

ΛBV AS A NON SEPARABLE DUAL SPACE*

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Abstract – Let C be a field of subsets of a set I . Also, let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a non-decreasing positive sequence of real numbers such that $\lambda_1 = 1$, $1/\lambda_i \rightarrow 0$ and $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. In this paper we prove that ΛBV of all the games of Λ -bounded variation on C is a non-separable and norm dual Banach space of the space of simple games on C . We use this fact to establish the existence of a linear mapping T from ΛBV onto FA (finitely additive set functions) which is positive, efficient and satisfies a weak form of symmetry, namely invariance under a semigroup of automorphisms of (I, C) .

Keywords – Set functions, duality, compactness, non separable

1. INTRODUCTION

Let C be a field of subsets of a nonempty set I . It is well-known that the space FA of all the finitely additive games of bounded variation on C , equipped with the total variation norm, is isometrically isomorphic to the norm dual of the space of all simple functions on C , endowed with the sup norm ([1]) (also see [2]). Maccheroni and Ruckle in [3] established a parallel result for the space BV of all the games of bounded variation on C . Indeed, they showed that BV , equipped with the total variation norm, is isometrically isometric to the norm dual of the space of all simple games endowed with a suitable norm where a simple game is a game which is non zero only on a finite number of elements of C . Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a non-decreasing positive sequence of real numbers such that $\lambda_1 = 1$, $1/\lambda_i \rightarrow 0$ and $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. We introduce space ΛBV which shares many properties of space BV . Here, we prove that space ΛBV of all the games of Λ bounded variation on C equipped with the total variation norm, is isometrically isometric to the norm dual of the space of all simple games, endowed with a suitable norm. We use this fact to establish the existence of a linear mapping T from ΛBV onto FA (finitely additive set functions) which is positive, efficient and satisfies a weak form of symmetry, namely invariance under a semigroup of automorphisms of (I, C) .

2. PRELIMINARIES

A set function $\nu : C \rightarrow R$ is a game if $\nu(\emptyset) = 0$. A game on C is monotone if $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. A chain $\{S_i\}_{i=0}^n$ in C is a finite strictly increasing sequence

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$$\phi = S_0 \subset S_1 \subset \dots \subset S_n = I$$

of the elements of C . ΛBV is the set of all games such that

$$\|u\| = \sup \left\{ \sum_{i=1}^n \frac{|u(S_i) - u(S_{i-1})|}{\lambda_i} : \{S_i\}_{i=0}^n \text{ is a chain in } C \right\} < \infty.$$

A game in ΛBV is said to be of Λ bounded variation. A game is called a simple game if it is non-zero only on a finite number of elements of C . A function u in ΛBV is called finitely additive if

$$u(A \cup B) = u(A) + u(B)$$

whenever A and B are in C and $A \cap B = \phi$.

The set FA of finitely additive functions in ΛBV forms a closed subspace of ΛBV . A function u in ΛBV is called increasing if $u(A) \leq u(B)$ whenever $A \subset B$. Each u in ΛBV has the form $u = u^+ + u^-$ when u^+ and u^- are increasing and $\|u\| = u^+(I) + u^-(I)$. A linear mapping T in $L(BV)$ is positive if Tu increases whenever u increases.

Let C' denote the group of automorphisms of (I, C) . A subspace X is called symmetric if $u \circ \pi$ is in X for each x in X and each π in C' . A value is a linear mapping T from a symmetric subspace X of ΛBV onto the space FA of finitely additive set functions which satisfies three conditions:

- (a) T is positive: i.e., Tu increases whenever u increases.
- (b) T is symmetric: i.e., $T(u \circ \pi) = (Tu) \circ \pi$ for each π in C' and u in X .
- (c) T is efficient: $(Tu)(I) = u(I)$ for each u in X .

In this note we establish the existence of linear operations from all of ΛBV onto FA which satisfy (a), (b) and a weaker form of (c), namely symmetry under a semigroup of C' . In addition, these linear operators are projections (i.e., $Tu = u$ for u in FA). Our main result is that, given any locally finite subgroup Φ of C' there is a projection T from ΛBV onto FA which is symmetric under Φ . Since ΛBV is a (proper) subspace of R^C , it inherits a topology from the product topology of R^C . This is the weak topology generated by the projection functional

$$P_A : \Lambda BV \rightarrow R$$

$$u \rightarrow u(A)$$

where $A \in C$. A net $\{u_\alpha\}$ converges to u in this topology if $u_\alpha(A) \rightarrow u(A)$ for all $A \in C$ (we write $u_\alpha \xrightarrow{C} u$). This topology is called Λ -vague topology for the analogy with the vague topology on the set of probability measures.

3. ΛBV AS A NON SEPARABLE DUAL SPACE

In [4], Aumann and Shapley proved that BV is a Banach space. Here, we show ΛBV is a Banach space too.

Let $\Omega = \{S_i\}_{i=0}^n$ be a chain. For any set function v we define

$$\|v\|_\Omega = \left\{ \sum_{i=1}^n \frac{|v(S_i) - v(S_{i-1})|}{\lambda_i} \right\} < \infty.$$

This shows that a necessary and sufficient condition, $\nu \in \Lambda BV$, is that $\|\nu\|_\Omega$ be bounded over all chain Ω . Then, $\nu \in \Lambda BV$ if and only if $\|\nu\| = \sup \|\nu\|_\Omega$, where the sup is taken over all chains Ω .

It is obvious that this defines a norm on ΛBV . Now, we show that with this norm, ΛBV is a complete space.

Theorem 3.1. ΛBV is complete, hence a Banach space.

Proof: Let $\{\nu_n\}$ be a Cauchy sequence of elements of ΛBV . For any subset S of I , we show that sequence $\{\nu_n(S)\}$ is a Cauchy sequence in R .

Let S be a subset of I . For the chain

$$\Phi \subset S \subset I;$$

We have

$$\begin{aligned} \|\nu_n - \nu_m\| &\geq \frac{|(\nu_n(S) - \nu_m(S)) - (\nu_n(\Phi) - \nu_m(\Phi))|}{\lambda_1} \\ &= |(\nu_n(S) - \nu_m(S))|. \end{aligned}$$

Then the sequence $\{\nu_n(S)\}$ is a Cauchy sequence in R and is convergent; denote it's limit by $\nu(S)$. We must first show that ν is Λ -bounded variation. Let N be such that $\|\nu_n - \nu_m\| \leq 1$ whenever $n \geq N$. Then for each chain Ω and each $n \geq N$ we have

$$\begin{aligned} \|\nu_n\|_\Omega - \|\nu_N\| &\leq \|\nu_n\|_\Omega - \|\nu_N\|_\Omega \\ &\leq \|\nu_n - \nu_N\|_\Omega \\ &\leq \|\nu_n - \nu_N\| \\ &\leq 1 \end{aligned}$$

letting $n \rightarrow \infty$, we deduce

$$\|\nu\|_\Omega \leq 1 + \|\nu_N\|.$$

Hence ν is Λ -bounded variation. That $\|\nu_n - \nu\| \rightarrow 0$ is now easily verified, so the theorem is proved.

Here, we show that ΛBV is a non separable space. So, the dual of ΛBV is non separable too.

Theorem 3.2. $\Lambda BV[a, b]$ is non separable.

Proof: For each a satisfying $a < s < b$ and subset A of $[a, b]$, let $\chi_s(A)$ be the set function defined by

$$\chi_s(A) = \begin{cases} 1 & \text{if } [a, s] \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

We see that χ_s is a monotone set function and belongs to the $\Lambda BV[a, b]$. For any s and r with $a < s < r < b$, let Ω be the chain $\emptyset \subseteq [a, s] \subseteq I$. Then

$$\begin{aligned} \|\chi_r - \chi_s\| &\geq \|\chi_r - \chi_s\|_{\Omega} \\ &\geq \frac{|(\chi_r - \chi_s)([a, s]) - (\chi_r - \chi_s)(\emptyset)|}{\lambda_1} \\ &\geq 1 \end{aligned}$$

This completes the proof.

4. ΛBV AS A DUAL SPACE

In [3], Maccheroni and Ruckle showed that BV is a dual Banach space. Indeed, they showed that BV is isometrically isomorphic to the norm dual of space of all simple games. Here, we establish this result for ΛBV .

We define the game $e_A : C \rightarrow R$ by

$$e_A(B) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{otherwise} \end{cases}$$

Let X be the space of all simple games. For all $A \in C - \{\emptyset\}$ and $e_{\emptyset} = 0$ being $x = \sum_{A \in C} x(A)e_A$ for all $x \in X$, we have $X = \langle e_A : A \in C \rangle$. For each chain $\Omega = \{S_i\}_{i=0}^n$ in C , define a semi norm on X by

$$\|x\|_{\Omega} = \max_{0 \leq k \leq n} \left| \sum_{i=k}^n x(S_i) \right|. \tag{1}$$

For all $x \in X$. Let $X_{\Omega} = \langle e_A : A \in \Omega \rangle$. If $x \in X_{\Omega}$, we say that x depends on the chain Ω . For all $x \in X$, set

$$\|x\| = \inf \sum_{e=1}^L \|x_e\|_{\Omega_e}$$

where the inf is taken over all finite decompositions $x = \sum_{e=1}^L x_e$ in which x_e depends on the chain Ω_e and $\|\cdot\|_{\Omega_e}$ is defined as in (1) for all $e = 1, 2, \dots, L$.

Lemma 4 of [3] showed that this equation defines a norm on X .

Lemma 4.1. The function $\|\cdot\| : X \rightarrow R$ is a norm on X .

Given a linear continuous functional $f : X \rightarrow R$, define the game G_f as follows

$$G_f(A) = f(e_A)$$

For all $A \in C$.

Theorem 4.2. Let X^* be the norm dual of $(X, \|\cdot\|)$. The operator

$$G : X^* \rightarrow \Lambda BV$$

$$f \mapsto G_f$$

is an isometric isomorphism from X^* onto ΛBV .

Proof: We first show that if $\Omega = \{S_i\}_{i=0}^n$ is a chain in C , then

$$\sum_{k=1}^n \frac{|G_f(S_k) - G_f(S_{k-1})|}{\lambda_k} \leq \|f\|,$$

which implies that $G_f \in \Lambda BV$ and $\|G_f\| \leq \|f\|$.

Define $x \in X_\Omega$ by

$$\begin{aligned} x(S_n) &= \text{Sgn}(f(eS_n) - f(eS_{n-1})), \\ x(S_n) + x(S_{n-1}) &= \text{Sgn}(f(eS_{n-1}) - f(eS_{n-2})), \\ &\vdots \\ x(S_n) + x(S_{n-1}) + \dots + x(S_1) &= \text{Sgn}(f(eS_1) - f(eS_0)), \\ x(S_0) &= 0. \end{aligned}$$

Obviously $\|x\|_\Omega \leq 1$, so that $\|x\| < 1$. Similar to proof of theorem 5 of [3], we have,

$$\begin{aligned} \|f\| \geq f(x) &= \sum_j^n |G_f(S_j) - G_f(S_{j-1})| \\ &\geq \sum_{j=1}^n \frac{|G_f(S_j) - G_f(S_{j-1})|}{\lambda_j} \end{aligned}$$

which implies that $\|f\| \geq \|G_f\|$. Then G is well defined and obviously linear and injective.

Given $u \in \Lambda BV$, we can define f_u on X by

$$f_u(x) = \sum_{A_j \in C} \frac{u(A_j)}{\lambda_j} x(A_j),$$

for all $x \in X$. It is trivial that f_u is linear.

If x depends on $\Omega = \{S_j\}_{j=0}^n$, then

$$\begin{aligned} f_u(x) &= \sum_{j=0}^n \frac{u(S_j)}{\lambda_j} x(S_j) \\ &= \frac{u(S_0)}{\lambda_0} \sum_{k=0}^n x(S_k) + \sum_{j=1}^n \left[\left(\frac{u(S_j) - u(S_{j-1})}{\lambda_j} \right) \sum_{k=j}^n x(S_k) \right] \\ &= \sum_{j=1}^n \left[\frac{u(S_j) - u(S_{j-1})}{\lambda_j} \sum_{j=1}^n x(S_k) \right] \end{aligned}$$

$$\begin{aligned} &\leq \sum_{j=1}^n \left[\frac{|u(S_j) - u(S_{j-1})|}{\lambda_j} \left| \sum_{k=j}^n x(S_k) \right| \right] \\ &\leq \|x\|_{\Omega} \|u\|. \end{aligned}$$

If $x = \sum_{e=1}^L x_e$ with $x_e \in X_{\Omega_e}$ for all $e = 1, 2, \dots, L$, then

$$\begin{aligned} f_u(x) &= \sum_{e=1}^L f_u(x_e) \\ &\leq \sum_{e=1}^L \|u\| \|x_e\|_{\Omega_e} \\ &\leq \|u\| \sum_{e=1}^L \|x_e\|_{\Omega_e}, \end{aligned}$$

and so

$$\begin{aligned} f_u(x) &\leq \inf \left\{ \|u\| \sum_{e=1}^L \|x_e\|_{\Omega_e} : x = \sum_{e=1}^L x_e, x_e \in X_{\Omega_e} \right\} \\ &= \|u\| \|x\|. \end{aligned}$$

We conclude that $f_u \in X^*$, $G(f_u) = u$ and G is onto. For all $u \in \Lambda BV$, $f_u = G_u^{-1}$ and $\|G_u^{-1}\| = \|f_u\| \leq \|u\|$.

Therefore, for all $f \in X^*$, $\|f\| = \|G_{(G_f)}^{-1}\| \leq \|G_f\|$ and G is an isometry.

Let G be similar to the previous theorem. We show that,

Theorem 4.3. G is weak* Λ -vague homeomorphism.

Proof: Let $\{f^a\}$ be a net in X^* . By using the notations of the previous theorem, we have that $f^a \xrightarrow{w^*} f$ iff $f^a(x) \rightarrow f(x)$ for all $x \in X$ iff $f^a(e_A) \rightarrow f(e_A)$ for all $A \in \mathcal{C}$ iff $G_{f^a}(A) \rightarrow G_f(A)$ for all $A \in \mathcal{C}$ iff $G_{f^a} \xrightarrow{\mathcal{C}} G_f$.

In Theorem 4.2, together with the Alaoghlu theorem, we have the compactness of the unit ball $U(BV)$ in the Λ -vague topology.

Theorem 4.4. The unit ball $U(BV)$ is compact with respect to the Λ -vague topology.

5. PROJECTIONS FROM ΛBV ONTO FA

Given I and \mathcal{C} as in §1, let Θ denote the set of one-to-one functions Θ from I into I such that $\pi(S) \in \mathcal{C}$ if and only if $S \in \mathcal{C}$. Then Θ forms a group under composition. For each π in Θ the function T_π defined by $T_\pi u = u \circ \pi$ is a linear operator from ΛBV into ΛBV with $\|T_\pi\| = 1$. A function u in ΛBV is called finitely additive if

$$u(A \cup B) = u(A) + u(B)$$

whenever A and B are in \mathcal{C} and $A \cap B = \emptyset$. The set FA of finitely additive functions in ΛBV forms a closed subspace of ΛBV . A function u in ΛBV is called increasing if $u(A) \leq u(B)$ whenever $A \subset B$. Each u in ΛBV has the form $u = u^+ - u^-$ when u^+ and u^- are increasing and $\|u\| = u^+(I) - u^-(I)$. A linear mapping T in $L(\Lambda BV)$ is positive if Tu is increasing whenever u is increasing.

Definition 5.1. Let Φ be a subgroup of Θ . A Φ -value is a projection P from ΛBV onto FA which fulfills the following conditions:

$$\|Pu\| \leq \|u\|, \quad u \in \Lambda BV. \tag{2}$$

$$Pu(I) = u(I), \quad u \text{ in } \Lambda BV. \tag{3}$$

$$PT_\pi = T_\pi P \text{ for all } \pi \text{ in } \Phi. \tag{4}$$

Definition 5.2. For each finite partition D of I into members of \mathcal{C} , $\Gamma_\Lambda - set[\Gamma_\Lambda(D)]$ is the set of all T in $L(\Lambda BV)$ for which

$$Tu(I) = u(I) \text{ for } u \text{ in } \Lambda BV; \tag{5}$$

$$\|Tu\| \leq \|u\|, \quad u \in \Lambda BV; \tag{6}$$

$$Tu \text{ is additive on the algebra of sets determined by } D; \tag{7}$$

$$Tu(B) = u(B) \text{ for } u \text{ in } FA, B \text{ in } D. \tag{8}$$

Lemma 5.3. No set $\Gamma_\Lambda(D)$ is empty.

Proof: Suppose $D = \{D_1, D_2, \dots, D_k\}$ (any order). Let $E_0 = \emptyset, E_1 = D_1, \dots, E_n = D_1 \cup D_2 \cup \dots \cup D_n, \dots, E_k = I$. For each D_j in \mathcal{C} let d_{D_j} be the function

$$d_{D_j}(A) = \begin{cases} \lambda_j & \text{if } D_j \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

Define Q_D from ΛBV into ΛBV by

$$Q_{D^u} = \sum_{j=1}^k \frac{(u(E_j) - u(E_{j-1}))d_{D_j}}{\lambda_j}.$$

It is clear that Q_D is linear and satisfied (5) since the sum for $Q_{D^u}(I)$ collapses to $u(I)$. Since each d_{D_j} is increasing, and each coefficient is positive when u is increasing it follows that Q_D is positive. If $u = u^+ - u^-$ when u^+ and u^- are increasing and $\|u\| = u^+(I) - u^-(I)$ we have

$$\begin{aligned} \|Q_{D^u}\| &\leq \|Q_D u^+\| + \|Q_D u^-\| \\ &= Q_D u^+(I) + Q_D u^-(I) \\ &= u^+(I) - u^-(I) \\ &= \|u\|. \end{aligned}$$

Thus (6) is valid. We omit the straightforward arguments which show Q_D satisfies (6) and (7). Now with a similar proposition 2.2 and theorem 2.3 of [5], one can prove that

Theorem 5.4. There exists a projection Q from ΛBV onto $F A$ satisfying (2) and (3).

Theorem 5.5. If Φ is a locally finite subgroup there is a Φ – value P from ΛBV onto $F A$.

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REFERENCES

1. Dunford, N. & Schwartz, J. T. (1958). *Linear operators*. New York, Interscience.
2. Esi, A. H. & Polat, H. (2006). On strongly Δ^n -summable sequence spaces. *Iran. J. Sci. Technol.*, 30(2), 229-234.
3. Maccheroni, F. & Ruckle, W. H. (2002). BV as a dual space. *Rendiconti del Seminario Matematico di Padova*, 107, 101-109.
4. Aumann, R. J. & Shapley, L. S. (1974). *Values of non-atomic games*. Princeton University Press.
5. Ruckle, W. H. (1982). Projection in Certain Spaces of set Functions. *Mathematics of Operations Research*, 7(2), 314-318.