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ON A SPECIAL CLASS OF FINSLER METRICS*

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Abstract – In this paper, we study projective Randers change and C-conformal change of P-reducible metrics. Then we show that every P-reducible generalized Landsberg metric of dimension $n > 2$ must be a Landsberg metric. This implies that on Randers manifolds the notions of generalized Landsberg metric and Berwald metric are equivalent.

Keywords – Randers metric, Landsberg metric, Berwald metric, Randers change, C-conformal change

1. INTRODUCTION

In [1], Matsumoto introduced the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, Matsumoto-Hōjō proves that the converse is true too [2]. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric α perturbated by a one form β. Randers metrics have important applications both in mathematics and physics [3].

As a generalization of C-reducible metrics, Matsumoto-Shimada introduced the notion of P-reducible metrics [4]. This class of Finsler metrics has some interesting physical means and contains the class of Randers metrics as a special case. Therefore the study of P-reducible spaces will enhance our understanding of the geometric meaning of Randers metrics.

By considering the special form of a Randers metric, Shibata has dealt with a change of Finsler metric which is called Randers change [5]. Adding a 1-form to a Finsler metric is called a Randers change or β-change. For a β-change of Finsler metric, the differential one-form β play very important roles.

In this paper, we consider the projective Randers change of a P-reducible Finsler metric. We assume that F be a P-reducible Finsler metric and $\overline{F}$ is a general relatively isotropic Landsberg metric obtained from F by a projective Randers change. Then we prove that $\overline{F}$ is C-reducible metric if and only if F is C-reducible metric.

Theorem 1. Let (M, F) be a P-reducible Finsler manifold. Suppose that $\overline{F}$ is a general relatively isotropic Landsberg metric $(\overline{\mathrm{L}} = \overline{\lambda}\,\overline{\mathrm{C}})$ obtained from F by a projective Randers change. Then F is Randers metric if and only if $\overline{F}$ is P-reducible.

Beside the Randers changes, we have another special transformation named C-conformal transformation. The notion of C-conformal transformation and its properties that are regarded as a special conformal transformation is introduced by Hashiguchi [6]. C-conformal transformations are a special form of conformal transformation that satisfies a condition on Cartan tensor and conformal factor. In section 4,

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we study C-conformal transformation of P-reducible metrics and prove the following.

Theorem 2. Let (M, F) be a P-reducible Finsler manifold. Suppose that $\overline{F}$ is obtained from F by C-conformal change. Then F is Randers metric if and only if $\overline{F}$ is P-reducible.

As a generalization of Landsberg metrics, we introduce the notion of generalized Landsberg metrics. A Finsler metric is called a generalized Landsberg metric if the Riemannian curvature of the Berwald and Chern connections coincides. It is easy to show that Landseberg metrics belong to this class of Finsler metrics. Here we show that:

Theorem 3. Let (M, F) be a P-reducible Finsler manifold of dimension $n > 2$. Suppose that F is a generalized Landsberg metric. Then F is a Landsberg metric.

There are many connections in Finsler geometry [7]. Throughout this paper, we set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by “ | " and “, " respectively.

2. PRELIMINARIES

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 := TM \setminus \{0\}$ the slit tangent bundle of M.

A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite,

$$\mathrm{g}_y(u,v) := \frac{1}{2}[F^2(y+su+tv)]_{s,t=0}, \qquad u,v \in T_xM .$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathrm{C}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by

$$\mathrm{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}[g_{y+tw}(u,v)]_{t=0}, \qquad u,\ v,\ w \in T_xM$$

The family $\mathrm{C} := \{\mathrm{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathrm{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define mean Cartan torsion I by $\mathrm{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i \partial/\partial x^i|_x$. By Diecke's Theorem, F is Riemannian if and only if $\mathrm{I} = 0$.

For $y \in T_xM_0$, define the Matsumoto torsion $\mathrm{M}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by $\mathrm{M}_y(u,v,w) := M_{ijk}(y)\, u^i v^j w^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_i\, h_{jk} + I_j\, h_{ki} + I_k\, h_{ij}\},$$

and $h_{ij} := g_{ij} - F^{-2}y_i y_j$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathrm{M} = 0$. This quantity is introduced by Matsumoto [1]. Matsumoto proves that every Randers metric satisfies $\mathrm{M} = 0$. Later on, Matsumoto-Hōjō proves that the converse is true too.

Lemma 1. ([2, 8]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathrm{M} = 0$.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to R$ defined by

$$\mathbf{L}_y(u,v,w) := L_{ijk}(y)\, u^i v^j w^k,$$

where $L_{ijk} := C_{ijk|s} y^s$, $u = u^i \partial/\partial x^i |_x$, $v = v^i \partial/\partial x^i |_x$ and $w = w^i \partial/\partial x^i |_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if **L**=0.

The quotient **L**/*C* is regarded as the relative rate of change of $\mathbf{C}$ along geodesics. A Finsler metric F on a manifold M is said to be general relatively isotropic Landsberg metric if $\mathbf{L} = \lambda\, \mathbf{C}$, where λ is a positively 1-homogeneous scalar function on TM_0 [9]. The generalized Funk metrics on the unit ball $B^n \subset R^n$ satisfy $\mathbf{L} + cF\,\mathbf{C} = 0$ for some constant $c \neq 0$ [10].

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}$, where $\mathbf{J}_y(u) := J_i(y) u^i$, where $J_i = I_{i|s} y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$.

Define $\overline{\mathbf{M}}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by $\overline{\mathbf{M}}_y(u,v,w) := \overline{M}_{ijk}(y)\, u^i v^j w^k$ where

$$\overline{M}_{ijk} := L_{ijk} - \frac{1}{n+1}\{J_i\, h_{jk} + J_j\, h_{ki} + J_k\, h_{ij}\}.$$

A Finsler metric F is said to be P-reducible if $\overline{\mathbf{M}}_y = 0$. The notion of P-reducibility was given by Matsumoto-Shimada [4]. It is obvious that every C-reducible metric is a P-reducible metric.

Given a Finsler manifold (M, F), then a global vector field **G** is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(y)$ are local functions on TM as follows

$$G^i(y) := \frac{1}{4} g^{il}(y)\{\frac{\partial^2 [F^2]}{\partial x^k \partial y^l}(y) y^k - \frac{\partial [F^2]}{\partial x^l}(y)\}, \quad y \in T_xM.$$

Then **G** is called the associated spray to (M, F). The projection of an integral curve of **G** is called a geodesic in M**.** In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by

$$\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)\, u^j v^k w^l \frac{\partial}{\partial x^i}|_x,$$

where

$$B^i{}_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^i \partial y^j \partial y^k}(y).$$

The quantity **B** is called the Berwald curvature. A Finsler metric is called a Berwald metric if **B**=0 [11, 12]. Every Berwald metric is a Landsberg metric.

3. PROOF OF THEOREM 1

Let $\beta = b_i(x)y^i$ be a 1-form and F a positive Finsler metric on a manifold *M*. Consider the following special transformation $\bar{F} := F + \beta$. Adding a 1-form to a Finsler metric is called a Randers change. Let $\|\beta\| := \sup_{F(y)=1} |\beta(y)| < 1$. Then $\bar{F}$ is still a positive definite Finsler metric [5].

A change $F \to \bar{F}$ of a Finsler metric on the same underlying manifold *M* is called projective, if any geodesic in (M, F) remains to be a geodesic in $(M, \bar{F})$ and vice versa. In this section, we restrict our consideration to special Randers change, called projective Randers change, which preserve all the geodesic curves. According to Hashiguchi-Ichijyō [13], a Randers change is projective if and only if $b_{i|j} = b_{j|i}$, that is, $b_i(x)$ is locally a gradient vector field. To avoid any calculation, we use the notion of [14]. Put

$$r_{ij} := \frac{1}{2}(\partial_j b_i + \partial_i b_j) - b_r F^r_{\ ij} \ , \ r_{00} := r_{ij}\, y^i y^j \ , \quad t := \frac{\bar{F}}{F}$$

and

$$q_k := r_{mk}\, y^m - \frac{r_{00}}{2F} + \{b_k + (1+t)\,\ell_k\},$$

where $F^r_{\ ij}$ are the coefficients of Cartan connection and $\ell_k := F^{-1} y_k$. Using these notations, we are ready to prove the Theorem 1.

Lemma 2. ([13]) Relation between Landsberg tensors of *F* and $\bar{F}$ is given by

$$\bar{L}_{ijk} = t\, L_{ijk} + \frac{r_{00}}{2F} C_{ijk} + \frac{1}{2F}\{q_i h_{jk} + q_j h_{ki} + q_k h_{ij}\}. \tag{1}$$

Now, we are going to prove the Theorem 1.

Proof of Theorem 1. By Lemma 2, we get

$$\lambda \bar{C}_{ijk} = t\, L_{ijk} + \frac{r_{00}}{2F} C_{ijk} + \frac{1}{2F}\{q_i h_{jk} + q_j h_{ki} + q_k h_{ij}\}. \tag{2}$$

Using $\bar{F}\,\bar{h}^{ij} = F\, h^{ij}$, and multiplying (3 .1) by $F\, h^{ij}$ from right and $\bar{F}\,\bar{h}^{ij}$ from left, one can obtains

$$q_k = \frac{2}{n+1}\{\bar{\lambda}\,\bar{F}\,\bar{I}_k - \bar{F}\, J_k - \frac{r_{00}}{2} I_k\}. \tag{3}$$

Substitution (3) in (2) leads to

$$\begin{aligned} \bar{\lambda}\,\bar{C}_{ijk} = {} & t\,\{L_{ijk} - \frac{1}{n+1}(J_i h_{jk} + J_j h_{ki} + J_k h_{ij})\} \\ & - \frac{r_{00}}{2F}\{C_{ijk} - \frac{1}{n+1}(I_i h_{jk} + I_j h_{ki} + I_k h_{ij})\} \\ & + \frac{t\,\bar{\lambda}}{n+1}\{\bar{I}_i h_{jk} + \bar{I}_j h_{ki} + \bar{I}_k h_{ij}\}. \end{aligned} \tag{4}$$

Put $t\, h_{ij} = \bar{h}_{ij}$ in (4), one can yields

$$\bar{\lambda}\{\bar{C}_{ijk} - \frac{1}{n+1}(\bar{I}_i \bar{h}_{jk} + \bar{I}_j \bar{h}_{ki} + \bar{I}_k \bar{h}_{ij})\} = t\{L_{ijk} - \frac{1}{n+1}(J_i h_{jk} + J_j h_{ki} + J_k h_{ij})\} - \frac{r_{00}}{2F}\{C_{ijk} - \frac{1}{n+1}(I_i h_{jk} + I_j h_{ki} + I_k h_{ij})\}. \tag{5}$$

Replacing the definition of P-reducibility of *F* in (5) implies that

$$\bar{\lambda}\{\bar{C}_{ijk} - \frac{1}{n+1}(\bar{I}_i \bar{h}_{jk} + \bar{I}_j \bar{h}_{ki} + \bar{I}_k \bar{h}_{ij})\} = \frac{r_{00}}{2F}\{C_{ijk} - \frac{1}{n+1}(I_i h_{jk} + I_j h_{ki} + I_k h_{ij})\}. \tag{6}$$

Then $\bar{F}$ is Randers metric if and only if *F* is Randers metric.

Corollary 1. Let (M, F) be a Finsler manifold with Randers metric $F = \alpha + \beta$. Suppose that $\bar{F}$ is obtained from F by a projective Randers change and satisfies $\bar{\mathrm{L}} = \bar{\lambda}\,\bar{\mathrm{C}}$. Then $\bar{F}$ is a Randers metric.

Corollary 2. Let (M, F) be a P-reducible Finsler manifold and F satisfies $\mathrm{L} = \lambda\,\mathrm{C}$. Suppose that $\bar{F}$ is obtained from F by a projective Randers change and satisfies $\bar{\mathrm{L}} = \bar{\lambda}\,\bar{\mathrm{C}}$ such that $r_{00} + 2\lambda\bar{F} \neq 0$. Then F is a Randers metric if and only if $\bar{F}$ is a Randers metric.

Proof: By (5) we have

$$2F\bar{\lambda}\{\bar{C}_{ijk} - \frac{1}{n+1}(\bar{I}_i \bar{h}_{jk} + \bar{I}_j \bar{h}_{ki} + \bar{I}_k \bar{h}_{ij})\} = (\lambda\bar{F} + r_{00})\{C_{ijk} - \frac{1}{n+1}(I_i h_{jk} + I_j h_{ki} + I_k h_{ij})\}. \tag{7}$$

By (7) and $r_{00} + 2\lambda\bar{F} \neq 0$, we get the proof.

4. PROOF OF THEOREM 2

Two Finsler metrics *F* and $\bar{F}$ on *M* are called *conformal* if $\bar{g}_{ij} = \varphi g_{ij}$, where φ is a positive scalar function on *TM*. If φ is a constant, they are called *homothetic*. The Knebelman's Theorem states that φ falls into, at most, a point function. Thus we can assume $\varphi = e^{2\alpha}$, where α is a scalar function on *M*.

Put

$$\alpha_i = \partial\alpha / \partial x^i, \quad C^i{}_j = C^{ir}{}_j \alpha_r, \quad \alpha_0 := \alpha_i y^i.$$

Then we have

$$\bar{F} = e^{\alpha} F, \quad \bar{g}^{ij} = e^{-2\alpha} g^{ij}.$$

Two Finsler metrics *F* and $\bar{F}$ on *M* are called *C-conformal* if their conformal transformation is non-homothetic and satisfies $C_{jk} = 0$ [6].

Theorem 4. C-conformal change of every Randers metric is a Randers metric.

Proof: Let $F = \alpha + \beta$ be a Randers metric and $\bar{F}$ is obtained from *F* by a C-conformal change. By Lemma 1, *F* is C-reducible

$$C_{ijk} = \frac{1}{n+1}\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}. \tag{8}$$

By assumption we have

$$h_{ij} = e^{-2\alpha} \bar{h}_{ij}, \tag{9}$$

$$C_{ijk} = e^{-2\alpha}\bar{C}_{ijk} \,. \tag{10}$$

Contraction (10) with $g^{ij} = e^{2\alpha}\bar{g}^{ij}$ yields

$$I_i = \bar{I}_i \tag{11}$$

Putting (9), (10) and (11) in (8), we get

$$\bar{C}_{ijk} = \frac{1}{n+1}\{\bar{I}_i \bar{h}_{jk} + \bar{I}_j \bar{h}_{ki} + \bar{I}_k \bar{h}_{ij}\}. \tag{12}$$

This means that $\bar{F}$ is a Randers metric.

Proof of Theorem 2. Relation between Landsberg tensors and mean Landsberg tensors are given by

$$\bar{L}_{ijk} = e^{2\alpha}\{L_{ijk} + \alpha_0 C_{ijk}\}, \tag{13}$$

$$\bar{J}_k = J_k + \alpha_0 I_k \,. \tag{14}$$

Since *F* is a P-reducible metric, we have

$$L_{ijk} = \frac{1}{n+1}\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\}. \tag{15}$$

Plugging (13) and (14) in (15) yields

$$e^{-2\alpha}\bar{L}_{ijk} - \alpha_0 C_{ijk} = \frac{e^{-2\alpha}}{n+1}\{\bar{h}_{ij}(\bar{J}_k - \alpha_0 I_k) + \bar{h}_{ik}(\bar{J}_j - \alpha_0 I_j) + \bar{h}_{jk}(\bar{J}_i - \alpha_0 I_i)\}. \tag{16}$$

By (9) and (16), we get:

$$e^{-2\alpha}\{\bar{L}_{ijk} - \frac{1}{n+1}(\bar{J}_i \bar{h}_{jk} + \bar{J}_j \bar{h}_{ki} + \bar{J}_k \bar{h}_{ij})\} = \alpha_0\{C_{ijk} - \frac{1}{n+1}(I_i h_{jk} + I_j h_{ki} + I_k h_{ij})\}. \tag{17}$$

It is obvious that F is C-reducible if and only if $\bar{F}$ is P-reducible. Since F is C-reducible, then it is P-reducible. Therefore by (4.10), $\bar{F}$ is P-reducible.

In continue, we consider C-conformal change of P-reducible Finsler metric F in the case that $\bar{F}$ is a general relatively isotropic Landsberg metric. We get the following.

Corollary 3. Let (M, F) be a P-reducible Finsler manifold. Suppose that $\bar{F}$ is obtained from F by a C-conformal change and satisfies $\bar{\mathrm{L}} = \bar{\lambda}\,\bar{\mathrm{C}}$. Then F is Randers metric if and only if $\bar{F}$ is Randers metric.

Proof: We have:

$$\bar{L}_{ijk} = \bar{\lambda}\,\bar{C}_{ijk} \,. \tag{18}$$

By (17) and (18) we get:

$$\bar{\lambda}e^{-2\alpha}\{\bar{C}_{ijk} - \frac{1}{n+1}(\bar{I}_i \bar{h}_{jk} + \bar{I}_j \bar{h}_{ki} + \bar{I}_k \bar{h}_{ij})\} = \alpha_0\{C_{ijk} - \frac{1}{n+1}(I_i h_{jk} + I_j h_{ki} + I_k h_{ij})\}. \tag{19}$$

Then F is C-reducible if and only if $\bar{F}$ is C-reducible. By Lemma 1, we get the proof.

5. PROOF OF THEOREM 3

We say that a Finsler metric F is generalized Landsberg metric if the h-curvature of the Berwald and Chern connections coincide. The relation between h-curvatures of Berwald and Chern connections is given by

$$\mathrm{R}^i{}_{jkl} = R^i{}_{jkl} + [L^i{}_{jl|k} - L^i{}_{jk|l} + L^i_{sk} L^s_{jl} - L^i_{sl} L^s_{jk}] \tag{20}$$

where **R** and R denote the Riemannian curvatures of Berwald and Chern connections respectively. By definition of generalized Landsberg metric we have

$$L^i{}_{jl|k} - L^i{}_{jk|l} + L^i_{sk} L^s_{jl} - L^i_{sl} L^s_{jk} = 0. \tag{21}$$

For some geometric meanings of generalized Landsberg metrics see [9].

Proof of Theorem 3. Contracting (21) by g_{it}, we conclude that **R**=R if and only if

$$L^i_{sk} L^s_{jl} - L^i_{sl} L^s_{jk} = 0, \tag{22}$$

$$L^i{}_{jl|k} - L^i{}_{jk|l} = 0. \tag{23}$$

On the other hand, we have

$$h_{ij} = h_{ir}\, h_{js}\, g^{rs}, \tag{24}$$

$$J_i = g^{rs} h_{ir}\, J_s. \tag{25}$$

F is a P-reducible metric

$$L_{ijk} = \frac{1}{n+1}\{J_i\, h_{jk} + J_j\, h_{ki} + J_k\, h_{ij}\}. \tag{26}$$

By taking (5.5), (5.6) and (5.7) in (5.3), it follows that

$$\{h_{ij}\, h_{lk} - h_{ik}\, h_{lj}\}\, J^s J_s - \{h_{lk} J_j - h_{lj} J_k\}\, J_i + \{h_{ij}\, J_k - h_{lj} J_k\}\, J_r = 0. \tag{27}$$

We have

$$g^{ij}\, h_{ij} = n-1. \tag{28}$$

Contracting (5.8) with $g^{ij}\, g^{tk}$ and using (5.5), (5.6) and (5.9), one can obtains the following

$$(n+1)(n-2)\, J^s\, J_s = 0. \tag{29}$$

By assumption, F is a positive definite metric and $n > 2$, we then conclude that:

$$J_s = 0. \tag{30}$$

By (5.7) and (5.11), we conclude that F is a Landsberg metric.

Here we investigate the Randers manifolds with vanishing generalized Landsberg curvature. We prove that Randers manifolds with vanishing generalized Landsberg curvature are Berwaldian manifolds.

Corollary 4. Let $F = \alpha + \beta$ be a Randers metric on a manifold M. Then F is a generalized Landsberg metric if and only if F is a Berwald metric.

Proof: By the Lemma 1, *F* is C-reducible and then is a P-reducible metric. Since *F* is a generalized Landsberg metric, then by Theorem 3, *F* is a Landsberg metric. In a 1974 paper [1], Matsumoto showed that $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel. In a 1977 paper [15], Hashiguchi-Ichijyo showed that for a Randers metric $F = \alpha + \beta$, if β is parallel, then *F* is a Berwald metric. This completes the proof.

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