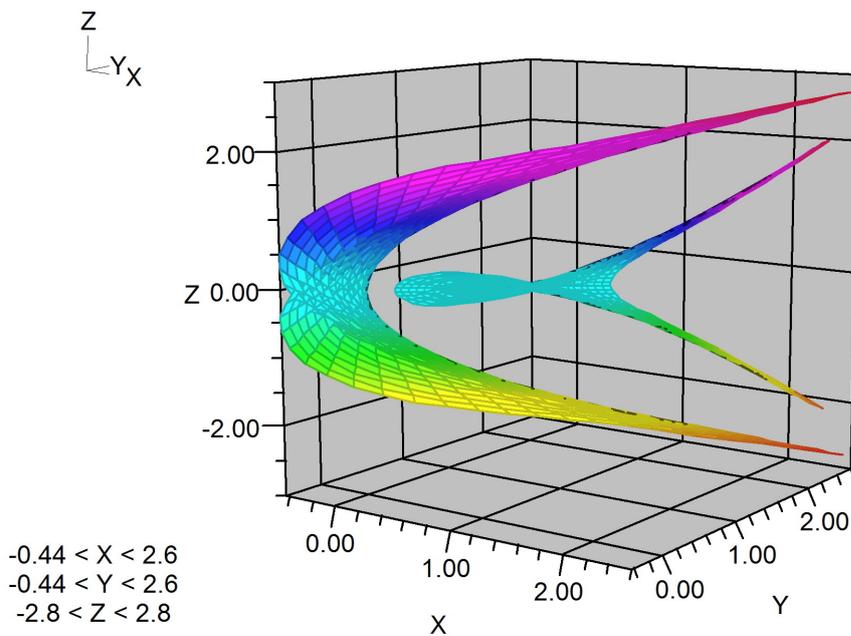


Fig. 4. Ruled surface φ_2 and its evolute offset

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