INEQUALITIES FOR MEROMORPHICALLY P-VALENT FUNCTIONS*

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Abstract – The aim of this paper is to prove some inequalities for p-valent meromorphic functions in the punctured unit disk \( \Delta^* \) and find important corollaries.

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1. INTRODUCTION

Let \( \Sigma_p \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k
\]

which are analytic meromorphic multivalent in the punctured unit disk

\( \Delta^* = \{ z : 0 < |z| < 1 \} \).

We say that \( f(z) \) is p-valently starlike of order \( \gamma (0 \leq \gamma < p) \) if and only if for \( z \in \Delta^* \)

\[
- \Re \left\{ \frac{zf''(z)}{f(z)} \right\} > \gamma,
\]

Also, \( f(z) \) is p-valently convex of order \( \gamma (0 \leq \gamma < p) \) if and only if

\[
- \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, \quad (z \in \Delta^*).
\]

Definition 1.1: A function \( f(z) \in \Sigma_p \) is said to be in the subclass \( X^*_p(j) \) if it satisfies the inequality

\[
\left| \frac{(p-1)!}{(-1)^j(p+j-1)!} \frac{f^{(j)}(z)}{z^{-p-j}} - 1 \right| < 1
\]

where

\[
f^{(j)}(z) = (-1)^j \frac{(p+j-1)!}{(p-1)!z^{p+j}} + \sum_{k=p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j}
\]

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is the $j$-th differential of $f(z)$ and a function $f(z) \in \Sigma_p$ is said to be in the subclass $Y_p^*(j)$ if it satisfies the inequality

$$\left| -\frac{zf^{(j)}(z)'}{f^{(j)}(z)} - (p + j) \right| < p + j. \quad (6)$$

To establish our main results we need the following lemma due to Jack [1].

**Lemma 1.2.** Let $w(z)$ be analytic in $\Delta = \{z : |z| < 1\}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0$, then

$$z_0w'(z_0) = cw(z_0)$$

where $c$ is a real number and $c \geq 1$.

Some different inequalities on $p$-valent holomorphic and $p$-valent meromorphic functions by using operators were studied in [2-5].

## 2. MAIN RESULTS

In the first theorem we give a sufficient condition for $f \in \Sigma_p$ to be in the class $X_p^*(j)$.

**Theorem 2.1.** If $f(z) \in \Sigma_p$ satisfies the inequality

$$\text{Re}\left\{ z\frac{f^{(j)}(z)' + p + j}{f^{(j)}(z)} \right\} > 1 - \frac{1}{2p} \quad (7)$$

then $f(z) \in X_p^*(j)$.

**Proof:** Let $f(z) \in \Sigma_p$, we define the function $w(z)$ by

$$\frac{(p-1)!}{(-1)^j(p+j-1)!} \frac{f^{(j)}(z)}{z^{-p-j}} = 1 - w(z), \quad (z \in \Delta^*). \quad (8)$$

It is easy to verify that $w(0) = 0$.

From (8) we obtain

$$f^{(j)}(z) = \frac{(p+j-1)!}{(p-1)!} z^{-p-j} - \frac{(p+j-1)!}{(p-1)!} z^{-p-j} w(z)$$

or

$$[f^{(j)}(z)]' = (-1)^{j+1} (p+j) z^{-p-j-1} \frac{(p+j-1)!}{(p-1)!} + (-1)^j (p+j) z^{-p-j-1} \frac{(p+j-1)!}{(p-1)!} z^{-p-j} w'(z).$$

After a simple calculation we obtain

$$\frac{zw'(z)}{1-w(z)} = \frac{z[f^{(j)}(z)']}{f^{(j)}(z)} + (p+j). \quad (9)$$
Inequalities for meromorphically \( p \)-valent function

Now, suppose that there exists a point \( z_0 \in \Delta^* \) such that

\[
\max_{|z|=\delta}|w(z)| = |w(z_0)| = 1.
\]

Then by letting \( w(z_0) = e^{i\theta} \) (\( w(z_0) \neq 1 \)) and using the Jack’s lemma in the equation (9), we have

\[
-Re\left\{\frac{zf^{(j)}(z)}{f^{(j)}(z)}\right\}' + p + j = Re\left\{\frac{z_0w'(z_0)}{1-w(z_0)}\right\} = Re\left\{\frac{cw(z_0)}{1-w(z_0)}\right\}
\]

\[
= c Re\left\{\frac{e^{i\theta}}{1-e^{i\theta}}\right\} = -\frac{c}{2} < -\frac{1}{2},
\]

which contradicts the hypothesis (7). Hence, we conclude that for all \( z \), \( |w(z)| < 1 \) and from (8) we have

\[
\left|\frac{(p-1)!f^{(j)}(z)}{(-1)^j(p + j - 1)!z^{-p-j}-1}\right| = |w(z)| < 1
\]

and this gives the result.

**Theorem 2.2.** If \( f(z) \in \Sigma_p \) satisfies the inequality

\[
Re\left\{\frac{zf^{(j)}(z)}{f^{(j)}(z)}\right\}' - \left(1 + \frac{zf^{(j)}(z)}{f^{(j)}(z)}\right)'' > \frac{2p + 1}{2(p + 1)},
\]

(10)

then \( f(z) \in Y_p^*(j) \).

**Proof:** Let \( f(z) \in \Sigma_p \). We consider the function \( w(z) \) as follows:

\[
-\frac{zf^{(j)}(z)}{f^{(j)}(z)} = (p + j)(1-w(z)).
\]

(11)

It is easy to see that \( w(0)=0 \). Furthermore, by differentiating both sides of (11) we get

\[
-\left[1 + \frac{zf^{(j)}(z)}{f^{(j)}(z)}\right]'' = (p + j)(1-w(z)) + \frac{zw'(z)}{1-w(z)}.
\]

Now suppose there exists a point \( z_0 \in \Delta^* \) such that \( \max_{|z|=\delta}|w(z)| = |w(z_0)| = 1 \). Then by letting \( w(z_0) = e^{i\theta} \) and using Jack’s lemma we have

\[
Re\left\{\frac{zf^{(j)}(z)}{f^{(j)}(z)}\right\}' - \left(1 + \frac{zf^{(j)}(z)}{f^{(j)}(z)}\right)'' = Re\left\{\frac{z_0w'(z_0)}{1-w(z_0)}\right\}
\]

\[
= c Re\left\{\frac{e^{i\theta}}{1-e^{i\theta}}\right\} = -\frac{c}{2} < -\frac{1}{2}
\]

which contradicts the condition (10). So we conclude that \( |w(z)| < 1 \) for all \( z \in \Delta^* \). Hence, from (11) we obtain

\[
-\frac{zf^{(j)}(z)}{f^{(j)}(z)} < (p + j)
\]

\[
\left|\frac{zf^{(j)}(z)}{f^{(j)}(z)}-(p+j)\right| < p + j.
\]
This completes the proof.

By taking $j = 0$ in Theorems 2.1 and 2.2, we obtain the following corollaries.

**Corollary 1.** If $f(z) \in \Sigma_p$ satisfies the inequality

$$- \text{Re} \left\{ \frac{zf'}{f} + p \right\} > 1 - \frac{1}{2p},$$

then

$$\left| \frac{f(z)}{z^{-p}} - 1 \right| < 1.$$

**Corollary 2.** If $f(z) \in \Sigma_p$ satisfies the inequality

$$- \text{Re} \left\{ \frac{zf'}{f} - \left(1 + \frac{zf''}{f'}\right) \right\} > \frac{2p + 1}{2(p + 1)},$$

then $\left| \frac{zf'}{f} - p \right| < p$ or equivalently $f(z)$ is meromorphically $p$-valent starlike with respect to the origin.

By taking $j = 1$ in theorems 2.1 and 2.2, we obtain the following corollaries.

**Corollary 3.** If $f(z) \in \Sigma_p$ satisfies the inequality

$$- \text{Re} \left\{ \frac{zf''}{f'} + p + 1 \right\} > 1 - \frac{1}{2p}.$$

Then $\left| \frac{f'(z)}{z^{-p-1}} - p \right| < p$ or equivalently $f(z)$ is meromorphically $p$-valent close-to-convex with respect to the origin.

**Corollary 4.** If $f(z) \in \Sigma_p$ satisfies the inequality

$$- \text{Re} \left\{ \frac{zf''}{f'} \left(1 + \frac{zf''}{f'}\right) \right\} > \frac{2p + 1}{2(p + 1)},$$

then

$$\left| - \frac{zf''}{f'} - (p + 1) \right| < p + 1$$

or equivalently $f(z)$ is meromorphically multivalent convex.

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