Iranian Journal of Science & Technology, Transaction A, Vol. 34, No. A3
Printed in the Islamic Republic of Iran, 2010


# $\Lambda BV$ AS A NON SEPARABLE DUAL SPACE*

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**Abstract** – Let $C$ be a field of subsets of a set $I$. Also, let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a non-decreasing positive sequence of real numbers such that $\lambda_1 = 1$, $1/\lambda_i \to 0$ and $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. In this paper we prove that $\Lambda BV$ of all the games of $\Lambda$-bounded variation on $C$ is a non-separable and norm dual Banach space of the space of simple games on $C$. We use this fact to establish the existence of a linear mapping $T$ from $\Lambda BV$ onto $FA$ (finitely additive set functions) which is positive, efficient and satisfies a weak form of symmetry, namely invariance under a semigroup of automorphisms of $(I, C)$.

**Keywords –** Set functions, duality, compactness, non separable

## 1. INTRODUCTION

Let $C$ be a field of subsets of a nonempty set $I$. It is well-known that the space $FA$ of all the finitely additive games of bounded variation on $C$, equipped with the total variation norm, is isometrically isomorphic to the norm dual of the space of all simple functions on $C$, endowed with the sup norm ([1]) (also see [2]). Maccheroni and Ruckle in [3] established a parallel result for the space $BV$ of all the games of bounded variation on $C$. Indeed, they showed that $BV$, equipped with the total variation norm, is isometrically isometric to the norm dual of the space of all simple games endowed with a suitable norm where a simple game is a game which is non zero only on a finite number of elements of $C$. Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a non-decreasing positive sequence of real numbers such that $\lambda_1 = 1$, $1/\lambda_i \to 0$ and $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. We introduce space $\Lambda BV$ which shares many properties of space $BV$. Here, we prove that space $\Lambda BV$ of all the games of $\Lambda$ bounded variation on $C$ equipped with the total variation norm, is isometrically isometric to the norm dual of the space of all simple games, endowed with a suitable norm. We use this fact to establish the existence of a linear mapping $T$ from $\Lambda BV$ onto $FA$ (finitely additive set functions) which is positive, efficient and satisfies a weak form of symmetry, namely invariance under a semigroup of automorphisms of $(I, C)$.

## 2. PRELIMINARIES

A set function $\nu : C \to R$ is a game if $\nu(\phi) = 0$. A game on C is monotone if $\nu(A) \le \nu(B)$ whenever $A \subseteq B$. A chain $\{S_i\}_{i=0}^{n}$ in $C$ is a finite strictly increasing sequence

*Received by the editor November 24, 2008 and in final revised form August 3, 2010
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$$\phi = S_0 \subset S_1 \subset \ldots \subset S_n = I$$

of the elements of $C$. $\Lambda BV$ is the set of all games such that

$$\| u \| = \sup \left\{ \sum_{i=1}^{n} \frac{|u(S_i) - u(S_{i-1})|}{\lambda_i} : \{S_i\}_{i=0}^{n} \ \ is\, a \ \ chain \ \ in \ \ C \right\} < \infty .$$

A game in $\Lambda BV$ is said to be of $\Lambda$ bounded variation. A game is called a simple game if it is non-zero only on a finite number of elements of $C$. A function $u$ in $\Lambda BV$ is called finitely additive if

$$u(A \cup B) = u(A) + u(B)$$

whenever $A$ and $B$ are in $C$ and $A \cap B = \phi$.

The set $F\,A$ of finitely additive functions in $\Lambda BV$ forms a closed subspace of $\Lambda BV$. A function $u$ in $\Lambda BV$ is called increasing if $u(A) \le u(B)$ whenever $A \subset B$. Each $u$ in $\Lambda BV$ has the form $u = u^+ + u^-$ when $u^+$ and $u^-$ are increasing and $\| u \| = u^+(I) + u^-(I)$. A linear mapping $T$ in $L(BV)$ is positive if $Tu$ increases whenever $u$ increases.

Let C* denote the group of automorphisms of $(I, C)$. A subspace $X$ is called symmetric if $u\, o\, \pi$ is in $X$ for each $x$ in $X$ and each $\pi$ in C*. A value is a linear mapping $T$ from a symmetric subspace $X$ of $\Lambda BV$ onto the space $F\,A$ of finitely additive set functions which satisfies three conditions:

(a) $T$ is positive: i.e., $Tu$ increases whenever $u$ increases.

(b) $T$ is symmetric: i.e., $T(u\,0\,\pi) = (Tu)\ \ o\,\pi$ for each $\pi$ in C* and $u$ in $X$.

(c) $T$ is efficient: $(Tu)(I) = u(I)$ for each $u$ in $X$.

In this note we establish the existence of linear operations from all of $\Lambda BV$ onto $F\,A$ which satisfy (a), (b) and a weaker form of (c), namely symmetry under a semigroup of C*. In addition, these linear operators are projections (i.e., $Tu = u$ for $u$ in $F\,A$). Our main result is that, given any locally finite subgroup $\Phi$ of C* there is a projection $T$ from $\Lambda BV$ onto $F\,A$ which is symmetric under $\Phi$. Since $\Lambda BV$ is a (proper) subspace of $R^C$, it inherits a topology from the product topology of $R^C$. This is the weak topology generated by the projection functional

$$P_A : \Lambda BV \to R$$
$$u \ \to \ u(A)$$

where $A \in C$. A net $\{u_\alpha\}$ converges to $u$ in this topology if $u_\alpha(A) \to u(A)$ for all $A \in C$ (we write $u_\alpha \xrightarrow{C} u$). This topology is called $\Lambda$-*vague* topology for the analogy with the vague topology on the set of probability measures.

### 3. $\Lambda BV$ AS A NON SEPARABLE DUAL SPACE

In [4], Aumann and Shapley proved that $BV$ is a Banach space. Here, we show $\Lambda BV$ is a Banach space too.

Let $\Omega = \{S_i\}_{i=0}^{n}$ be a chain. For any set function $\nu$ we define

$$\| \nu \|_\Omega = \left\{ \sum_{i=1}^{n} \frac{|\nu(S_i) - \nu(S_{i-1})|}{\lambda_i} \right\} < \infty .$$

This shows that a necessary and sufficient condition, $\nu \in \Lambda BV$, is that $\|\nu\|_{\Omega}$ be bounded over all chain $\Omega$. Then, $\nu \in \Lambda BV$ if and only if $\|\nu\| = \sup \|\nu\|_{\Omega}$, where the $\sup$ is taken over all chains $\Omega$.

It is obvious that this defines a norm on $\Lambda BV$. Now, we show that with this norm, $\Lambda BV$ is a complete space.

**Theorem 3.1.** $\Lambda BV$ is complete, hence a Banach space.

**Proof:** Let $\{\nu_n\}$ be a Cauchy sequence of elements of $\Lambda BV$. For any subset $S$ of $I$, we show that sequence $\{\nu_n(S)\}$ is a Cauchy sequence in $R$.
Let $S$ be a subset of $I$. For the chain

$$\Phi \subset S \subset I;$$

We have

$$\|\nu_n - \nu_m\| \geq \frac{|(\nu_n(S) - \nu_m(S)) - (\nu_n(\Phi) - \nu_m(\Phi))|}{\lambda_1}$$

$$= |(\nu_n(S) - \nu_m(S))|.$$

Then the sequence $\{\nu_n(S)\}$ is a Cauchy sequence in $R$ and is convergent; denote it's limit by $\nu(S)$. We must first show that $\nu$ is $\Lambda$-bounded variation. Let $N$ be such that $\|\nu_n - \nu_m\| \leq 1$ whenever $n \geq N$. Then for each chain $\Omega$ and each $n \geq N$ we have

$$\begin{aligned} \|\nu_n\|_{\Omega} - \|\nu_N\| &\leq \|\nu_n\|_{\Omega} - \|\nu_N\|_{\Omega} \\ &\leq \|\nu_n - \nu_N\|_{\Omega} \\ &\leq \|\nu_n - \nu_N\| \\ &\leq 1 \end{aligned}$$

letting $n \to \infty$, we deduce

$$\|\nu\|_{\Omega} \leq 1 + \|\nu_N\|.$$

Hence $\nu$ is $\Lambda$-bounded variation. That $\|\nu_n - \nu\| \to 0$ is now easily verified, so the theorem is proved.

Here, we show that $\Lambda BV$ is a non separable space. So, the dual of $\Lambda BV$ is non separable too.

**Theorem 3.2.** $\Lambda BV[a,b]$ is non separable.

**Proof:** For each a satisfying $a < s < b$ and subset $A$ of $[a,b]$, let $\chi_s(A)$ be the set function defined by

$$\chi_s(A) = \begin{cases} 1 & if \quad [a,s] \subseteq A \\ 0 & otherwise. \end{cases}$$

We see that $\chi_s$ is a monotone set function and belongs to the $\Lambda BV[a,b]$. For any $s$ and $r$ with $a < s < r < b$, let $\Omega$ be the chain $\emptyset \subseteq [a,s] \subseteq I$. Then

$$\|\chi_r - \chi_s\| \geq \|\chi_r - \chi_s\|_\Omega$$
$$\geq \frac{|(\chi_r - \chi_s)([a,s]) - (\chi_r - \chi_s)(\emptyset)|}{\lambda_1}$$
$$\geq 1$$

This completes the proof.

## 4. $\Lambda BV$ AS A DUAL SPACE

In [3], Maccheroni and Ruckle showed that $BV$ is a dual Banach space. Indeed, they showed that $BV$ is isometrically isomorphic to the norm dual of space of all simple games. Here, we establish this result for $\Lambda BV$.

We define the game $e_A: \ C \to R$ by

$$e_A(B) = \begin{cases} 1 & if \quad B = A \\ 0 & otherwise \end{cases}$$

Let $X$ be the space of all simple games. For all $A \in C - \{\phi\}$ and $e_\phi = 0$ being $x = \sum_{A\in C} x(A)e_A$ for all $x \in X$, we have $X = \langle e_A : \ A \in C \rangle$. For each chain $\Omega = \{S_i\}_{i=0}^n$ in $C$, define a semi norm on $X$ by

$$\|x\|_\Omega = \max_{0\leq k\leq n} \left| \sum_{i=k}^{n} x(s_i) \right|. \tag{1}$$

For all $x \in X$. Let $X_\Omega = \langle e_A \ : \ A \in \Omega \rangle$. If $x \in X_\Omega$, we say that $x$ depends on the chain $\Omega$. For all $x \in X$, set

$$\|x\| = \inf \sum_{e=1}^{L} \|x_e\| \Omega_e$$

where the inf is taken over all finite decompositions $x = \sum_{e=1}^{L} x_e$ in which $x_e$ depends on the chain $\Omega_e$ and $\|.\|_{\Omega_e}$ is defined as in (1) for all $e = 1, 2, ..., L$.

Lemma 4 of [3] showed that this equation defines a norm on $X$.

**Lemma 4.1.** The function $\|.\|: X \to R$ is a norm on $X$.

Given a linear continuous functional $f: X \to R$, define the game $G_f$ as follows

$$G_f(A) = f(e_A)$$

For all $A \in C$.

**Theorem 4.2.** Let $X^*$ be the norm dual of $(X, \|.\|)$. The operator

$$G: X^* \to \Lambda BV$$

$$f \mapsto G_f$$

is an isometric isomorphism from $X^*$ onto $\Lambda BV$.

**Proof:** We first show that if $\Omega = \{S_i\}_{i=0}^{n}$ is a chain in $C$, then

$$\sum_{k=1}^{n} \frac{\left|G_f(S_k) - G_f(S_{k-1})\right|}{\lambda_k} \leq \| f \|,$$

which implies that $G_f \in \Lambda BV$ and $\| G_f \| \leq \| f \|$.

Define $x \in X_\Omega$ by

$$x(S_n) = Sgn(f(eS_n) - f(eS_{n-1})),$$
$$x(S_n) + x(S_{n-1}) = Sgn(f(eS_{n-1}) - f(eS_{n-2})),$$
$$\vdots$$
$$x(S_n) + x(S_{n-1}) + \cdots + x(S_1) = Sgn(f(eS_1) - f(eS_0)),$$
$$x(S_0) = 0\,.$$

Obviously $\| x \|_\Omega \leq 1$, so that $\| x \| < 1$. Similar to proof of theorem 5 of [3], we have,

$$\| f \| \geq f(x) = \sum_{j}^{n} \left| G_f(S_j) - G_f(S_{j-1}) \right|$$

$$\geq \sum_{j=1}^{n} \frac{\left| G_f(S_j) - G_f(S_{j-1}) \right|}{\lambda_j}$$

which implies that $\| f \| \geq \| G_f \|$. Then $G$ is well defined and obviously linear and injective.

Given $u \in \Lambda BV$, we can define $f_u$ on $X$ by

$$f_u(x) = \sum_{A_j \in C} \frac{u(A_j)}{\lambda_j} x(A_j),$$

for all $x \in X$. It is trivial that $f_u$ is linear.

If $x$ depends on $\Omega = \{S_j\}_{j=0}^{n}$, then

$$f_u(x) = \sum_{j=0}^{n} \frac{u(S_j)}{\lambda_j} x(S_j)$$

$$= \frac{u(S_0)}{\lambda_0} \sum_{k=0}^{n} x(S_k) + \sum_{j=1}^{n} \left[ \left( \frac{u(S_j) - u(S_{j-1})}{\lambda_j} \right) \sum_{k=j}^{n} x(S_k) \right]$$

$$= \sum_{j=1}^{n} \left[ \frac{u(S_j) - u(S_{j-1})}{\lambda_j} \sum_{j=1}^{n} x(S_k) \right]$$

$$\leq \sum_{j=1}^{n}\left[\frac{\left|u(S_j)-u(S_{j-1})\right|}{\lambda_j}\ \left|\sum_{k=j}^{n} x(S_k)\right|\right]$$

$$\leq \| x \|_{\Omega} \| u \| .$$

If $x=\sum_{e=1}^{L} x_e$ with $x_e \in X_{\Omega_e}$ for all $e=1,2,\dots,L,$ then

$$f_u(x)=\sum_{e=1}^{L} f_u(x_e)$$

$$\leq \sum_{e=1}^{L} \| u \| \, \| x_e \|_{\Omega_e}$$

$$\leq \| u \| \sum_{e=1}^{L} \| x_e \|_{\Omega_e} \ ,$$

and so

$$f_u(x) \leq \inf\left\{ \ \| u \| \sum_{e=1}^{L} \| x_e \|_{\Omega_e} : \ x=\sum_{e=1}^{L} x_e , \ \ x_e \in X_{\Omega_e} \right\}$$

$$= \| u \| \, \| x \| .$$

We conclude that $f_u \in X^*$, $G(f_u)=u$ and $G$ is onto. For all $u \in \Lambda BV$, $f_u = G_u^{-1}$ and $\| G_u^{-1} \| = \| f_u \| \leq \| u \|$.

Therefore, for all $f \in X^*$, $\| f \| = \| G^{-1}_{(G_f)} \| \leq \| G_f \|$ and $G$ is an isometry.

Let $G$ be similar to the previous theorem. We show that,

**Theorem 4.3.** $G$ is $weak^* \ \Lambda - vague$ homeomorphism.

**Proof:** Let $\{f^a\}$ be a net in $X^*$. By using the notations of the previous theorem, we have that $f^a \xrightarrow{w^*} f$ iff $f^a(x) \to f(x)$ for all $x \in X$ iff $f^a(e_A) \to f(e_A)$ for all $A \in C$ iff $G_{f^a}(A) \to G_f(A)$ for all $A \in C$ iff $G_{f^a} \xrightarrow{C} G_f$ .

In Theorem 4.2, together with the Alaoghlu theorem, we have the compactness of the unit ball $U(BV)$ in the $\Lambda - vague$ topology.

**Theorem 4.4.** The unit ball $U(BV)$ is compact with respect to the $\Lambda - vague$ topology.

## 5. PROJECTIONS FROM $\Lambda BV$ ONTO $FA$

Given $I$ and C as in §1, let $\Theta$ denote the set of one-to-one functions $\Theta$ from $I$ into $I$ such that $\pi(S) \in$ C if and only if $S \in$ C. Then $\Theta$ forms a group under composition. For each $\pi$ in $\Theta$ the function $T_\pi$ defined by $T_\pi u = u \, o \, \pi$ is a linear operator from $\Lambda BV$ into $\Lambda BV$ with $\| T_\pi \| = 1$. A function $u$ in $\Lambda BV$ is called finitely additive if

$$u(A \cup B) = u(A) + u(B)$$

whenever $A$ and $B$ are in $C$ an $A \cap B = \phi$. The set $FA$ of finitely additive functions in $\Lambda BV$ forms a closed subspace of $\Lambda BV$. A function $u$ in $\Lambda BV$ is called increasing if $u(A) \le u(B)$ whenever $A \subset B$. Each $u$ in $\Lambda BV$ has the form $u = u^+ - u^-$ when $u^+$ and $u^-$ are increasing and $\| u \| = u^+(I) - u^-(I)$. A linear mapping $T$ in $L(\Lambda BV)$ is positive if $Tu$ is increasing whenever $u$ is increasing.

**Definition 5.1.** Let $\Phi$ be a subgroup of $\Theta$. A $\Phi$-value is a projection $P$ from $\Lambda BV$ onto $FA$ which fulfills the following conditions:

$$\| Pu \| \le \| u \|, \qquad u \in \Lambda BV. \tag{2}$$

$$Pu(I) = u(I), \qquad u \ \ in \ \ \Lambda BV. \tag{3}$$

$$PT_\pi = T_\pi P \quad for \ \ all \ \ \pi \ \ in \ \ \Phi. \tag{4}$$

**Definition 5.2.** For each finite partition $D$ of $I$ into members of $C$, $\Gamma_\Lambda - set[\Gamma_\Lambda(\mathrm{D})]$ is the set of all $T$ in $L(\Lambda BV)$ for which

$$Tu(I) = u(I) \quad for \ \ u \ \ in \ \ \Lambda BV; \tag{5}$$

$$\| Tu \| \le \| u \|, \qquad u \in \Lambda BV; \tag{6}$$

*Tu is additive on the algebra of sets determined by D*; (7)

$$Tu(B) = u(B) \quad for \ \ u \ \ in \ \ FA, \ \ B \ \ in \ \ D. \tag{8}$$

**Lemma 5.3.** No set $\Gamma_\Lambda(D)$ is empty.

**Proof:** Suppose $D = \{D_1, D_2, ..., D_k\}$ (any order). Let $E_0 = \phi$, $E_1 = D_1$, …, $E_n = D_1 \cup D_2 \cup ... \cup D_n$, …, $E_k = I$. For each $D_j$ in $C$ let $d_{D_j}$ be the function

$$d_{D_j}(A) = \begin{cases} \lambda_i & if \quad D_j \subseteq A \\ 0 & otherwise. \end{cases}$$

Define $Q_D$ from $\Lambda BV$ into $\Lambda BV$ by

$$Q_{D^u} = \sum_{j=1}^{k} \frac{\left(u(E_j) - u(E_{j-1})\right) d_{D_j}}{\lambda_j}.$$

It is clear that $Q_D$ is linear and satisfied (5) since the sum for $Q_{D^u}(I)$ collapses to $u(I)$. Since each $d_{D_j}$ is increasing, and each coefficient is positive when $v$ is increasing it follows that $Q_D$ is positive. If $u = u^+ - u^-$ when $u^+$ and $u^-$ are increasing and $\| u \| = u^+(I) + u^-(I)$ we have

$$\begin{aligned} \| Q_{D^u} \| &\le \| Q_D u^+ \| + \| Q_D u^- \| \\ &= Q_D u^+(I) + Q_D u^-(I) \\ &= u^+(I) + u^-(I) \\ &= \| u \|. \end{aligned}$$

Thus (6) is valid. We omit the straightforward arguments which show $Q_D$ satisfies (6) and (7). Now with a similar proposition 2.2 and theorem 2.3 of [5], one can prove that

**Theorem 5.4.** There exists a projection $Q$ from $\Lambda BV$ onto $FA$ satisfying (2) and (3).

**Theorem 5.5.** If $\Phi$ is a locally finite subgroup there is a $\Phi - value$ $P$ from $\Lambda BV$ onto $FA$.

***Acknowledgements-*** The authors express their sincere thanks to Professor Fabio Maccheroni and Professor William H. Ruckle for their valuable suggestions and comments which led to the improvement of this paper.

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