ALMOST CONVERGENCE THROUGH THE GENERALIZED DE LA VALLÉE-POUSSIN MEAN*

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Abstract – Lorentz characterized the almost convergence through the concept of uniform convergence of de la Vallée-Poussin mean. In this paper, we generalize the notion of almost convergence by using the concept of invariant mean and the generalized de la Vallée-Poussin mean. We determine the bounded linear operators for the generalized $\sigma$-conservative, $\sigma$-regular and $\sigma$-coercive matrices.

Keywords – Sequence spaces; invariant mean; matrix transformation; bounded linear operators

1. INTRODUCTION AND PRELIMINARIES

We shall write $w$ for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let $\emptyset, l, c$ and $c_0$ denote the sets of all finite, bounded, convergent and null sequences respectively; and $cs$ be the set of all convergent series. We write $l^p := \{x \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty \}$ for $1 \leq p < \infty$. By $e$ and $e_n(n \in N)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, and $e_n(n) = 1$ and $e_k(n) = 0$ ($k \neq n$). For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x[n] = \sum_{k=0}^{n} x_k e(k)$ be its $n$-section.

Note that $l, c$ and $c_0$ are Banach spaces with the sup-norm $\|x\|_\infty = \sup_k |x_k|$, and $l^p$ ($1 \leq p < \infty$) are Banach spaces with the norm $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$; while $\emptyset$ is not a Banach space with respect to any norm.

A sequence $(b(n))_{n=0}^{\infty}$ in a linear metric space $X$ is called Schauder basis if for every $x \in X$, there is a unique sequence $(\beta(n))_{n=0}^{\infty}$ of scalars such that $x = \sum_{k=0}^{\infty} \beta_k b(n)$.

A sequence space $X$ with a linear topology is called a K-space if each of the maps $p_i(x) = x_i$ is continuous for all $i \in N$. A K-space is called an FK-space if $X$ is complete linear metric space; a BK-space is a normed FK-space. An FK-space $X \ni \emptyset$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e(k)$, that is, $x = \lim_{n \to \infty} x[n]$.

Let $X$ and $Y$ be two sequence spaces and $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of real or complex numbers. We write $A_n(x) = \sum_{k=0}^{n} a_{nk} x_k$, provided that the series on the right converges for each $n$. If $x = (x_k) \in X$ implies that $Ax \in Y$, then we say that $A$ defines a matrix transformation from $X$ into $Y$, and by $(X, Y)$ we denote the class of such matrices [1].

The following is a very important result:

Lemma 1.1. ([2], Theorem 1.23). Let $X$ and $Y$ be BK spaces and $B(X, Y)$ denote the set of all bounded linear operators from $X$ into $Y$.

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(a) Then \((X, Y) \subset B(X, Y)\), that is every \(A \in (X, Y)\) defines an operator \(L_A \in B(X, Y)\) by \(L_A(x) = Ax\) for all \(x \in X\).

(b) If \(X\) has \(AK\), then \(B(X, Y) \subset (X, Y)\).

(c) We have \(A \in (X, l_w)\) if and only if
\[
\|A\|_{(X, l_w)} = \sup_n \|A_n\|_X < \infty;
\]
and further studied recently in [7]. In [8-12] we study various classes of four dimensional matrices, e.g. almost convergent sequences [5]. The idea of [5] characterized the almost convergence (and hence the 

Remark 2.1. It is easy to see that \(V^A_\sigma\) is a BK space with \(\|x\| = \sup m, n |tmn(x)|\).
Remark 2.2. Note that a convergent sequence is \((\sigma, \lambda)\)-convergent but converse need not hold, e.g. let \(\lambda_m = m, \sigma(n) = n + 1\) and the sequence \(x = (x_k)\) be defined by

\[
x_k = \begin{cases} 
1 & \text{if } k \text{ is odd}, \\
-1 & \text{if } k \text{ is even},
\end{cases}
\]

then \(x = (x_k)\) is \((\sigma, \lambda)\)-convergent to 0 but not convergent.

3. \((\sigma, \lambda)\)-CONSERVATIVE MATRICES

Definition 3.1. An infinite matrix \(A = (a_{nk})\) is said to be \((\sigma, \lambda)\)-conservative if and only if \(Ax \in V^\lambda_\sigma\) for all \(x = (x_k) \in c\). We denote this by \(A \in (c, V^\lambda_\sigma)\).

Definition 3.2. We say that infinite matrix \(A = (a_{nk})\) is said to be \((\sigma, \lambda)\)-regular if and only if \(A = (a_{nk})\) is \((\sigma, \lambda)\)-conservative and \((\sigma, \lambda)\)-\(\lim Ax = \lim x\) for all \(x = (x_k) \in c\). We denote this by \(A \in (c, V^\lambda_\sigma)\) \(\text{reg}\).

Remark 3.1. If we take \(\lambda_n = n\), then \(V^\lambda_\sigma\) is reduced to the space and \((\sigma, \lambda)\)-conservative and \((\sigma, \lambda)\)\-regular matrices are respectively reduced to \(\sigma\)-conservative and \(\sigma\)-regular matrices (cf [4]); and in addition, if \(\sigma(n) = n + 1\) then the space \(V^\lambda_\sigma\) is reduced to the space \(f\) of almost convergent sequences (cf [5]) and these matrices are reduced to the almost conservative and almost regular matrices (cf [14]) respectively.

In the following theorem we characterize \((\sigma, \lambda)\)-conservative and \((\sigma, \lambda)\)-regular matrices and find the associated bounded linear operators.

Theorem 3.1. (a) A matrix \(A = (a_{nk})\) is \((\sigma, \lambda)\)-conservative, i.e. \(A \in (c, V^\lambda_\sigma)\) if and only if it satisfies the following conditions

\[
(i) \quad \|A\|_{(l_\infty, l_\infty)} = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty; \\
(ii) \quad a(k) = \left(a_{nk}\right)_{n=1}^{\infty} \in V^\lambda_\sigma, \text{ for each } k; \\
(iii) \quad a = \left(\sum_{k=1}^{\infty} a_{nk}\right)_{n=1}^{\infty} \in V^\lambda_\sigma.
\]

In this case, the \((\sigma, \lambda)\)-limit of \(Ax\) is

\[
\lim x \left[u - \sum_k u_k\right] + \sum_k x_k u_k,
\]

Where \(u = (\sigma, \lambda)\)-\(\lim a\) and \(u_k = (\sigma, \lambda)\)-\(\lim a_k, k = 1, 2, \ldots\)

(b) \(A \in (c, V^\lambda_\sigma)\) defines an operator \(L_A \in \mathcal{B}(c, V^\lambda_\sigma)\) by \(L_A(x) = Ax\) for all \(x \in c\), and \(\|L_A\| = \|A\|_{(l_\infty, l_\infty)}\).

Proof: (a) It is quite similar to that of Theorem 1 of Schaefer [4] once we take

\[
t_{m,n}(x) = \frac{1}{\lambda_m} \sum_{k=1}^{\infty} \sum_{j \in l_m} a_{\sigma j(n)} x_k.
\]

(b) It follows directly from Lemma 1.1(a). Since \(V^\lambda_\sigma\) is a BK space and \((c, V^\lambda_\sigma) \subset (c, l_\infty) \subset (l_\infty, l_\infty)\), we get by Lemma 1.1(c) \(\|L_A\| = \|A\|_{(l_\infty, l_\infty)}\).
This completes the proof of the theorem.
Now, we deduce the following.

**Corollary 3.2.** $A = (a_{nk})$ is $(\sigma, \lambda)$-regular if and only if the conditions (i), (ii) with $(\sigma, \lambda)$-limit zero for each $k$, and (iii) with $(\sigma, \lambda)$-limit 1 of Theorem 3.1 hold. In this case, the $(\sigma, \lambda)$-limit of $Ax$ is $\sum_k x_k u_k$.

4. $(\sigma, \lambda)$-COERCIVE MATRICES

**Definition 4.1.** A matrix $A = (a_{nk})$ is said to be $(\sigma, \lambda)$-coercive if and only if $Ax \in V_\sigma^\lambda$ for all $x = (x_k) \in l_\infty$, and this is denoted by $A \in (l_\infty, V_\sigma^\lambda)$.

**Remark 4.1.** If we then take $\lambda_n = n$, $(\sigma, \lambda)$-coercive matrices are reduced to $\sigma$-coercive matrices (cf [4]); and in addition, if $\sigma(n) = n + 1$ then these matrices are reduced to the almost coercive matrices (cf [15]).

We prove the following lemma which will be used in our next theorem.

**Lemma 4.1.** Let $B(n) = (b_{mk}(n))$, $n = 0, 1, 2, \ldots$ be a sequence of infinite matrices such that
(i) $\|B(n)\| < H < +\infty$ for all $n$; and
(ii)$\lim_{n} b_{mk}(n) = 0$ for each $k$, uniformly in $n$.

Then
$$\lim_n \sum_k b_{mk}(n)x_k = 0 \text{ uniformly in } n \text{ for each } x \in l_\infty$$

If and only if
$$\lim_n \sum_k |b_{mk}(n)| = 0 \text{ uniformly in } n . \tag{2}$$

**Proof:** Let (2) hold and $x \in l_\infty$. Then, since
$$\left| \sum_k b_{mk}(n)x_k \right| \leq \|x\|_\infty \sum_k |b_{mk}(n)|,$$
condition (1) holds clearly.

Conversely suppose that (1) holds but (2) does not hold. Let
$$\lim_n \sum_k |b_{mk}(n)| = \lambda > 0 \text{ for all } n .$$

For fixed $n$, let us write $b(m, k)$ in place of $b_{mk}(n)$. Let for a given $\epsilon > 0$,
$$N(\epsilon) = \left\{ m \in \mathbb{N} : \sum_k |b(m, k)| > \lambda - \epsilon \right\} .$$

Then by (i) and (ii) there exist increasing sequences of integers $m_r \in N(1/r)$ and $k_r$ such that
$$\begin{cases}
\sum_{k \leq k_r-1} b(m_r, k) < \frac{1}{r}, \\
\sum_{k > k_r} |b(m_r, k)| < \frac{1}{r}.
\end{cases} \tag{3}$$

Now define $x \in l_\infty$ such that $k_{r-1} < k < k_r$,
$$x_k = \begin{cases}
1 & \text{if } b(m_r, k) \geq 0, \\
-1 & \text{if } b(m_r, k) < 0.
\end{cases}$$

Then for all $m_r \in N \left( \frac{1}{r} \right)$. 

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\[
\sum_k b(m_r, k)x_k = \sum_{k \leq k_{r-1}} b(m_r, k)x_k + \sum_{k_{r-1} < k \leq k_r} b(m_r, k)x_k + \sum_{k > k_r} b(m_r, k)x_k
\]

\[
\geq \sum_{k_{r-1} < k \leq k_r} b(m_r, k)x_k - \|x\|_\infty \sum_{k \leq k_{r-1}} |b(m_r, k)| - \|x\|_\infty \sum_{k > k_r} |b(m_r, k)|
\]

\[
\geq \sum_{k_{r-1} < k \leq k_r} b(m_r, k)x_k - \frac{2}{r}
\]

\[
= \sum_{k_{r-1} < k \leq k_r} |b(m_r, k)| - \frac{2}{r}
\]

\[
= \sum_k |b(m_r, k)| + \sum_{k \leq k_{r-1}} |b(m_r, k)| + \sum_{k > k_r} |b(m_r, k)| - \frac{2}{r}
\]

\[
\geq \sum_k |b(m_r, k)| - \frac{4}{r}
\]

Therefore,

\[
\lim_r \sum_k b(m_r, k)x_k \geq \lim_r \sum_k |b(m_r, k)|
\]

and (1) implies that

\[
\lim_m \sum_k b_{mk}(n) = 0 \quad \text{uniformly in } n.
\]

This completes the proof of the lemma.

Now, we characterize \((\sigma, \lambda)\)-coercive matrices and obtain the bounded linear operator for these matrices.

**Theorem 4.1.** (a) A matrix \(A = a_{nk}\) is \((\sigma, \lambda)\)-coercive, i.e. \(A \in (l_\infty, V^{\lambda}_\sigma)\) if and only if (i) and (ii) of Theorem 3.1 hold, and

\[
(iii) \quad \lim_m \left| \sum_{k \geq 1} \sum_{j \in I_m} a_{\sigma(j), k} - u_k \right| \quad \text{uniformly in } n.
\]

In this case, the \((\sigma, \lambda)\)-limit of \(Ax\) is

\[
\sum_k x_k u_k \quad \forall x \in l_\infty,
\]

Where \(u_k = (\sigma, \lambda)-\lim a_{nk}\).

(b) \(A \in (l_\infty, V^{\lambda}_\sigma)\) defines an operator \(L_A \in B(l_\infty, V^{\lambda}_\sigma)\) by \(L_A(x) = Ax\) for all \(x \in l_\infty\), and \(\|L_A\| = \|A\|_{(l_\infty, l_\infty)}\).

**Proof:** (a) **Sufficiency.** Same as in Theorem 3 of [4].

**Necessity.** Let \(A\) be \((\sigma, \lambda)\)-coercive matrix. This implies that \(A\) is \((\sigma, \lambda)\)-conservative, then we have condition (i) and (ii) from Theorem 3.1. Now we have to show that (iii) holds.

Suppose that for some \(n\), we have
\[
\limsup_n \sum_{k=1}^{\infty} \left| \sum_{j \in I_m} [a_{\sigma(j,n),k} - u_k] \right| / \lambda_m = N > 0.
\]

Since \( \|A\| \) is finite, \( N \) is also finite. We observe that since \( \sum_{k=1}^{\infty} |u_k| < \infty \) and \( A \) is \((\sigma, \lambda)\)-coercive, the matrix \( B = (b_{nk}) \), where \( b_{nk} = a_{nk} - u_k \), is also a \((\sigma, \lambda)\)-coercive matrix. By an argument similar to that of Theorem 2.1 in [15], one can find \( x \in l_\infty \) for which \( Bx \notin V_\sigma^A \). This contradiction implies the necessity of (iii).

Now, we use Lemma 4.1 to show that this convergence is uniform in \( n \). Let
\[
h_{mk}(n) = \sum_{j \in I_m} [a_{\sigma(j,n),k} - u_k] / \lambda_m
\]
and let \( H(n) \) be the matrix \((h_{mk}(n))\). It is easy to see that \( \|H(n)\| \leq 2\|A\|_{(l_\infty, l_\infty)} \) for every \( n \); and from condition (ii)
\[
\lim_m h_{mk}(n) = 0 \text{ for each } k, \text{ uniformly in } n.
\]
For any \( x \in l_\infty \)
\[
\lim_m \sum_{j \in I_m} h_{mk}(n)x_k = (\sigma, \lambda)\)-Lim \( Ax - \sum_{k=1}^{\infty} u_k x_k
\]
and the limit exists uniformly in \( n \), since \( Ax \in V_\sigma^A \). Moreover, this limit is zero since
\[
\lim_m \sum_{k=1}^{\infty} \left| h_{mk}(n)x_k \right| \leq \|x\|_\infty \sum_{k=1}^{\infty} \left| \sum_{j \in I_m} [a_{\sigma(j,n),k} - u_k] \right| / \lambda_m.
\]
Hence
\[
\lim_m \sum_{k=1}^{\infty} \left| h_{mk}(n) \right| = 0 \text{ uniformly in } n;
\]
i.e. the condition (iii) holds.

(b) Observe that \( (l_\infty, V_\sigma^A) \subset (l_\infty, l_\infty) \) and the proof is the same as that of Theorem 3.1 (b). This completes the proof of the theorem

5. \((l_1, V_\sigma^A)\)-MATRICES

We prove the following theorem:

**Theorem 5.1.** (a) We have \((l_1, V_\sigma^A) = B(l_1, V_\sigma^A) \) and \( A \in (l_1, V_\sigma^A) \) if and only if
\[
(i) \, \|A\| = \sup_{m,n,k} \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma(j,n),k} \right| < \infty, \text{ and}
\]
the condition (ii) of Theorem 3.1 hold.
(b) If \( A \in (l_1, V_\sigma^A) \) then \( \|L_A\| = \|A\| \).

**Proof:** Since \( l_1 \) has \( AK \), Lemma 1.1 (b) yields the first part.

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Sufficiency. Let the conditions hold. For \( x = (x_k) \in l_1 \), we see that

\[
\lim_{m \to \infty} \frac{1}{\lambda} \sum_{k=1}^{\infty} \sum_{j \in I_m} a_{\sigma_j(n),k} x_k = \sum_{k=1}^{\infty} a_k x_k , \text{ uniformly in } n
\]

it also converges absolutely. Furthermore, \( \frac{1}{\lambda} \sum_{k=1}^{\infty} \sum_{j \in I_m} a_{\sigma_j(n),k} x_k \) converges absolutely for each \( m, n \). Given \( \varepsilon > 0 \), there exists \( k_0 = k_0(\varepsilon) \) such that

\[
\sum_{k > k_0} |x_k| < \varepsilon.
\]

By (ii), we can find \( m_0 \in \mathbb{N} \) such that

\[
\left| \sum_{k > k_0} \left[ \frac{1}{\lambda} \sum_{j \in I_m} a_{\sigma_j(n),k} x_k - a_k \right] x_k \right| < \infty,
\]

for all \( m > m_0 \), uniformly in \( n \). Now

\[
\left| \sum_{k=1}^{\infty} \left[ \frac{1}{\lambda} \sum_{j \in I_m} a_{\sigma_j(n),k} x_k - a_k \right] x_k \right| \leq \left| \sum_{k \geq k_0} \left[ \frac{1}{\lambda} \sum_{j \in I_m} a_{\sigma_j(n),k} x_k - a_k \right] x_k \right| + \left| \sum_{k > k_0} \left[ \frac{1}{\lambda} \sum_{j \in I_m} a_{\sigma_j(n),k} x_k - a_k \right] x_k \right|,
\]

for all \( m > m_0 \), uniformly in \( n \), by (5), (6) and (i). Hence (4) holds.

Necessity. Let us define a continuous linear functional \( Q_{mn} \) on \( l_1 \) by

\[
Q_{mn}(x) = \frac{1}{\lambda_m} \sum_{k} \sum_{j \in I_m} a_{\sigma_j(n),k} x_k.
\]

Now

\[
|Q_{mn}(x)| \leq \sup_k \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma_j(n),k} \right| \|x_k\|_1
\]

and hence

\[
\|Q_{mn}\| \leq \sup_k \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma_j(n),k} \right|.
\]

For any fixed \( k \in \mathbb{N} \), define \( x = (x_i) \) by

\[
x_i = \begin{cases} 
s \left( \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma_j(n),k} \right) ; & \text{for } i = k \\
0 ; & \text{for } i \neq k.
\end{cases}
\]

Then \( \|x\|_1 = 1 \), and

\[
|Q_{mn}(x)| = \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma_j(n),k} x_k \right| = \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma_j(n),k} \right| \|x\|_1,
\]

So that

\[
\|Q_{mn}\| \geq \sup_k \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma_j(n),k} \right|.
\]

Now, by (7) and (8)
Since $A \in (l_1, V^2_\sigma)$, we have

$$\sup_{m,n}|Q_{mn}(x)| = \sup_{m,n} \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} x_k \right| < \infty.$$ 

Therefore, by the uniform boundedness principle, we have

$$\sup_{m,n}\|Q_{mn}(x)\| = \sup_{m,n,k} \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} \right| < \infty.$$ 

(b) If $A \in (l_1, V^2_\sigma)$ then

$$\|L_A(x)\| = \sup_{m,n}|t_{mn}(Ax)| \leq \|A\|\|x\|_1,$$

which implies that $\|L_A(x)\| \leq \|A\|$. Also, $L_A \in B(l_1, V^2_\sigma)$ implies that

$$\|L_A(x)\| = \|L_A\|\|x\|_1,$$

and it follows from $\|e^{(k)}\|_1 = 1$ for all $k$ that $\|L_A\| \leq \|L_A\|$. Hence $\|L_A\| = \|A\|$. 

This completes the proof of the theorem.

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