\( \Lambda BV \) AS A NON SEPARABLE DUAL SPACE*

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Abstract – Let \( C \) be a field of subsets of a set \( I \). Also, let \( \Lambda = \{\lambda_i\}_{i=1}^{\infty} \) be a non-decreasing positive sequence of real numbers such that \( \lambda_i = 1, \ 1/\lambda_i \to 0 \) and \( \sum_{i=1}^{\infty} 1/\lambda_i = \infty \). In this paper we prove that \( \Lambda BV \) of all the games of \( \Lambda \)-bounded variation on \( C \) is a non-separable and norm dual Banach space of the space of simple games on \( C \). We use this fact to establish the existence of a linear mapping \( T \) from \( \Lambda BV \) onto \( FA \) (finitely additive set functions) which is positive, efficient and satisfies a weak form of symmetry, namely invariance under a semigroup of automorphisms of \((I, C)\).

Keywords – Set functions, duality, compactness, non separable

1. INTRODUCTION

Let \( C \) be a field of subsets of a nonempty set \( I \). It is well-known that the space \( FA \) of all the finitely additive games of bounded variation on \( C \), equipped with the total variation norm, is isometrically isomorphic to the norm dual of the space of all simple functions on \( C \), endowed with the sup norm ([1]) (also see [2]). Maccheroni and Ruckle in [3] established a parallel result for the space \( BV \) of all the games of bounded variation on \( C \). Indeed, they showed that \( BV \), equipped with the total variation norm, is isometrically isometric to the norm dual of the space of all simple games endowed with a suitable norm where a simple game is a game which is non zero only on a finite number of elements of \( C \). Let \( \Lambda = \{\lambda_i\}_{i=1}^{\infty} \) be a non-decreasing positive sequence of real numbers such that \( \lambda_i = 1, \ 1/\lambda_i \to 0 \) and \( \sum_{i=1}^{\infty} 1/\lambda_i = \infty \). We introduce space \( \Lambda BV \) which shares many properties of space \( BV \). Here, we prove that space \( \Lambda BV \) of all the games of \( \Lambda \) bounded variation on \( C \) equipped with the total variation norm, is isometrically isometric to the norm dual of the space of all simple games, endowed with a suitable norm. We use this fact to establish the existence of a linear mapping \( T \) from \( \Lambda BV \) onto \( FA \) (finitely additive set functions) which is positive, efficient and satisfies a weak form of symmetry, namely invariance under a semigroup of automorphisms of \((I, C)\).

2. PRELIMINARIES

A set function \( \nu : C \to R \) is a game if \( \nu(\emptyset) = 0 \). A game on \( C \) is monotone if \( \nu(A) \leq \nu(B) \) whenever \( A \subseteq B \). A chain \( \{S_i\}_{i=0}^{n} \) in \( C \) is a finite strictly increasing sequence

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of the elements of $C$. $\Lambda BV$ is the set of all games such that

$$
\|u\| = \sup \left\{ \sum_{i=1}^{n} \frac{u(S_i) - u(S_{i-1})}{\lambda_i} : \{S_i\}_{i=0}^{n} \text{ is a chain in } C \right\} < \infty.
$$

A game in $\Lambda BV$ is said to be of $\Lambda$ bounded variation. A game is called a simple game if it is non-zero only on a finite number of elements of $C$. A function $u$ in $\Lambda BV$ is called finitely additive if

$$
u(A \cup B) = u(A) + u(B)
$$

whenever $A$ and $B$ are in $C$ and $A \cap B = \phi$.

The set $FA$ of finitely additive functions in $\Lambda BV$ forms a closed subspace of $\Lambda BV$. A function $u$ in $\Lambda BV$ is called increasing if $u(A) \leq u(B)$ whenever $A \subseteq B$. Each $u$ in $\Lambda BV$ has the form $u = u^* + \nu$ where $u^*$ and $\nu$ are increasing and $\|u\| = u^*(I) + \nu(I)$. A linear mapping $T$ in $L(BV)$ is positive if $Tu$ increases whenever $u$ increases.

Let $C$ denote the group of automorphisms of $(I, C)$. A subspace $X$ is called symmetric if $u \circ \pi$ is in $X$ for each $x$ in $X$ and each $\pi$ in $C$. A value is a linear mapping $T$ from a symmetric subspace $X$ of $\Lambda BV$ onto the space $FA$ of finitely additive set functions which satisfies three conditions:

(a) $T$ is positive: i.e., $Tu$ increases whenever $u$ increases.

(b) $T$ is symmetric: i.e., $T(u \circ \pi) = (Tu) \circ \pi$ for each $\pi$ in $C$ and $u$ in $X$.

(c) $T$ is efficient: $(Tu)(I) = u(I)$ for each $u$ in $X$.

In this note we establish the existence of linear operations from all of $\Lambda BV$ onto $FA$ which satisfy (a), (b) and a weaker form of (c), namely symmetry under a semigroup of $C$. In addition, these linear operators are projections (i.e., $Tu = u$ for $u$ in $FA$). Our main result is that, given any locally finite subgroup $\Phi$ of $C$ there is a projection $T$ from $\Lambda BV$ onto $FA$ which is symmetric under $\Phi$. Since $\Lambda BV$ is a (proper) subspace of $R^C$, it inherits a topology from the product topology of $R^C$. This is the weak topology generated by the projection functional

$$
P_a : \Lambda BV \to R
$$

where $A \in C$. A net $\{u_a\}$ converges to $u$ in this topology if $u_a(A) \to u(A)$ for all $A \in C$ (we write $u_a \overset{c}{\to} u$). This topology is called \textit{\Lambda-vague} topology for the analogy with the vague topology on the set of probability measures.

3. $\Lambda BV$ AS A NON SEPARABLE DUAL SPACE

In [4], Aumann and Shapley proved that $BV$ is a Banach space. Here, we show $\Lambda BV$ is a Banach space too.

Let $\Omega = \{S_i\}_{i=0}^{n}$ be a chain. For any set function $\nu$ we define

$$
\|\nu\|_\Omega = \left\{ \sum_{i=1}^{n} \frac{\nu(S_i) - \nu(S_{i-1})}{\lambda_i} \right\} < \infty.
$$
This shows that a necessary and sufficient condition, $\nu \in \Lambda BV$, is that $\|\nu\|_\Omega$ be bounded over all chain $\Omega$. Then, $\nu \in \Lambda BV$ if and only if $\|\nu\| = \sup\|\nu\|_\Omega$, where the $\sup$ is taken over all chains $\Omega$.

It is obvious that this defines a norm on $\Lambda BV$. Now, we show that with this norm, $\Lambda BV$ is a complete space.

**Theorem 3.1.** $\Lambda BV$ is complete, hence a Banach space.

**Proof:** Let $\{\nu_n\}$ be a Cauchy sequence of elements of $\Lambda BV$. For any subset $S$ of $I$, we show that sequence $\{\nu_n(S)\}$ is a Cauchy sequence in $R$.

Let $S$ be a subset of $I$. For the chain $\Omega$,

$$\Omega \subseteq S \subseteq I;$$

We have

$$\|\nu_n - \nu_m\| \geq \frac{\left| (\nu_n(S) - \nu_m(S)) - (\nu_n(\Phi) - \nu_m(\Phi)) \right|}{\lambda_1} = \left| (\nu_n(S) - \nu_m(S)) \right|.$$

Then the sequence $\{\nu_n(S)\}$ is a Cauchy sequence in $R$ and is convergent; denote it’s limit by $\nu(S)$.

We must first show that $\nu$ is $\Lambda$-bounded variation. Let $N$ be such that $\|\nu_n - \nu_m\| \leq 1$ whenever $n \geq N$. Then for each chain $\Omega$ and each $n \geq N$ we have

$$\|\nu_n\|_\Omega - \|\nu_N\| \leq \|\nu_n\|_\Omega - \|\nu_N\|_\Omega \leq \|\nu_n - \nu_N\|_\Omega \leq \|\nu_n - \nu_N\| \leq 1$$

letting $n \to \infty$, we deduce

$$\|\nu\|_\Omega \leq 1 + \|\nu_N\|.$$

Hence $\nu$ is $\Lambda$-bounded variation. That $\|\nu_n - \nu\| \to 0$ is now easily verified, so the theorem is proved.

Here, we show that $\Lambda BV$ is a non separable space. So, the dual of $\Lambda BV$ is non separable too.

**Theorem 3.2.** $\Lambda BV[a,b]$ is non separable.

**Proof:** For each a satisfying $a < s < b$ and subset $A$ of $[a,b]$, let $\chi_s(A)$ be the set function defined by

$$\chi_s(A) = \begin{cases} 1 & \text{if } [a,s] \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\chi_s$ is a monotone set function and belongs to the $\Lambda BV[a,b]$. For any $s$ and $r$ with $a < s < r < b$, let $\Omega$ be the chain $\emptyset \subseteq [a,s] \subseteq I$. Then
This completes the proof.

4. ABV AS A DUAL SPACE

In [3], Maccheroni and Ruckle showed that $BV$ is a dual Banach space. Indeed, they showed that $BV$ is isometrically isomorphic to the norm dual of space of all simple games. Here, we establish this result for $ΛBV$.

We define the game $e_A : C \to R$ by

$$e_A(B) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{otherwise} \end{cases}$$

Let $X$ be the space of all simple games. For all $A \in C - \{\emptyset\}$ and $e_{\emptyset} = 0$ being $x = \sum_{A \in C} x(A)e_A$ for all $x \in X$, we have $X = \{e_A : A \in C \}$. For each chain $Ω = \{S_i\}_{i=0}^n$ in $C$, define a semi norm on $X$ by

$$\| x \|_{Ω} = \max_{0 \leq k \leq n} \left| \sum_{i=k}^n x(S_i) \right|.$$ (1)

For all $x \in X$. Let $X_Ω = \{e_A : A \in Ω\}$. If $x \in X_Ω$, we say that $X$ depends on the chain $Ω$. For all $x \in X$, set

$$\| x \| = \inf \sum_{e=1}^L \| x_e \|_{Ω_e}$$

where the inf is taken over all finite decompositions $x = \sum_{e=1}^L x_e$ in which $x_e$ depends on the chain $Ω_e$ and $\| . \|_{Ω_e}$ is defined as in (1) for all $e = 1, 2, ..., L$.

Lemma 4 of [3] showed that this equation defines a norm on $X$.

Lemma 4.1. The function $\| . \| : X \to R$ is a norm on $X$.

Given a linear continuous functional $f : X \to R$, define the game $G_f$ as follows

$$G_f(A) = f(e_A)$$

For all $A \in C$.

Theorem 4.2. Let $X^*$ be the norm dual of $(X, \| . \|)$. The operator

$$G : X^* \to ΛBV$$
is an isometric isomorphism from $X^*$ onto $\Lambda BV$.

**Proof:** We first show that if $\Omega = \{S_j\}_{j=0}^n$ is a chain in $C$, then

$$
\sum_{k=1}^n \left| \frac{G_f(S_k) - G_f(S_{k-1})}{\lambda_k} \right| \leq \|f\|
$$

which implies that $G_f \in \Lambda BV$ and $\|G_f\| \leq \|f\|$.

Define $x \in X_\Omega$ by

$$
x(S_n) = Sgn(f(eS_n) - f(eS_{n-1})),
$$

$$
x(S_n) + x(S_{n-1}) = Sgn(f(eS_{n-1}) - f(eS_{n-2})),
$$

$$
\vdots
$$

$$
x(S_n) + x(S_{n-1}) + \cdots + x(S_1) = Sgn(f(eS_1) - f(eS_0)),
$$

$$
x(S_0) = 0.
$$

Obviously $\|x\|_\Omega \leq 1$, so that $\|x\| < 1$. Similar to proof of theorem 5 of [3], we have,

$$
\|f\| \geq f(x) = \sum_{j} \left| G_f(S_j) - G_f(S_{j-1}) \right|
$$

$$
\geq \sum_{j=1}^n \left| \frac{G_f(S_j) - G_f(S_{j-1})}{\lambda_j} \right|
$$

which implies that $\|f\| \geq \|G_f\|$. Then $G$ is well defined and obviously linear and injective.

Given $u \in \Lambda BV$, we can define $f_u$ on $X$ by

$$
f_u(x) = \sum_{A_j \in C} \frac{u(A_j)}{\lambda_j} x(A_j),
$$

for all $x \in X$. It is trivial that $f_u$ is linear.

If $x$ depends on $\Omega = \{S_j\}_{j=0}^n$, then

$$
f_u(x) = \sum_{j=0}^n \frac{u(S_j)}{\lambda_j} x(S_j)
$$

$$
= \frac{u(S_0)}{\lambda_0} \sum_{k=0}^n x(S_k) + \sum_{j=1}^n \left[ \frac{u(S_j) - u(S_{j-1})}{\lambda_j} \sum_{k=j}^n x(S_k) \right]
$$

$$
= \sum_{j=1}^n \left[ \frac{u(S_j) - u(S_{j-1})}{\lambda_j} \sum_{j=1}^n x(S_k) \right]
$$
\[
\sum_{j=1}^{n} \left[ \frac{u(S_j) - u(S_{j-1})}{\lambda_j} \sum_{k=j}^{n} x(S_k) \right] \leq \| x \|_{\Omega} \| u \|.
\]

If \( x = \sum_{e=1}^{L} x_e \) with \( x_e \in X_{\Omega_e} \) for all \( e = 1, 2, \ldots, L \), then

\[
f_u(x) = \sum_{e=1}^{L} f_u(x_e)
\]

\[
\leq \sum_{e=1}^{L} \| u \| \| x_e \|_{\Omega_e}
\]

\[
\leq \| u \| \sum_{e=1}^{L} \| x_e \|_{\Omega_e},
\]

and so

\[
f_u(x) \leq \inf \left\{ \| u \| \sum_{e=1}^{L} \| x_e \|_{\Omega_e} : x = \sum_{e=1}^{L} x_e, \ x_e \in X_{\Omega_e} \right\}
\]

\[
= \| u \| \| x \|.
\]

We conclude that \( f_u \in X^* \), \( G(f_u) = u \) and \( G \) is onto. For all \( u \in \Lambda BV \), \( f_u = G_u^{-1} \) and \( \| G_u^{-1} \| = \| f_u \| \leq \| u \| \).

Therefore, for all \( f \in X^* \), \( \| f \| = \| G_u^{-1} \| \leq \| G_f \| \) and \( G \) is an isometry.

Let \( G \) be similar to the previous theorem. We show that,

**Theorem 4.3.** \( G \) is weak\(^*\) \( \Lambda - \) vague homeomorphism.

**Proof:** Let \( \left\{ f^a \right\} \) be a net in \( X^* \). By using the notations of the previous theorem, we have that \( f^a \xrightarrow{w} f \) iff \( f^a(x) \to f(x) \) for all \( x \in X \) iff \( f^a(e_A) \to f(e_A) \) for all \( A \in C \) iff \( G_{f^a}(A) \to G_f(A) \) for all \( A \in C \) iff \( G_{f^a} \xrightarrow{C} G_f \).

In Theorem 4.2, together with the Alaoghlu theorem, we have the compactness of the unit ball \( U(BV) \) in the \( \Lambda - \) vague topology.

**Theorem 4.4.** The unit ball \( U(BV) \) is compact with respect to the \( \Lambda - \) vague topology.

## 5. PROJECTIONS FROM \( \Lambda BV \) ONTO \( FA \)

Given \( I \) and \( C' \) as in §1, let \( \Theta \) denote the set of one-to-one functions \( \Theta \) from \( I \) into \( I \) such that \( \pi(S) \in C \) if and only if \( S \in C \). Then \( \Theta \) forms a group under composition. For each \( \pi \) in \( \Theta \) the function \( T_\pi \) defined by \( T_\pi u = u \circ \pi \) is a linear operator from \( \Lambda BV \) into \( \Lambda BV \) with \( \| T_\pi \| = 1 \). A function \( u \) in \( \Lambda BV \) is called finitely additive if

\[
u(A \cup B) = \nu(A) + \nu(B)
\]
whenever $A$ and $B$ are in $C$ an $A \cap B = \phi$. The set $F \Lambda$ of finitely additive functions in $\Lambda BV$ forms a closed subspace of $\Lambda BV$. A function $u$ in $\Lambda BV$ is called increasing if $u(A) \leq u(B)$ whenever $A \subseteq B$. Each $u$ in $\Lambda BV$ has the form $u = u^+ - u^-$ when $u^+$ and $u^-$ are increasing and $\|u\| = u^+(I) - u^-(I)$. A linear mapping $T$ in $L(\Lambda BV)$ is positive if $Tu$ is increasing whenever $u$ is increasing.

**Definition 5.1.** Let $\Phi$ be a subgroup of $\Theta$. A $\Phi$-value is a projection $P$ from $\Lambda BV$ onto $F \Lambda$ which fulfills the following conditions:

$$\|Pu\| \leq \|u\|, \quad u \in \Lambda BV. \quad (2)$$

$$Pu(I) = u(I), \quad u \text{ in } \Lambda BV. \quad (3)$$

$$PT_\pi = T_\pi P \quad \text{for all } \pi \text{ in } \Phi. \quad (4)$$

**Definition 5.2.** For each finite partition $D$ of $I$ into members of $C$, $\Gamma_\Lambda = \{\Gamma_\Lambda(D)\}$ is the set of all $T$ in $L(\Lambda BV)$ for which

$$Tu(I) = u(I) \quad \text{for } u \text{ in } \Lambda BV; \quad (5)$$

$$\|Tu\| \leq \|u\|, \quad u \in \Lambda BV; \quad (6)$$

$Tu$ is additive on the algebra of sets determined by $D$;

$$Tu(B) = u(B) \quad \text{for } u \text{ in } F \Lambda, \ B \text{ in } D. \quad (7)$$

**Lemma 5.3.** No set $\Gamma_\Lambda(D)$ is empty.

**Proof:** Suppose $D = \{D_1, D_2, \ldots, D_k\}$ (any order). Let $E_0 = \phi$, $E_1 = D_1$, $\ldots$, $E_n = D_1 \cup D_2 \cup \ldots \cup D_n$, $\ldots$, $E_k = I$. For each $D_j$ in $C$ let $d_{D_j}$ be the function

$$d_{D_j}(A) = \begin{cases} \lambda_i & \text{if } D_j \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

Define $Q_D$ from $\Lambda BV$ into $\Lambda BV$ by

$$Q_D = \sum_{j=1}^k \left( u(E_j) - u(E_{j-1}) \right) d_{D_j}. \quad (8)$$

It is clear that $Q_D$ is linear and satisfied (5) since the sum for $Q_D(I)$ collapses to $u(I)$. Since each $d_{D_j}$ is increasing, and each coefficient is positive when $V$ is increasing it follows that $Q_D$ is positive. If $u = u^+ - u^-$ when $u^+$ and $u^-$ are increasing and $\|u\| = u^+(I) - u^-(I)$ we have

$$\|Q_D\| \leq \|Q_D u^+\| + \|Q_D u^-\| = Q_D u^+(I) + Q_D u^-(I) = u^+(I) + u^-(I) = \|u\|. \quad (9)$$
Thus (6) is valid. We omit the straightforward arguments which show $Q_D$ satisfies (6) and (7). Now with a similar proposition 2.2 and theorem 2.3 of [5], one can prove that

**Theorem 5.4.** There exists a projection $Q$ from $\Lambda BV$ onto $FA$ satisfying (2) and (3).

**Theorem 5.5.** If $\Phi$ is a locally finite subgroup there is a $\Phi$–value $P$ from $\Lambda BV$ onto $FA$.

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