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A CLASS OF LIFT METRICS ON FINSLER MANIFOLDS*

E. PEYGHAN[1] AND A. TAYEBI[2]**

[1]Department of Mathematics and Computer Science, Arak University, Arak, I. R. of Iran
Email: epeyghan@gmail.com
[2]Department of Mathematics, Faculty of Science, Qom University, Qom, I. R. of Iran
Email: akbar.tayebi@gmail.com

Abstract – In this paper, we are going to study the g-natural metrics on the tangent bundle of Finsler manifolds. We concentrate on the complex and Kählerian and Hermitian structures associated with Finsler manifolds via g-natural metrics. We prove that the almost complex structure induced by this metric is a complex structure on tangent bundle if and only if the Finsler metric is of scalar flag curvature. Then we show that the complex structure is Hermitian if and only if the Finsler metric is of constant flag curvature.

Keywords – Finsler manifold, G-natural metrics, Kähler structure

1. INTRODUCTION

Almost complex structures are some important structures in differential geometry [1]. These structures obtained many applications in physics. H. E. Brandt has shown that the spacetime tangent bundle, in the case of Finsler spacetime manifold, is almost complex [2, 3]. Also, he demonstrated that in this case, the spacetime tangent bundle is complex provided that the gauge curvature field vanishes [4].

The Sasaki-Matsumoto lift $\widetilde{G}_{SM}$ defined on the manifold $T\widetilde{M} := TM - \{0\}$ of a Finsler metric tensor g is extremely important in the analysis of the geometry of a Finsler space $F^n = (M, F(x, y))$ [5-7]. $\widetilde{G}_{SM}$ determines a Riemannian structure on $T\widetilde{M}$, which only depends on the fundamental function F. It is not difficult to see that $\widetilde{G}_{SM}$ does not have a Finslerian meaning. More precisely, $\widetilde{G}_{SM}$ is not homogeneous with respect to the vertical variables y^i. Consequently, we cannot study global properties - as the Gauss-Bonnet Theorem – for the Finsler space F^n by means of this lift [2], [3], [8]. Also, since the two terms of the metric $\widetilde{G}_{SM}$ do not have the same physical dimensions, it does not satisfy the principles of the Post-Newtonian Calculus and so it is not convenient for a gauge theory. For these reasons, R. Miron introduce a new lift $\widetilde{G}_{SM}$ to $T\widetilde{M}$, which only depends on the fundamental function F of the Finsler space F^n and is 0-homogeneous on the fibers of the tangent bundle TM [6].

Kähler and para-Kähler structures associated with Finsler spaces and their relations with flag curvature were studied by M. Crampin and B.Y. Wu [5, 9]. They have found some interesting results on this matter. In [9], Wu gives some equivalent statements to the Kählerity of $(TM, \widetilde{G}_{SM}, \widetilde{J}_{SM})$. Also, in [6] Miron proves that the space $(TM, \widetilde{G}_M, \widetilde{J}_M)$ is in fact conformal, almost Kählerian.

In this paper, by using Finsler metric F on a manifold M, we introduced a lift metric $\widetilde{G}$ on TM named by the generalized Sasaki metric that Miron's Metric is a special case of this metric. To continue, we define an almost complex structure $\widetilde{J}$ on the slit tangent bundle $(T\widetilde{M}, \widetilde{G})$, by

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**Corresponding author

$$\tilde{J}(\frac{\delta}{\delta x^i}) = -\frac{1}{\sqrt{a(F^2)}}\frac{\partial}{\partial y^i} \quad \text{and} \quad \tilde{J}(\frac{\partial}{\partial y^i}) = \sqrt{a(F^2)}\frac{\delta}{\delta x^i}.$$

We prove that $\tilde{J}$ is a complex structure on $\tilde{TM}$ if and only if the Riemannian curvature of the Finsler metric F gets a special form. Then we show that $(TM, \tilde{G}, \tilde{J})$ is Hermitian if and only if F is of constant flag curvature k whenever $a(F^2) = \frac{1}{kF^2 + c}$. The integrability condition of $\tilde{J}$ follows that the base manifold M has a zero flag curvature and $(TM, \tilde{G}, \tilde{J})$ is Kählerian.

There are many connections in Finsler geometry [10, 11]. Throughout this paper, we use the Chern connection on Finsler manifolds.

2. PRELIMINARIES

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $\tilde{TM} := TM \setminus \{0\}$ the slit tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties: (i) F is C^∞ on $\tilde{TM}$; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the quadratic form $g_y : T_xM \otimes T_xM \to R$ defined by $g_y(u,v) := g_{ij}(y)\, u^i v^j$ is positive definite:

$$g_{ij}(x,y) := \tfrac{1}{2}[F^2(x,y)]_{y^i y^j}$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$, and $v = v^i \frac{\partial}{\partial x^i}|_x$.

Lemma 1. (Euler's Lemma) Let H be a real-valued function on R of positively homogeneous of degree r. If H is differentiable away from the origin of R, then

$$y^i \frac{\partial}{\partial y^i} H(y) = r\, H(y).$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathrm{C}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by $\mathrm{C}_y(u,v,w) := C_{ijk}(y)\, u^i v^j w^k$ where

$$C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}.$$

The family $\mathrm{C} := \{\mathrm{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. By using the notion of Cartan torsion, for $y \in T_xM_0$ define mean Cartan torsion $\mathrm{I}_y : T_xM \to R$ by $\mathrm{I}_y(u) := I_i(y)\, u^i$, where $I_i(y) := g^{jk} C_{ijk}(y)$. It is well known that **I=0** if and only if F is Riemannian [10].

The horizontal covariant derivatives of **C** and **I** along geodesics raises the Landsberg curvature $\mathrm{L}_y : T_xM \otimes T_xM \otimes T_xM \to R$ and mean Landsberg curvature $\mathrm{J}_y : T_xM \to R$ defined by $\mathrm{L}_y(u,v,w) := L_{ijk}(y)\, u^i v^j w^k$ and $\mathrm{J}_y(u) := J_i(y)\, u^i$ where

$$L_{ijk} := C_{ijk|s}\, y^s \text{ and } J_i := I_{i|s}\, y^y$$

The families $\mathrm{L} := \{\mathrm{L}_y\}_{y\in \tilde{TM}}$ and $\mathrm{J} := \{\mathrm{J}_y\}_{y\in \tilde{TM}}$ are called the Landsberg curvature and mean Landsberg curvature. A Finsler metric is called Landsberg metric and weakly Landsberg metric if **L=0** and $\mathrm{J} = 0$, respectively [10].

Let us consider the pull-back tangent bundle π^*TM over $\tilde{TM}$ defined by

$$\pi^* TM := \{(u,v) \in T\widetilde{M} \times T\widetilde{M} \mid \pi(u) = \pi(v)\}.$$

Take a local coordinate system (x^i) in *M*, the local natural frame $\{\frac{\partial}{\partial x^i}\}$ of $T_x M$ determines a local natural frame $\partial_i|_v$ for $\pi_v^* TM$ the fibers of $\pi^* TM$, where $\partial_i|_v = (v, \frac{\partial}{\partial x^i}|_x)$, and $v = y^i \frac{\partial}{\partial x^i}|_x \in T\widetilde{M}$. The fiber $\pi_v^* TM$ is isomorphic to $T_{\pi(v)} M$ where $\pi(v) = x$. There is a canonical section ℓ of $\pi^* TM$ defined by $\ell_v = (v,v)/F(v)$.

Using the coefficients g_{ij} and C_{ijk}, we define $C^i{}_{jk} := g^{is} C_{sjk}$ where (g^{js}) is the inverse matrix of g_{ij}. The formal Christoffel symbols of the second kind are

$$\gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\},$$

They are functions on $T\widetilde{M}$. We can also define some other quantities on $T\widetilde{M}$ by

$$N^i_j(x,y) := \gamma^i{}_{jk} y^k - C^i{}_{jk} \gamma^k{}_{rs} y^r y^s,$$

where $y \in T\widetilde{M}$. The above N^i_j are called the nonlinear connection coefficients on $T\widetilde{M}$.

Theorem 2. (Chern connection) The pull-back tangent bundle $\pi^* TM$ admits a unique linear connection ∇, which is torsion-free and almost metric-compatible. The coefficients of this connection in the standard coordinate system are given by

$$\Gamma^l{}_{jk} = \gamma^l{}_{jk} - g^{li}\{C_{ijs} N^s_k - C_{jks} N^s_i + C_{kis} N^s_j\}.$$

Let ∇ be the Chern connection on $\pi^* TM$ and $\{e_i\}_{i=1}^n$ be a local orthonormal frame field for $\pi^* TM$ such that $e_n := \ell$. Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. Put $\nabla e_i = \omega_i^j \otimes e_j$ and $\Omega e_i = 2\Omega_i{}^j \otimes e_j$, where $\{\Omega_i^j\}$ and $\{\omega_i^j\}$ are called respectively, the curvature forms and connection forms of ∇ with respect to $\{e_i\}$. Put $\omega^{n+i} := \omega_n^i + d(logF)\delta_n^i$. Then $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(T\widetilde{M})$. The curvature 2-forms of Chern connection are

$$\Omega_i^j = d\,\omega_i^j - \omega_i^k \wedge \omega_k^j.$$

Since Ω^i_j are 2-forms on $T\widetilde{M}$, they can be expanded as

$$\Omega_i^j = \frac{1}{2} R_i{}^j{}_{kl} \omega^k \wedge \omega^l + P_i{}^j{}_{kl} \omega^k \wedge \omega^{n+l} + \frac{1}{2} Q_i{}^j{}_{kl} \omega^{n+k} \wedge \omega^{n+l},$$

Since ∇ is torsion free then $Q_i{}^j{}_{kl} = 0$.

Flag curvature: A flag curvature is a geometrical invariant that generalizes what in Riemannian geometry is called the sectional curvature. For all $x \in M$ and $0 \neq y \in T_x M$, $V := V^i \frac{\partial}{\partial x^i}$ is called the transverse edge. Flag curvature is obtained by carrying out the following computation at the point $(x,y) \in T\widetilde{M}$, and viewing y and V as sections of $\pi^* TM$:

$$K(y,V) := \frac{V^i (y^j R_{jikl}\, y^l) V^k}{g(y,y)g(V,V) - [g(y,V)]^2},$$

where $R_{jikl} := g_{is} R^{s}_{j\,kl}$. If *K* is independent of the transverse edge V, then (M, F) is called scalar flag curvature. The scalar denoted by $\lambda = \lambda(x, y)$, if it has no dependence on either x or y, then the Finsler manifold is said to be of constant flag curvature [12].

3. LIFT METRIC

For a given Finsler manifold (M, F), we can endow its slit tangent bundle $T\widetilde{M}$ with a Riemannian metric, known as the generalized Sasaki metric [8]. It can be described in local coordinates as follows. Let $(x, y) = (x^i, y^i)$ be the local coordinates on $T\widetilde{M}$. It is well known that the tangent space to $T\widetilde{M}$ at (x, y) splits into the direct sum of the vertical subspace $VT\widetilde{M}_{(x,y)} := span\{\frac{\partial}{\partial y^i}\}$ and the horizontal subspace $HT\widetilde{M}_{(x,y)} := span\{\frac{\delta}{\delta x^i}\}$:

$$TT\widetilde{M}_{(x,y)} = VT\widetilde{M}_{(x,y)} \oplus HT\widetilde{M}_{(x,y)}$$

where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}$.

M. Matsumoto [13] extended to Finsler spaces F^n the notion of Sasaki lift, considering the tensor field

$$\tilde{G}_{SM}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + g_{ij}(x, y)\delta y^i \otimes \delta y^j, \qquad \forall (x, y) \in T\tilde{M} \tag{1}$$

It easily follows that $\widetilde{G}_{SM}$ is a Riemannian metric globally defined on $T\widetilde{M}$ and dependent only on the fundamental function *F* of the Finsler space F^n.

Next we consider the $F(T\widetilde{M})$-linear mapping $\widetilde{J}_{SM} : \chi(T\widetilde{M}) \to \chi(T\widetilde{M})$, defined by

$$\tilde{J}_{SM}(\frac{\delta}{\delta x^i}) = -\frac{\partial}{\partial y^i}, \quad \tilde{J}_{SM}(\frac{\partial}{\partial y^i}) = \frac{\delta}{\delta x^i} \qquad (i = 1, \cdots, n) \tag{2}$$

It is known that $\widetilde{J}_{SM}$ is an almost complex structure on $T\widetilde{M}$ depending only on the fundamental function *F* which becomes a complex structure on $T\widetilde{M}$ if and only if the horizontal distribution $HT\widetilde{M}$ is integrable.

Remarking that the pair $(\widetilde{G}_{SM}, \widetilde{J}_{SM})$ is an almost Hermitian structure, we recall the known result that $(T\widetilde{M}, \widetilde{G}_{SM}, \widetilde{J}_{SM})$ is an almost Kählerian space.

The G-natural metric $\widetilde{G}$ on $T\widetilde{M}$ is defined by

$$\tilde{G}_{(a,b)}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + \big(a(F^2)g_{ij}(x, y) + b(F^2)y_i y_j\big)\delta y^i \otimes \delta y^j \tag{3}$$

where $F^2 = g_{ij}(x, y)y^i y^j$ and $a : \mathrm{Im}(F^2) \subset R^+ \to R^+$

For $b = 0$ and $a = \frac{c^2}{F^2}$ for any constant *c*, the metrical structure (3) was studied by R. Miron in [11] as a homogeneous lift of $g_{ij}(x, y)$ to $T\widetilde{M}$.

We are looking for a new almost complex structure paired with $G_{a,b}$ to provide a complex structure. We modify $\widetilde{J}_{SM}$ to a $F(T\widetilde{M})$-linear map $\widetilde{J}_{a,b}$ given in the basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^k})$ as follows:

$$\tilde{J}_{a,b}(\frac{\delta}{\delta x^i}) = (\alpha \delta^k_i + \beta y_i y^k)\frac{\partial}{\partial y^k}, \quad \tilde{J}_{a,b}(\frac{\partial}{\partial y^i}) = (\gamma \delta^h_i + \delta y_i y^h)\frac{\delta}{\delta x^h} \tag{4}$$

where α, β, γ and δ are functions on $T\widetilde{M}$ to be determined. The condition $\widetilde{J}^2_{a,b} = -I$ leads to

$$\alpha\gamma = -1, \ \alpha\gamma + \beta\gamma + \beta\delta F^2 = 0. \tag{5}$$

Then the condition

$$\widetilde{G}_{(a,b)}(\widetilde{J}_{a,b}(X),\ \widetilde{J}_{a,b}(Y)) = -\widetilde{G}_{(a,b)}(X,Y)$$

gives

$$a\,\alpha^2 = -1, \quad \gamma^2 = a, \quad 2\gamma\delta + \delta^2 F^2 = b, \quad (2\,\alpha\,\beta + \beta^2 F^2)(a + F^2) + b\alpha^2 = 0. \tag{6}$$

The solution of the system of equation (5), (6) is

$$\alpha = -\frac{1}{\sqrt{a}}, \quad \beta = \frac{\sqrt{a} + \sqrt{a + b\,F^2}}{F^2\sqrt{a(a + b\,F^2)}}, \quad \gamma = \sqrt{a}, \quad \delta = \frac{\sqrt{a} + \sqrt{a + b\,F^2}}{F^2}. \tag{7}$$

We notice that for *b=0*, besides the solution provided by (7), that is

$$\alpha = -\frac{1}{\sqrt{a}}, \quad \gamma = \sqrt{a}, \quad \beta = \frac{2}{F^2\sqrt{a}}, \quad \delta = \frac{2\sqrt{a}}{F^2}. \tag{8}$$

There also exists the solution

$$\alpha = -\frac{1}{\sqrt{a}}, \quad \gamma = \sqrt{a}, \quad \beta = 0, \quad \delta = 0. \tag{9}$$

Let $\widetilde{J}$ be the almost complex structure given by (4), (9). Then we get, without difficulty, the following theorem:

Theorem 1. The pair $(\widetilde{G}, \widetilde{J})$ is almost Hermitian structure on $T\widetilde{M}$, where

$$\tilde{G}(x,y) = g_{ij}(x,y)\,dx^i \otimes dx^j + a(F^2)\,g_{ij}(x,y)\,\delta y^i \otimes \delta y^j. \tag{10}$$

Also, by a simple calculation, we can get the following lemma:

Lemma 2. Let (M, F) be a Finsler manifold with Chern connection. Then we have:

$$[\delta_i, \delta_j] = -R^{\,k}_{\;i\,j}\,\partial_{\bar{k}} \tag{11}$$

$$[\partial_{\bar{i}}, \partial_{\bar{j}}] = 0 \tag{12}$$

$$[\delta_i, \partial_{\bar{j}}] = (\Gamma^{\,k}_{\;i\,j} + L^{\,k}_{\;i\,j})\,\partial_{\bar{k}}, \tag{13}$$

where $\partial_{\bar{i}} := \dfrac{\partial}{\partial y^i}$ and $\delta_i = \dfrac{\delta}{\delta x^i}$.

Theorem 3. The Levi-Civita connection $\tilde{\nabla}$ on $T\widetilde{M}$ with respect to $\widetilde{G}$ is given by the following equations:

$$\tilde{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}} = a(F^2)\,L^{k}_{i\,j}\,\delta_k + [C^{k}_{i\,j} + \frac{a'(F^2)}{a(F^2)}(y_i\,\delta^k_j + y_j\,\delta^k_i - g_{ij}\,y^k)]\partial_{\bar{k}} \tag{14}$$

$$\tilde{\nabla}_{\partial_{\bar{i}}}\delta_j = [C^{k}_{i\,j} + \frac{1}{2}a(F^2)\,y^l\,R^{\,k}_{l\,ij}]\delta_k - L^{k}_{i\,j}\;\partial_{\bar{k}} \tag{15}$$

$$\tilde{\nabla}_{\delta_{\bar{i}}} \partial_{\bar{j}} = [C^k_{i\,j} + \frac{1}{2} a(F^2)\, y^l\, R^k_{l\,ij}]\; \delta_k - \Gamma^k_{i\,j}\; \partial_{\bar{k}} \tag{16}$$

$$\tilde{\nabla}_{\delta_{\bar{i}}} \delta_j = \Gamma^k_{i\,j}\; \delta_k - [\frac{1}{a(F^2)} C^k_{i\,j} + \frac{1}{2}\; R^k_{l\,ij})] \partial_{\bar{k}} \tag{17}$$

Proof: By the definition of $\tilde{G}$ and Lemma 2 and Koszul formula

$$2\,\tilde{G}(\tilde{\nabla}^{\tilde{Y}}_{\tilde{X}}, \tilde{Z}) = \tilde{X}\,\tilde{G}(\tilde{Y}, \tilde{Z}) + \tilde{Y}\,\tilde{G}(\tilde{Z}, \tilde{X}) - \tilde{Z}\,\tilde{G}(\tilde{X}, \tilde{Y})$$
$$+ \tilde{G}([\tilde{X}, \tilde{Y}],\; \tilde{Z}) - \tilde{G}([\tilde{Y}, \tilde{Z}],\; \tilde{X}\;) + \tilde{G}([\tilde{Z}, \tilde{X}],\; \tilde{Y}\;)$$

we have:

$$2\,M^r_{i\,j}\; g_{rk} = -\frac{\delta}{\delta x^k}\big(a(F^2) g_{ij}(x,y)\big) + a(F^2)(\Gamma^h_{jk} + L^h_{jk}) g_{hi}(x,y)$$
$$+\, a(F^2)(\Gamma^h_{ik} + L^h_{ik}) g_{hj}(x,y), \tag{18}$$

where $M^{\;r}_{ij}$ is the horizontal coefficient of $\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \dfrac{\partial}{\partial y^j}$. Since $\dfrac{\delta}{\delta x^k}(F^2) = 0$ and

$$\frac{\delta}{\delta x^k} g_{ij}(x,y) - \Gamma^h_{jk}\; g_{hi}(x,y) - \Gamma^h_{ik}\; g_{hj}(x,y) = 0,$$

then (18) change to

$$2\,M^r_{i\,j}\; g_{rk} = a(F^2) L^h_{jk} g_{hi} + a(F^2) L^h_{ik} g_{hj} \;=\; 2a(F^2) L_{ijk}\,.$$

By multiplying g^{lk} in the above equation we get $M^{\,l}_{i\,j} = a(F^2) L^{\,l}_{i\,j}$. Similarly, we get

$$N^l_{ij} = C^k_{i\,j} + \frac{a'(F^2)}{a(F^2)}(y_i\; \delta^k_j + y_j\; \delta^k_i - g_{ij}\; y^k).$$

Then, we prove (14). In a similar way, we get (15), (16), (17).

4. MAIN RESULTS

In this section, we prove that the almost complex structure $\tilde{J}$ on the slit tangent bundle is a complex structure if and only if F is of scalar flag curvature. In the case $a(F^2) = \dfrac{1}{kF^2 + c}$, we show that $(TM, \tilde{G}, \tilde{J})$ is Hermitian if and only if $(M,\, F)$ has a constant flag curvature. Also, for $a(F^2) = c$, we conclude that almost complex structure on the slit tangent bundle is integrable if and only if the base manifold has a zero flag curvature. In that case, the slit tangent bundle is Kählerian.

Theorem 1. Let $(M,\, F)$ be a Finsler manifold. Then $\tilde{J}$ is complex structure on $T\tilde{M}$ (integrable) if and only if F is of scalar flag curvature.

Proof: Using the definition of the Nijenhuis tensor field N_J of J, that is,

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \qquad \forall\, X, Y \in \Gamma(TM),$$

we get:

$$N_J(\delta_i, \delta_j) = \{R^{\ k}_{ij} - \frac{a'(F^2)}{a^2(F^2)} y_i\, \delta^k_j + \frac{a'(F^2)}{a^2(F^2)} y_j\, \delta^k_i\}\, \partial_{\bar{k}}. \tag{19}$$

It follows by a straightforward computation that $N_J(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0$, $N_J(\partial_{\bar{i}}, \delta_j) = 0$, whenever $N_J(\delta_i, \delta_j) = 0$. Therefore, from relations (19), we conclude that *J* is a complex structure if and only if the following relation holds:

$$R^{\ k}_{ij} = -\frac{a'(F^2)}{a^2(F^2)}(y_j \delta^k_i - y_i \delta^k_j). \tag{20}$$

This means that *F* is of the scalar flag curvature $-\frac{a'(F^2)}{a^2(F^2)}$.

The equation (20) suggests that we look for the function *a* such that $-\frac{a'(F^2)}{a^2(F^2)} = k$, where *k* is a constant. For $t = F^2$, solving the equation $-a' = k\,a^2$ one gets $a(F^2) = \frac{1}{kF^2 + c}$ where *c* is a constant of integration. For this function *a*, the equation (20) becomes

$$R^k_{ij} = k(y_j \delta^k_i - y_i \delta^k_j). \tag{21}$$

which implies that the Finsler metric *F* is of the constant flag curvature *k*. By considering the condition $a(F^2) = \frac{1}{kF^2 + c}$, we have proved the following theorem.

Theorem 2. Let (M, F) be a Finsler manifold and $a(F^2) = \frac{1}{kF^2 + c}$. Then $(TM, \widetilde{G}, \widetilde{J})$ is Hermitian if and only if (M, F) has constant flag curvature *k*.

Now let us consider the Cartan forms

$$\theta = \frac{1}{2}\frac{\partial F^2}{\partial y^i} dx^i, \qquad \omega = g_{ij}(x, y)\,\delta y^i \wedge dx^j. \tag{22}$$

Evidently, θ and ω are globally defined on $\widetilde{TM}$ and ω is an almost symplectic structure on $\widetilde{TM}$. As is known, between θ and ω there is the relation

$$d\theta = \omega, \tag{23}$$

where *d* is the exterior differential operator. So ω is a closed 2-form. In other words, ω is a symplectic structure.

Let $\Omega(X, Y) = \widetilde{G}(\widetilde{J}X, Y)$ for all *X* and *Y* in $\widetilde{TM}$ be the almost symplectic structure associated to the almost Hermitian structure $(\widetilde{G}, \widetilde{J})$. With a straightforward computation we obtained

$$\Omega = \sqrt{a}\, g_{ij}(x, y)\,\delta y^i \wedge dx^j = \sqrt{a}\,\omega \tag{24}$$

Note that $(\widetilde{G}, \widetilde{J})$ is called a locally conformal almost Kählerian structure if $d\Omega = \Omega \wedge \alpha$, where α is a 1-form on $\widetilde{TM}$. In the following theorem we consider almost Kählerian and Kählerian structures of Finsler manifolds.

Theorem 3. Let (M, F) be a Finsler manifold. Then we have:

1. $(\widetilde{G}, \widetilde{J})$ is a locally conformal almost Kählerian structure.
2. $(TM, \widetilde{G}, \widetilde{J})$ is almost Kählerian if and only if $a(F^2) = c$, where *c* is a constant.

Proof: Since $d\,\omega = d^2\theta = 0$, then from (24) we get

$$d\Omega = \frac{2L\,a'}{a}\,dL \wedge \Omega. \tag{25}$$

Therefore, statement (i) is proved. Again, with attention to equation (25), Ω is closed if and only if $a'(F^2) = 0$.

Corollary 4. Let (M, F) be a Finsler manifold. If $a(F^2) \neq c$, then $(TM, \widetilde{G}, \widetilde{J})$ can not be Kählerian. If $a(F^2) = c$, then from (19) we get $N_J(\delta_i, \delta_j) = R^k{}_{ij}\,\partial_{\bar{k}}$. Therefore, we have the following theorem:

Theorem 5. Let (M, F) be a Finsler manifold. If $a(F^2)$ is a scalar function, then the following statements are mutually equivalent:
(i) (M, F) has zero flag curvature,
(ii) $\widetilde{J}$ is integrable,
(iii) $\widetilde{\nabla}\widetilde{J} = 0$,
(iv) $(TM, \widetilde{G}, \widetilde{J})$ is Kählerian.

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