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# SECTIONAL CURVATURE OF TIMELIKE RULED SURFACE PART I: LORENTZIAN BELTRAMI-EULER FORMULA*

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**Abstract –** In this paper, the Lorentzian version of Beltrami-Euler formula is investigated in $\Re_1^n$. Initially, the first fundamental form and the metric coefficients of generalized timelike ruled surface are calculated and by the help of the Christoffel Symbols, Riemann-Christoffel curvatures are obtained. Thus, the curvatures of spacelike and timelike tangential sections of generalized timelike ruled surface with timelike generating space and central ruled surface are found to be related to the determinant of the first fundamental form of the surface. In addition to this, the relation between the sectional curvature and the distribution parameter of this ruled surface is obtained. Finally, paying attention to the spacelike and timelike central ruled surface of the generalized timelike ruled surface one by one, four different types of Lorentzian Beltrami-Euler formulas are constituted for generalized timelike ruled surface with timelike generating space.

**Keywords –** Sectional curvature, ruled surface, Beltrami-Euler formula

## 1. INTRODUCTION

The fundamentals of the curvature theory go back to at least as far as the 3rd century BC. In ancient Greece Apollonius of Perga studied normals, centers of the curvature and the evolutes of elementary curves, [1]. The last three centuries have seen the theory of curvature bloom. L. Euler introduced the theory of curvature of surface in his first study. Euler's sectional curvatures were basically curvatures of curves obtained as intersections of a normal plane with the curve. Thus, the Euler theorem (Euler-curvature formula) related to normal and principal curvatures entered the literature. A short time later Meusnier also considered the intersections of planes that were not necessarily normal and gave a theorem with his name, called the Meusnier theorem. Various differential geometry books give the Euler and Meusnier theorem for $2-$dimensional surfaces in 3-dimensional Euclidean space, (see for example, [2, 3]).

Generalized ruled surface theory was put forward by M. Juza in [4] and the studies on this area reached ever higher during the last century. H. Frank, O. Giering, and C. Thas studied the properties of the ruled surface in $n-$ dimensional Euclidean space.

Applying the Euler and Meusnier theorems, which are the well-known theorems in the classical surface theory to the tangential sections of generalized ruled surface was performed by H. Frank and O. Giering, [5]. The sectional curvatures of the generalized ruled surfaces were evaluated in $n-$ dimensional Euclidean space $E^n$ and the obtained relations were entered into the literature as the Beltrami-Euler formula and Beltrami-Meusnier formula in [5].

In this work, we have studied the sectional curvatures of generalized timelike ruled surfaces with timelike generating space in $n-$dimensional Minkowski space $\Re_1^n$, taking into account the generalized

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timelike ruled surfaces given in [6] and [7]. Furthermore, we have also investigated the Beltrami Euler formula in the sense of Lorentzian in $n-$dimensional Minkowski space $\Re_1^n$.

## 2. PRELIMINARIES

Let $\vec{X}=\left(x_1,x_2,\ldots,x_n\right)$ and $\vec{Y}=\left(y_1,y_2,\ldots,y_n\right)$ be vectors in $\Re^n$ with $n>1$. The Lorentzian inner product of $\vec{X}$ and $\vec{Y}$ is defined to be the real number

$$\left\langle \vec{X},\vec{Y}\right\rangle = x_1y_1+x_2y_2+\ldots+x_{n-1}y_{n-1}-x_ny_n.$$

The inner product space consisting of the vector space $\Re^n$ together with the Lorentzian inner product is called $n-$dimensional Minkowski space, and is denoted by $\Re_1^n$. Since $\langle\ ,\ \rangle$ is an indefinite metric, recall that a vector $\vec{X}\in\Re_1^n$ can have one of three Lorentzian causal characters: it can be spacelike if $\left\langle \vec{X},\vec{X}\right\rangle>0$ or $\vec{X}=0$, timelike if $\left\langle \vec{X},\vec{X}\right\rangle<0$ and null (lightlike) if $\left\langle \vec{X},\vec{X}\right\rangle=0$ and $\vec{X}\neq 0$, [8].

Similarly, an arbitrary curve $\alpha=\alpha\left(t\right)\subset\Re_1^n$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\dot{\alpha}\left(t\right)$ are respectively spacelike, timelike or null (lightlike), [9]. The norm of $\vec{X}\in\Re_1^n$ is defined as $\left\|\vec{X}\right\|=\sqrt{\left|\left\langle \vec{X},\vec{X}\right\rangle\right|}$.

Let $W$ be a subspace of $\Re_1^n$ and denote $\langle\ ,\ \rangle\big|_W$ as the reduced metric in subspace $W$ of $\Re_1^n$. A subspace $W$ of $\Re_1^n$ can be spacelike, timelike or null (lightlike) if $\langle\ ,\ \rangle\big|_W$ is positive definite, $\langle\ ,\ \rangle\big|_W$ is nondegenerate of index 1 or $\langle\ ,\ \rangle\big|_W$ is degenerate, respectively, [9].

Let the set of all timelike vectors in $\Re_1^n$ be $\Gamma$. For $\vec{X}\in\Gamma$, we call

$$C\left(\vec{X}\right)=\left\{\vec{Y}\in\Gamma\,\middle|\,\left\langle \vec{X},\vec{Y}\right\rangle<0\right\}$$

the time-conic of Minkowski space $\Re_1^n$ including vector $\vec{X}$, [9].

Let $\vec{X}$ and $\vec{Y}$ be two timelike vectors in Minkowski space $\Re_1^n$. In this case the following inequality exists:

$$\left|\left\langle \vec{X},\vec{Y}\right\rangle\right|\geq\left\|\vec{X}\right\|\left\|\vec{Y}\right\|.$$

With equality if and only if $\vec{X}$ and $\vec{Y}$ are linear dependent.

If timelike vectors $\vec{X}$ and $\vec{Y}$ stay inside the same time-conic then there is a unique non-negative real number of $\theta\geq 0$ such that

$$\left\langle \vec{X},\vec{Y}\right\rangle=-\left\|\vec{X}\right\|\left\|\vec{Y}\right\|\cosh\theta \tag{1}$$

where the number $\theta$ is called an angle between the timelike vectors, [9].

Let $\vec{X}$ and $\vec{Y}$ be spacelike vectors in $\Re_1^n$ that span a spacelike subspace. We have that

$$\left|\left\langle \vec{X},\vec{Y}\right\rangle\right|\leq\left\|\vec{X}\right\|\left\|\vec{Y}\right\|$$

with equality if and only if $\vec{X}$ and $\vec{Y}$ are linearly dependent. Hence, there is a unique $0\leq\theta\leq\pi$ such that

$$\left\langle \vec{X},\vec{Y}\right\rangle=\left\|\vec{X}\right\|\left\|\vec{Y}\right\|\cos\theta. \tag{2}$$

The Lorentzian spacelike angle between $\vec{X}$ and $\vec{Y}$ is defined as $\theta$, [10].

Let $\vec{X}$ and $\vec{Y}$ be spacelike vectors in $\Re_1^n$ that span a timelike subspace. We have that

$$\left|\left\langle \vec{X},\vec{Y}\right\rangle\right| > \left\|\vec{X}\right\|\left\|\vec{Y}\right\|.$$

Hence, there is a unique real number $\theta > 0$ such that

$$\left\langle \vec{X},\vec{Y}\right\rangle = \left\|\vec{X}\right\|\left\|\vec{Y}\right\|\cosh\theta. \tag{3}$$

The Lorentzian timelike angle between $\vec{X}$ and $\vec{Y}$ is defined as $\theta$, [10].

Let $\vec{X}$ be a spacelike vector and $\vec{Y}$ be a timelike vector in $\Re_1^n$. Then there is a unique real number $\theta \geq 0$ such that

$$\left\langle \vec{X},\vec{Y}\right\rangle = \left\|\vec{X}\right\|\left\|\vec{Y}\right\|\sinh\theta. \tag{4}$$

The Lorentzian timelike angle between $\vec{X}$ and $\vec{Y}$ is defined to be $\theta$, [10].

## 3. GENERALIZED TIMELIKE RULED SURFACE WITH TIMELIKE GENERATING SPACE IN $n-$DIMENSIONAL MINKOWSKI SPACE $\Re_1^n$

Let $\{e_1(t),...,e_k(t)\}$ be an orthonormal vector field, which is defined at each point $\alpha(t)$ of a spacelike curve of an $n-$dimensional Minkowski space $\Re_1^n$. At the point $\alpha(t)\in\Re_1^n$ this system spans a $k-$dimensional subspace and is denoted by $E_k(t)$. It is given by $E_k(t)=Sp\{e_1(t),...,e_k(t)\}$. If the timelike subspace $E_k(t)$ moves along timelike curve $\alpha$, we obtain a $(k+1)-$dimensional surface in $\Re_1^n$. This surface is called a $(k+1)-$dimensional timelike ruled surface of the $n-$dimensional Minkowski space $\Re_1^n$ and is denoted by $M$, [7].

The subspace $E_k(t)$ and the curve $\alpha$ are called the generating space and the base curve, respectively. A parametrization of the ruled surface is the following:

$$\phi(t,u_1,...u_k) = \alpha(t) + \sum_{i=1}^{k} u_i e_i(t). \tag{5}$$

Throughout this paper we assume that $\left\{\dot{\alpha}(t)+\sum_{i=1}^{k} u_i \dot{e}_i(t), e_1(t),...,e_k(t)\right\}$ is linear independent.

$$A(t) = Sp\left\{e_1(t),...,e_k(t),\dot{e}_1(t),...,\dot{e}_k(t)\right\}$$

is called asymptotic bundle of $M$ with respect to $E_k(t)$. It is clear that $A(t)$ is a timelike subspace. If $\dim A(t) = k+m$, $0 \leq m \leq k$, then one can find an orthonormal base for $A(t)$ containing $E_k(t)$ such as $\{e_1(t),...e_k(t),a_{k+1}(t),...a_{k+m}(t)\}$. Furthermore, for the orthonormal base $\{e_1(t),\ldots,e_k(t)\}$, the following equations hold [7]

$$\begin{aligned} \dot{e}_\sigma &= \sum_{\mu=1}^{k} \alpha_{\sigma\mu} e_\mu + \kappa_\sigma a_{k+\sigma} \quad , \quad 1 \leq \sigma \leq m \\ \dot{e}_{m+\rho} &= \sum_{\mu=1}^{k} \alpha_{(m+\rho)\mu} e_\mu \quad , \quad 1 \leq \rho \leq k-m \end{aligned} \tag{6}$$

where

$$\varepsilon_\mu \alpha_{\nu\mu} = -\varepsilon_\nu \alpha_{\mu\nu} \tag{7}$$

and

$$\kappa_1 > \kappa_2 > \ldots > \kappa_m > 0. \tag{8}$$

The subspace

$$Sp\{e_1(t),...,e_k(t),\dot{e}_1(t),...,\dot{e}_k(t),\dot{\alpha}(t)\}$$

is called tangential bundle of $M$ with respect to $E_k(t)$ and is denoted as $T(t)$. It is obvious that

$$k+m \leq \dim T(t) \leq k+m+1 \quad , \quad 0 \leq m \leq k.$$

In what follows, we examine two cases separately. If $\dim T(t) = k+m$, then $\{e_1(t),...e_k(t),a_{k+1}(t),...a_{k+m}(t)\}$ is the base for both asymptotic and tangential bundles. If $\dim T(t) = k+m+1$, then one can find an orthonormal base for $T(t)$ as $\{e_1(t),...e_k(t),a_{k+1}(t),...a_{k+m}(t),a_{k+m+1}(t)\}$. For both cases tangential bundle $T(t)$ is a timelike subspace [7].

If $\dim T(t) = k+m+1$, then $M$, $(k+1)-$dimensional timelike ruled surface, has $(k-m)-$dimensional subspace called the central space of $M$ and is denoted by $Z_{k-m}(t) \subset E_k(t)$. The subspace $Z_{k-m}(t)$ is either spacelike or timelike subspace. If the base curve $\alpha$ of $M$ is chosen as the base curve and $Z_{k-m}(t)$ is the generating space, we get a $(k-m+1)-$dimensional ruled surface contained by $M$ in $\Re_1^n$. This is denoted by $\Omega$ and called the central ruled surface. If $Z_{k-m}(t)$ is spacelike (timelike) then central ruled surface $\Omega$ becomes a spacelike (timelike) ruled surface [6].

Taking $\Omega$ to be $(k-m+1)-$dimensional central ruled surface of $M$ $(m>0)$, we can write

$$\dot{\alpha}(t) = \sum_{\nu=1}^{k} \zeta_\nu e_\nu + \eta_{m+1} a_{k+m+1} \quad , \quad \eta_{m+1} \neq 0 \tag{9}$$

for the base curve $\alpha(t)$ [6]. Tangential space of $M$ is perpendicular to the asymptotic bundle $A(t)$ at the central points. Considering the equation (5) at the central point of central ruled surface $\Omega \subset M$, we see that [5]

$$u_\sigma = 0 \quad , \quad 1 \leq \sigma \leq m. \tag{10}$$

For the spacelike base curve $\alpha$ of $(k+1)-$dimensional timelike ruled surface $M$, if $\eta_{m+1} \neq 0$ the equality

$$P_\sigma = \frac{\eta_{m+1}}{\kappa_\sigma}, \; 1 \leq \sigma \leq m \tag{11}$$

is called the $\sigma^{\text{th}}$ principal distribution parameter of $M$ [6].

Taking the canonical base of the tangential bundle of $M$ in $\Re_1^n$ to be

$$\left\{ \sum_{\nu=1}^{k} \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) e_\nu + \sum_{\sigma=1}^{m} u_\sigma \kappa_\sigma a_{k+\sigma} + \eta_{m+1} a_{k+m+1}, e_1, e_2, \ldots, e_k \right\}. \tag{12}$$

We can evaluate the first fundamental form of $M$ and the metric coefficients with respect to this canonical base. In conventional notation, we choose $u_0 = t$ and calculate the metric coefficients of $M$ as follows

$$
\begin{aligned}
g_{00} &= \langle \varphi_t, \varphi_t \rangle = \sum_{\nu=1}^{k} \varepsilon_\nu \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right)^2 + \sum_{\sigma=1}^{m} \left( u_\sigma \kappa_\sigma \right)^2 + \left( \eta_{m+1} \right)^2, \\
g_{\nu 0} &= \left\langle \varphi_{u_\nu}, \varphi_t \right\rangle = \varepsilon_\nu \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) \quad , \quad 1 \le \nu \le k, \\
g_{\nu\mu} &= \left\langle \varphi_{u_\nu}, \varphi_{u_\mu} \right\rangle = \varepsilon_\nu \delta_{\nu\mu} \qquad ,1 \le \nu, \mu \le k.
\end{aligned} \tag{13}
$$

Therefore, the matrix of the first fundamental form of $M$ is expressed as

$$
\left[ g_{ij} \right] = \begin{bmatrix}
\sum_{\nu=1}^{k} \varepsilon_\nu \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right)^2 + \sum_{\sigma=1}^{m} \left( u_\sigma \kappa_\sigma \right)^2 + \left( \eta_{m+1} \right)^2 & \varepsilon_1 \left( \zeta_1 + \sum_{\mu=1}^{k} \alpha_{1\mu} u_\mu \right) & \varepsilon_2 \left( \zeta_2 + \sum_{\mu=1}^{k} \alpha_{2\mu} u_\mu \right) & \cdots & \varepsilon_k \left( \zeta_k + \sum_{\mu=1}^{k} \alpha_{k\mu} u_\mu \right) \\
\varepsilon_1 \left( \zeta_1 + \sum_{\mu=1}^{k} \alpha_{1\mu} u_\mu \right) & \varepsilon_1 & 0 & \cdots & 0 \\
\varepsilon_2 \left( \zeta_2 + \sum_{\mu=1}^{k} \alpha_{2\mu} u_\mu \right) & 0 & \varepsilon_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\varepsilon_k \left( \zeta_k + \sum_{\mu=1}^{k} \alpha_{k\mu} u_\mu \right) & 0 & 0 & \cdots & \varepsilon_k
\end{bmatrix}.
$$

It is easily seen that

$$
g = \det \left[ g_{ij} \right] = -\sum_{\sigma=1}^{m} \left( u_\sigma \kappa_\sigma \right)^2 - \eta_{m+1}^2 \quad , \quad 0 \le i, j \le k . \tag{14}
$$

Since $-\sum_{\sigma=1}^{m} \left( u_\sigma \kappa_\sigma \right)^2 - \eta_{m+1}^2 \ne 0$, $\left[ g_{ij} \right]$ is a regular matrix. Hence, from equations (13) and (14) we see that

$$
\begin{aligned}
g_{00} &= -g + \sum_{\nu=1}^{k} \varepsilon_\nu \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right)^2 , \\
g_{\nu 0} &= \varepsilon_\nu \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) \qquad , \qquad 1 \le \nu \le k, \\
g_{\nu\mu} &= \varepsilon_\nu \delta_{\nu\mu} \qquad , \qquad 0 \le \nu, \mu \le k, \\
g &= \det \left[ g_{ij} \right] = -\sum_{\sigma=1}^{m} \left( u_\sigma \kappa_\sigma \right)^2 - \eta_{m+1}^2 , \qquad 0 \le i, j \le k.
\end{aligned} \tag{15}
$$

In addition to these, the inverse matrix elements of $\left[ g_{ij} \right]$ are obtained as

$$g^{00} = -g^{-1},$$
$$g^{\nu 0} = \left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu\right) g^{-1} \quad , \quad 1 \le \nu \le k, \tag{16}$$
$$g^{\nu\lambda} = \left(\varepsilon_\nu \delta_{\nu\lambda} g - \left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu\right)\left(\zeta_\lambda + \sum_{\mu=1}^{k} \alpha_{\lambda\mu} u_\mu\right)\right) g^{-1} \quad , \quad 1 \le \nu, \lambda \le k.$$

Substituting the equations (15) and (16) into the Koszul equation (given in [8])

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_m g^{km} \left[ \frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} \right] \tag{17}$$

the Christoffel symbols are reached

$$\Gamma_{00}^{0} = \frac{1}{2g}\left[\frac{\partial g}{\partial u_0} + \sum_{\nu=1}^{k}\left(\zeta_\nu + \sum_{\mu=1}^{k}\alpha_{\nu\mu}u_\mu\right)\frac{\partial g}{\partial u_\nu}\right],$$
$$\Gamma_{00}^{\lambda} = \frac{1}{2g}\left[-\left(\zeta_\lambda + \sum_{\mu=1}^{k}\alpha_{\lambda\mu}u_\mu\right)\left(\frac{\partial g}{\partial u_0} + \sum_{\nu=1}^{k}\left(\zeta_\nu + \sum_{\mu=1}^{k}\alpha_{\nu\mu}u_\mu\right)\frac{\partial g}{\partial u_\nu}\right)\right.$$
$$\left. + 2g\left(\left(\dot{\zeta}_\lambda + \sum_{\mu=1}^{k}\dot{\alpha}_{\lambda\mu}u_\mu\right) + \sum_{\nu=1}^{k}\left(\zeta_\nu + \sum_{\mu=1}^{k}\alpha_{\nu\mu}u_\mu\right)\alpha_{\lambda\nu} + \frac{1}{2}\varepsilon_\lambda\frac{\partial g}{\partial u_\lambda}\right)\right],$$
$$\Gamma_{\nu\mu}^{0} = \Gamma_{\mu\nu}^{0} = 0, \tag{18}$$
$$\Gamma_{\nu\mu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} = 0,$$
$$\Gamma_{\lambda 0}^{0} = \Gamma_{0\lambda}^{0} = \frac{1}{2g}\frac{\partial g}{\partial u_\lambda},$$
$$\Gamma_{\nu 0}^{\lambda} = \Gamma_{0\nu}^{\lambda} = \frac{1}{2g}\left[-\left(\zeta_\lambda + \sum_{\mu=1}^{k}\alpha_{\lambda\mu}u_\mu\right)\frac{\partial g}{\partial u_\nu} + 2g\left(\alpha_{\lambda\nu}\right)\right].$$

Let $\{u_0, u_1, \ldots, u_k\}$ be a base of tangent space at the neighborhood of the coordinate systems $\{\partial_0, \partial_1, \ldots, \partial_k\}$, ($\frac{\partial}{\partial u_i} = \partial_i, 0 \le i \le k$) of $M$, then the Riemannian curvature tensor of $M$ becomes

$$R_{\partial_i \partial_j}\left(\partial_l\right) = \sum_{r=0}^{k} R_{lij}^{r} \partial_r$$

where the Riemannian curvature tensor's coefficients are

$$R_{lij}^{r} = \frac{\partial}{\partial u_i}\Gamma_{jl}^{r} - \frac{\partial}{\partial u_j}\Gamma_{il}^{r} - \sum_{s=0}^{k}\Gamma_{il}^{s}\Gamma_{js}^{r} + \sum_{s=0}^{k}\Gamma_{jl}^{s}\Gamma_{is}^{r}.$$

Therefore, the Riemannian-Christoffel curvature tensor of $M$ is

$$R_{hlij} = \sum_{r=0}^{k} g_{rh}\left(\frac{\partial}{\partial u_i}\Gamma_{jl}^{r} - \frac{\partial}{\partial u_j}\Gamma_{il}^{r} - \sum_{s=0}^{k}\Gamma_{il}^{s}\Gamma_{js}^{r} + \sum_{s=0}^{k}\Gamma_{jl}^{s}\Gamma_{is}^{r}\right). \tag{19}$$

In addition, there exist the following relations, [8]

$$R_{hlij} = R_{ijhl}$$
$$R_{ijhl} = -R_{jihl}. \tag{20}$$

Considering the equations (18) and (19), $R_{ij00}$, $R_{ij\nu\mu}$, $R_{\nu 0\mu 0}$ are found to be

$$\begin{aligned} &R_{ij00} = 0 &&, \quad 0 \le i, j \le k, \\ &R_{ij\nu\mu} = 0 &&, \quad 0 \le i, j \le k \ , \ 1 \le \nu, \mu \le k, \\ &R_{\nu 0\mu 0} = \frac{1}{2}\frac{\partial^2 g}{\partial u_\nu \partial u_\mu} - \frac{1}{4g}\frac{\partial g}{\partial u_\nu}\frac{\partial g}{\partial u_\mu} &&, \quad 1 \le \nu, \mu \le k. \end{aligned} \tag{21}$$

## 4. LORENTZIAN BELTRAMI-EULER FORMULA FOR GENERALIZED TIMELIKE RULED SURFACES WITH TIMELIKE GENERATING SPACE IN $n-$DIMENSIONAL MINKOWSKI SPACE, $\mathfrak{R}_1^n$

Two-dimensional subspace $\Pi$ of $(k+1)-$dimensional timelike ruled surface at the point $\xi \in T_M(\xi)$ is called the tangent section of $M$ at point $\xi$. If $\vec{v}$ and $\vec{w}$ form a basis of the tangent section $\Pi$, then $Q(\vec{v},\vec{w}) = \langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2$ is a nonzero quantity if and only if $\Pi$ is nondegenerate. This quantity represents the square of the Lorentzian area of the parallelogram determined by $\vec{v}$ and $\vec{w}$. Using the square of the Lorentzian area of the parallelogram determined by the basis vectors $\{\vec{v},\vec{w}\}$, one has the following classification for the tangent sections of the timelike ruled surfaces:

$$\begin{aligned} &Q(\vec{v},\vec{w}) = \langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2 < 0 \quad , \quad \text{(timelike plane)}, \\ &Q(\vec{v},\vec{w}) = \langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2 = 0 \quad , \quad \text{(degenerate plane)}, \\ &Q(\vec{v},\vec{w}) = \langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2 > 0 \quad , \quad \text{(spacelike plane)}. \end{aligned}$$

For the nondegenerate tangent section $\Pi$ given by the basis $\{\vec{v},\vec{w}\}$ of $M$ at the point $\xi$, the definition

$$K_\xi(\vec{v},\vec{w}) = \frac{\langle R_{\vec{v}\vec{w}}\vec{v},\vec{w}\rangle}{\langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2} \tag{22}$$

or

$$K(\vec{v},\vec{w}) = \frac{\sum R_{ijkm} w_i v_j w_k v_m}{\sum g_{ij} v_i v_j g_{km} w_k w_m - \left[\sum g_{ij} v_i w_j\right]^2} \tag{23}$$

is called the sectional curvature of $M$ at the point $\xi$, where $\vec{v} = \sum \beta_i \frac{\partial}{\partial x_i}$ and $\vec{w} = \sum \gamma_j \frac{\partial}{\partial x_j}$. Here the coordinates of the basis vectors $\vec{v}$ and $\vec{w}$ are $(\beta_0, \beta_1, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_k)$, respectively, [8].

Let the base curve of timelike ruled surface $M$ with the timelike generating space in $\mathfrak{R}_1^n$ be the base curve of the central ruled surface $\Omega$ of $M$. In this case, the normal tangential vector $n$ of $M$ that is orthogonal to $E_k(t)$ is defined as

$$n = \sum_{\sigma=1}^{m} u_\sigma \kappa_\sigma(t) a_{k+\sigma}(t) + \eta_{m+1} a_{k+m+1}(t), \qquad (\eta_{m+1} \ne 0) \tag{24}$$

at the point $\forall \xi(t,u_\nu)$ so that this normal tangential vector field is always spacelike since it is orthogonal to the generating space $E_k(t)$. In the principal frame $\{e_1(t), e_2(t), \ldots, e_{s-1}(t), e_s(t), e_{s+1}(t) \ldots, e_k(t)\}$ of generating space $E_k(t)$ there is only one timelike vector as follows

$$\langle e_s(t), e_s(t)\rangle = -1 \quad , \quad 1 \le s \le k.$$

Therefore,

$$\langle e_\nu, e_\nu\rangle\langle n,n\rangle - \langle e_\nu, n\rangle^2 > 0 \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne s,$$
$$\langle e_s, e_s\rangle\langle n,n\rangle - \langle e_s, n\rangle^2 < 0.$$

That is, the $\nu^{\text{th}}$ principal tangential section $(e_\nu, n)$, $1 \le \nu \le k$, $\nu \ne s$ with respect to the principal frame of $E_k(t)$ is the spacelike plane and $s^{\text{th}}$ principal tangential section $(e_s, n)$ is the timelike plane.

Considering these two cases separately, from equations (21), (23) and (24) the following theorem can be given related to the spacelike and timelike principal sectional curvatures at the point $\xi \in M$ .

**Theorem 4.1.** Let $M$ be a generalized timelike ruled surface with timelike generating space and central ruled surface in $n-$dimensional Minkowski space $\Re_1^n$. The $\nu^{\text{th}}$ principal sectional curvature of the spacelike section $(e_\nu, n)$, $1 \le \nu \le k$, $\nu \ne s$ and the $s^{\text{th}}$ principal sectional curvature of the timelike section $(e_s, n)$ at the point $\forall \xi \in M$ are given by

$$K_\xi(e_\nu, n) = -\frac{1}{2g}\frac{\partial^2 g}{\partial u_\nu^{\;2}} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\nu}\right)^2 \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne s \tag{25}$$

and

$$K_\xi(e_s, n) = \frac{1}{2g}\frac{\partial^2 g}{\partial u_s^{\;2}} - \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_s}\right)^2, \tag{26}$$

respectively, where $n$ is the spacelike normal tangential vector and $e_s$, $1 \le s \le k$, is timelike base vector in timelike generating space $E_k(t)$.

**Proof:** Let the coordinates of $e_\nu$ $(1 \le \nu \le k)$ and $n$, which form the basis of principal section $(e_\nu, n)$ with respect to the canonical basis given by equation (12) of tangential bundle of timelike ruled surface $M$ be $(\beta_0, \beta_1, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_k)$, respectively. Considering equations (22) we find that the $\nu^{\text{th}}$ principal sectional curvature of $(e_\nu, n)$ $(1 \le \nu \le k)$ becomes

$$K_\xi(e_\nu, n) = \frac{\beta_\nu \beta_\nu \gamma_0 \gamma_0 R_{\nu 0 \nu 0}}{\langle e_\nu, e_\nu\rangle\langle n,n\rangle - \langle e_\nu, n\rangle^2}.$$

If we substitute equations (21) and (24) into the last equation, we obtain

$$K_\xi(e_\nu, n) = \frac{\frac{1}{2}\frac{\partial^2 g}{\partial u_\nu^{\;2}} - \frac{1}{4g}\left(\frac{\partial g}{\partial u_\nu}\right)^2}{\langle e_\nu, e_\nu\rangle\langle n,n\rangle - \left\langle e_\nu, \sum_{\sigma=1}^{m} u_\sigma \kappa_\sigma a_{k+\sigma} + \eta_{m+1} a_{k+m+1}\right\rangle^2}.$$

Taking $e_s$ $(1 \le s \le k)$ to be the timelike vector in the generating space $E_k(t) = Sp\{e_1(t), e_2(t), \dots, e_s(t), \dots, e_k(t)\}$, the $\nu^{\text{th}}$ spacelike principal sectional curvature and the $s^{\text{th}}$ timelike principal sectional curvature are found to be

$$K_\xi(e_\nu, n) = \frac{\frac{1}{2}\frac{\partial^2 g}{\partial u_\nu^2} - \frac{1}{4g}\left(\frac{\partial g}{\partial u_\nu}\right)^2}{\sum_{\sigma=1}^{m}(u_\sigma \kappa_\sigma)^2 + \eta_{m+1}^2} \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne s$$

and

$$K_\xi(e_s, n) = \frac{\frac{1}{2}\frac{\partial^2 g}{\partial u_s^2} - \frac{1}{4g}\left(\frac{\partial g}{\partial u_s}\right)^2}{-\sum_{\sigma=1}^{m}(u_\sigma \kappa_\sigma)^2 - \eta_{m+1}^2},$$

respectively. Therefore, considering the last equation together with the equation (14) completes the proof.

**Corollary 4.1.** The sectional curvature of the $\nu^{\text{th}}$ nondegenerate (spacelike or timelike) principal section $(e_\nu, n)$ $(1 \le \nu \le k)$ of timelike ruled surface $M$ at the point $\forall \xi \in M$ is

$$K_\xi(e_\nu, n) = \varepsilon_\nu \left( -\frac{1}{2g}\frac{\partial^2 g}{\partial u_\nu^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\nu}\right)^2 \right) \quad , \quad 1 \le \nu \le k \tag{27}$$

where $\varepsilon_\nu = \langle e_\nu, e_\nu \rangle = \pm 1$.

From this point on we are going to name the sectional curvature of nondegenerate $\nu^{\text{th}}$ principal section $(e_\nu, n)$, $1 \le \nu \le k$, to be the $\nu^{\text{th}}$ principal sectional curvature timelike ruled surface $M$.

**Theorem 4.2.** Let $M$ be a generalized timelike ruled surface with timelike generating space and central ruled surface and $n$ be the spacelike normal tangent vector of $M$ in $n-$dimensional Minkowski space $\Re_1^n$. The $\sigma^{\text{th}}$ principal sectional curvature and the $(m+\rho)^{\text{th}}$ principal sectional curvature of $M$ at the point $\forall \xi \in M$ are

$$K_\xi(e_\sigma, n) = -\frac{\varepsilon_\sigma (\kappa_\sigma)^2 \left[ \sum_{\iota=1}^{m}(u_\iota \kappa_\iota)^2 + \eta_{m+1}^2 - (u_\sigma \kappa_\sigma)^2 \right]}{\left( \sum_{\iota=1}^{m}(u_\iota \kappa_\iota)^2 + \eta_{m+1}^2 \right)^2} \quad , \quad 1 \le \sigma \le m \tag{28}$$

and

$$K_\xi(e_{m+\rho}, n) = 0 \quad , \quad 1 \le \rho \le k - m \tag{29}$$

respectively, where $\varepsilon_\sigma = \langle e_\sigma, e_\sigma \rangle = \pm 1$.

**Proof:** Considering the equation (14) we see that

$$\frac{\partial g}{\partial u_\sigma} = -2u_\sigma \kappa_\sigma^2 \quad , \quad \frac{\partial^2 g}{\partial u_\sigma^2} = -2\kappa_\sigma^2 \quad , \quad \left(\frac{\partial g}{\partial u_\sigma}\right)^2 = 4\left(u_\sigma \kappa_\sigma^2\right)^2 \kappa_\sigma^2 \quad , \quad 1 \le \sigma \le m.$$

And

$$\frac{\partial g}{\partial u_{m+\rho}} = \frac{\partial^2 g}{\partial u_{m+\rho}^2} = \left(\frac{\partial g}{\partial u_{m+\rho}}\right)^2 = 0 \quad , \quad 1 \le \rho \le k-m.$$

Substituting these equations into the equation (27) we find the $\sigma^{\text{th}}$ nondegenerate principal sectional curvature to be

$$K_\xi\left(e_\sigma, n\right) = -\varepsilon_\sigma \frac{-2\left(\kappa_\sigma\right)^2}{-2\left(\sum_{\iota=1}^{m}\left(u_\iota \kappa_\iota\right)^2 + \eta_{m+1}^2\right)} + \varepsilon_\sigma \frac{4\left(u_\sigma \kappa_\sigma\right)^2\left(\kappa_\sigma\right)^2}{4\left(\sum_{\iota=1}^{m}\left(u_\iota \kappa_\iota\right)^2 + \eta_{m+1}^2\right)^2} \quad , \quad 1 \le \sigma \le m.$$

After simple calculations we reach equation (28). Similarly, when we calculate the $\left(m+\rho\right)^{\text{th}}$ principal sectional curvature, we find

$$K_\xi\left(e_{m+\rho}, n\right) = 0 \quad , \quad 1 \le \rho \le k-m.$$

Therefore, we give the following corollary.

**Corollary 4.2.** Let $M$ be a generalized timelike ruled surface with timelike generating space and central ruled surface in $n-$dimensional Minkowski space $\Re_1^n$. At the point $\forall \xi \in M$, the $\left(m+\rho\right)^{\text{th}}$ $\left(1 \le \rho \le k-m\right)$ principal sectional curvature of $M$ is equal to zero.

**Theorem 4.3.** Let $M$ a be generalized timelike ruled surface with timelike generating space and central ruled surface and $P_\sigma$, $1 \le \sigma \le m$, be the $\sigma^{\text{th}}$ principal distribution parameter of $M$ in $n-$dimensional Minkowski space $\Re_1^n$. At the central point $\forall \zeta \in M$, $\sigma^{\text{th}}$ principal sectional curvature and $\left(m+\rho\right)^{\text{th}}$ principal sectional curvature of $M$ are

$$K_\zeta\left(e_\sigma, n\right) = -\varepsilon_\sigma \frac{1}{P_\sigma^2} \quad , \quad 1 \le \sigma \le m$$

and

$$K_\zeta\left(e_{m+\rho}, n\right) = 0 \quad , \quad 1 \le \rho \le k-m$$

respectively.

**Proof:** Taking $\zeta \in \Omega$ to be the central point and considering equations (25) and (26) and after simplification we find the $\sigma^{\text{th}}$ sectional curvature of $M$ to be

$$K_\zeta\left(e_\sigma, n\right) = -\varepsilon_\sigma \left(\frac{\kappa_\sigma}{\eta_{m+1}}\right)^2 \quad , \quad 1 \le \sigma \le m.$$

Considering the last equation together with the equation (11) and Corollary 4.2 completes the proof.

Now, we take $e$ to be a unit vector in generating space $E_k(t)$ of ruled surface $M$, and $n$ to be a spacelike normal tangent vector orthogonal to $E_k(t)$. Let us elaborate on the curvature of tangential section $(e,n)$.
Since unit vector $e(t)$ is in $E_k(t)$, we write

$$e(t)\in Sp\{e_1(t),\ldots,e_m(t),e_{m+1}(t),\ldots,e_k(t)\}.$$

So

$$e(t)=\sum_{\sigma=1}^{m}\lambda_\sigma e_\sigma(t)+\sum_{\rho=m+1}^{k}\lambda_\rho e_\rho(t) \quad , \quad \|e(t)\|=1.$$

As $E_k(t)$ is a timelike subspace it contains a timelike vector. Since $F_m(t)=Sp\{e_1(t),\ldots,e_m(t)\}$ and $Z_{k-m}(t)=Sp\{e_{m+1}(t),\ldots,e_k(t)\}$, this timelike vector stays either in the subspace $F_m(t)$ or in the central space $Z_{k-m}(t)$. That is, if the central space $Z_{k-m}(t)$ is spacelike, the space $F_m(t)$ is timelike or if $Z_{k-m}(t)$ is timelike then the space $F_m(t)$ is spacelike. In this case the central ruled surface $\Omega$ of timelike ruled surface $M$ is either spacelike or timelike. Furthermore, the unit vector $e(t)$ is either spacelike or timelike.

Therefore, there exist the following cases (a1, a2, b1, b2). Now we consider these situations separately.

***(a1) Central space $Z_{k-m}(t)$ and unit vector $e(t)$ are spacelike***

Let $e_s$, $1\le s\le m$, be a timelike vector in $F_m(t)$. In this case the spacelike unit vector $e$ can be written as follows:

$$\begin{aligned} e&=\sum_{\sigma=1}^{s-1}\cosh\theta_\sigma e_\sigma+\sinh\theta_s e_s+\sum_{\sigma=s+1}^{m}\cosh\theta_\sigma e_\sigma+\sum_{\rho=m+1}^{k}\cosh\theta_\rho e_\rho \\ &=\sum_{\substack{\nu=1\\ \nu\neq s}}^{k}\cosh\theta_\nu e_\nu+\sinh\theta_s e_s. \end{aligned} \tag{30}$$

It is clear here that

$$\sum_{\substack{\nu=1\\ \nu\neq s}}^{k}\cosh^2\theta_\nu-\sinh^2\theta_s=1$$

where the angles $\theta_1,\theta_2,\ldots,\theta_s,\ldots,\theta_k$ are the hyperbolic angles between spacelike unit vector $e$ and base vectors $e_1,e_2,\ldots,e_s,\ldots,e_k$, respectively.

***(a2) Central space $Z_{k-m}(t)$ is spacelike and unit vector $e(t)$ is timelike***

Let $e_s$, $1\le s\le m$, be a timelike vector in $F_m(t)$. In this case the timelike unit vector $e$ can be written as

$$\begin{aligned} e&=\sum_{\sigma=1}^{s-1}\sinh\theta_\sigma e_\sigma+\cosh\theta_s e_s+\sum_{\sigma=s+1}^{m}\sinh\theta_\sigma e_\sigma+\sum_{\rho=m+1}^{k}\sinh\theta_\rho e_\rho \\ &=\sum_{\substack{\nu=1\\ \nu\neq s}}^{k}\sinh\theta_\nu e_\nu+\cosh\theta_s e_s. \end{aligned} \tag{31}$$

In addition,

$$\sum_{\substack{\nu=1\\ \nu\neq s}}^{k}\sinh^2\theta_\nu-\cosh^2\theta_s=-1$$

where the angles $\theta_1,\theta_2,\ldots,\theta_s,\ldots,\theta_k$ are the hyperbolic angles between timelike unit vector $e$ and base vectors $e_1,e_2,\ldots,e_s,\ldots,e_k$, respectively.

***(b1) Central space $Z_{k-m}(t)$ is timelike and unit vector $e(t)$ is spacelike***

Let $e_{m+s}$, $1\le s\le k-m$, be a timelike vector in the central space $Z_{k-m}(t)$. In this situation the spacelike unit vector $e$ can be defined as

$$\begin{aligned} e&=\sum_{\sigma=1}^{m}\cosh\theta_\sigma e_\sigma+\sum_{\rho=m+1}^{m+s-1}\cosh\theta_\rho e_\rho+\sinh\theta_{m+s}e_{m+s}+\sum_{\rho=m+s+1}^{k}\cosh\theta_\rho e_\rho\\ &=\sum_{\substack{\nu=1\\ \nu\neq m+s}}^{k}\cosh\theta_\nu e_\nu+\sinh\theta_{m+s}e_{m+s}\end{aligned}\tag{32}$$

and

$$\sum_{\substack{\nu=1\\ \nu\neq m+s}}^{k}\cosh^2\theta_\nu-\sinh^2\theta_{m+s}=1$$

so that the angles $\theta_1,\theta_2,\ldots,\theta_{m+s},\ldots,\theta_k$ are hyperbolic angles between spacelike unit vector $e$ and base vectors $e_1,e_2,\ldots,e_{m+s},\ldots,e_k$, respectively.

***(b2) Central space $Z_{k-m}(t)$ and unit vector $e(t)$ are timelike***

Let $e_{m+s}$, $1\le s\le k-m$, be a timelike vector in the central space $Z_{k-m}(t)$. In this case write $e$ timelike unit vector can be written as

$$\begin{aligned} e&=\sum_{\sigma=1}^{m}\sinh\theta_\sigma e_\sigma+\sum_{\rho=m+1}^{m+s-1}\sinh\theta_\rho e_\rho+\cosh\theta_{m+s}e_{m+s}+\sum_{\rho=m+s+1}^{k}\sinh\theta_\rho e_\rho\\ &=\sum_{\substack{\nu=1\\ \nu\neq m+s}}^{k}\sinh\theta_\nu e_\nu+\cosh\theta_{m+s}e_{m+s}\end{aligned}\tag{33}$$

and

$$\sum_{\substack{\nu=1\\ \nu\neq m+s}}^{k}\sinh^2\theta_\nu-\cosh^2\theta_{m+s}=-1$$

where the hyperbolic angles between timelike vector $e$ and base vectors $e_1,e_2,\ldots,e_{m+s},\ldots,e_k$ are $\theta_1,\theta_2,\ldots,\theta_{m+s},\ldots,\theta_k$, respectively.

Thus, we can give the following theorems for curvatures of the tangential section $(e,n)$ for (a1, a2, b1, b2) cases, respectively.

**Theorem 4.4.** Let $M$ be generalized timelike ruled surface with the spacelike central ruled surface in $n-$dimensional Minkowski space $\Re_1^n$ and $e$ be a spacelike unit vector in $E_k(t)$, taking $n$ to be a normal tangential vector orthogonal to $E_k(t)$ of $M$. The following relation exists between the sectional curvature of spacelike section $(e,n)$ and principal sectional curvatures at the point $\zeta \in \Omega \subset M$

$$K_\zeta(e,n) = \sum_{\substack{\sigma=1 \\ \sigma\neq s}}^{m} \cosh^2\theta_\sigma K_\zeta(e_\sigma,n) - \sinh^2\theta_s K_\zeta(e_s,n) \tag{34}$$

where $e_s$, $1 \le s \le m$, is a timelike vector in subspace $F_m(t)$ and the hyperbolic angles between spacelike unit vector $e$ and base vectors $e_1, e_2, \ldots, e_s, \ldots, e_k$ are $\theta_1, \theta_2, \ldots, \theta_s, \ldots, \theta_k$ ,respectively, so that

$$e = \sum_{\substack{\nu=1 \\ \nu\neq s}}^{k} \cosh\theta_\nu e_\nu + \sinh\theta_s e_s \quad \textit{and} \quad \sum_{\substack{\nu=1 \\ \nu\neq s}}^{k} \cosh^2\theta_\nu - \sinh^2\theta_s = 1.$$

**Proof:** Let the coordinates of the spacelike unit vector $e$ within the generating space $E_k(t)$ be $(\beta_0, \beta_1, \ldots, \beta_k)$ and the coordinates of spacelike normal tangent vector $n$ be $(\gamma_0, \gamma_1, \ldots, \gamma_k)$. From equations (3), (4), (22) and (23), we see that at the central point $\zeta \in \Omega$

$$K_\zeta(e,n) = \frac{\sum_{\substack{\sigma=1 \\ \sigma\neq s}}^{m} \cosh^2\theta_\sigma R_{\sigma 0\sigma 0} + \sinh^2\theta_s R_{s0s0}}{\langle e,e\rangle\langle n,n\rangle - \langle e,n\rangle^2}$$

with the last equation in mind we find from equation (21)

$$K_\zeta(e,n) = \sum_{\substack{\sigma=1 \\ \sigma\neq s}}^{m} \cosh^2\theta_\sigma \left[ -\frac{1}{2g}\frac{\partial^2 g}{\partial {u_\sigma}^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2 \right] + \sinh^2\theta_s \left[ -\frac{1}{2g}\frac{\partial^2 g}{\partial {u_s}^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_s}\right)^2 \right].$$

If one considers the equations (25) and (26) at the central point $\forall \zeta \in \Omega$, there exists a relation between sectional curvature of spacelike section $(e,n)$ and the principal sectional curvatures of $M$ which we gave in the equation (34).

This relation is called **I. type Lorentzian Beltrami-Euler formula** for the spacelike section of generalized timelike ruled surface with spacelike central ruled surface at the central point $\zeta \in \Omega$.

**Theorem 4.5.** Let $M$ be generalized timelike ruled surface with spacelike central ruled surface in $n-$dimensional Minkowski space $\Re_1^n$ and $e$ be a timelike unit vector in $E_k(t)$, taking $n$ to be a normal tangential vector orthogonal to $E_k(t)$ of $M$. In this case the relation between the sectional curvature of timelike section $(e,n)$ and principal sectional curvatures is

$$K_\zeta(e,n) = -\sum_{\substack{\sigma=1 \\ \sigma\neq s}}^{m} \sinh^2\theta_\sigma K_\zeta(e_\sigma,n) + \cosh^2\theta_s K_\zeta(e_s,n) \tag{35}$$

at the point $\zeta \in \Omega \subset M$, where $e_s$, $1 \le s \le m$, is a timelike vector in the subspace $F_m(t)$ and the angles $\theta_1, \theta_2, \ldots, \theta_s, \ldots, \theta_k$ represent the hyperbolic angles between the timelike unit vector $e$ and the base vectors $e_1, e_2, \ldots, e_s, \ldots, e_k$, respectively, so that the following equations hold

$$e=\sum_{\substack{\nu=1\\ \nu\neq s}}^{k}\sinh\theta_\nu e_\nu+\cosh\theta_s e_s \quad and \quad \sum_{\substack{\nu=1\\ \nu\neq s}}^{k}\sinh^2\theta_\nu-\cosh^2\theta_s=-1.$$

**Proof:** Let the coordinates of the timelike unit vector $e$ in the generating space $E_k(t)$ of $M$ and the coordinates of the spacelike normal tangent vector $n$ be $(\beta_0,\beta_1,\ldots,\beta_k)$ and $(\gamma_0,\gamma_1,\ldots,\gamma_k)$, respectively, in $\Re_1^n$. Thus, if we consider the equations (3) and (4) we see that

$$\beta_0=0 \quad , \quad \beta_s=\cosh\theta_s \quad , \quad 1\le s\le m$$
$$\beta_\nu=\sinh\theta_\nu \quad , \quad 1\le\nu\le k \quad , \quad \nu\neq s$$

and

$$\gamma_0=\langle n,e_0\rangle=1 \quad , \quad \gamma_\nu=\langle n,e_\nu\rangle=0 \quad , \quad 1\le\nu\le k.$$

If we substitute these equations into equation (23) for the central point $\zeta\in\Omega$ and consider equation (21) we reach

$$K_\zeta(e,n)=\sum_{\substack{\sigma=1\\ \sigma\neq s}}^{m}\sinh^2\theta_\sigma\left[\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma{}^2}-\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2\right]+\cosh^2\theta_s\left[\frac{1}{2g}\frac{\partial^2 g}{\partial u_s{}^2}-\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_s}\right)^2\right].$$

Taking into consideration the equations (25) and (26) at the central point $\forall\zeta\in\Omega$ completes the proof.

The equation (35) is called **II. type Lorentzian Beltrami-Euler formula** for the timelike sectional curvature of generalized timelike ruled surface with spacelike central ruled surface at the central point $\zeta\in\Omega$.

**Theorem 4.6.** Let $M$ be generalized timelike ruled surface with timelike central ruled surface, $e$ be a spacelike unit vector within $E_k(t)$ and $n$ be a normal tangential vector orthogonal to $E_k(t)$ of $M$ in $n-$ dimensional Minkowski space $\Re_1^n$. In this case the following relation between the sectional curvature of spacelike section $(e,n)$ and the principal sectional curvatures at the central point $\zeta\in\Omega\subset M$ are held

$$K_\zeta(e,n)=\sum_{\sigma=1}^{m}\cosh^2\theta_\sigma K_\zeta(e_\sigma,n) \tag{36}$$

where $e_{m+s}$, $1\le s\le k-m$, is a timelike vector in the central space $Z_{k-m}(t)$ and the hyperbolic angles between spacelike unit vector $e$ and base vectors $e_1,e_2,\ldots,e_{m+s},\ldots,e_k$ are $\theta_1,\theta_2,\ldots,\theta_{m+s},\ldots,\theta_k$, respectively. In this case the following relations are held

$$e=\sum_{\substack{\nu=1\\ \nu\neq m+s}}^{k}\cosh\theta_\nu e_\nu+\sinh\theta_{m+s}e_{m+s} \quad and \quad \sum_{\substack{\nu=1\\ \nu\neq m+s}}^{k}\cosh^2\theta_\nu-\sinh^2\theta_{m+s}=1.$$

**Proof:** Let the coordinates of the spacelike unit vector $e$ in the generating space of $M$ and the coordinates of the spacelike normal tangent vector $n$ be $(\beta_0,\beta_1,\ldots,\beta_k)$ and $(\gamma_0,\gamma_1,\ldots,\gamma_k)$, respectively in $\Re_1^n$. In a similar manner, using equations (3), (4), (21), (23) and (25) at the point $\forall\zeta\in\Omega$ we reach the relation

$$K_\zeta(e,n)=\sum_{\sigma=1}^{m}\cosh^2\theta_\sigma K_\zeta(e_\sigma,n)$$

in between the sectional curvature of spacelike section $(e,n)$ and the principal sectional curvature of $M$. This relation is called **III. type Lorentzian Beltrami-Euler formula** for the spacelike sectional curvature of generalized timelike ruled surface with timelike central ruled surface $M$ at the central point $\zeta\in\Omega$.

**Theorem 4.7.** Let $M$ be generalized timelike ruled surface with timelike central ruled surface, $e$ be a timelike unit vector in $E_k(t)$ and $n$ be a normal tangential vector orthogonal to $E_k(t)$ of $M$ in $n-$dimensional Minkowski space $\Re_1^n$. The relation

$$K_\zeta(e,n)=-\sum_{\sigma=1}^{m}\sinh^2\theta_\sigma K_\zeta(e_\sigma,n) \tag{37}$$

holds between the sectional curvature of timelike section $(e,n)$ and the principal sectional curvatures at the central point $\zeta\in\Omega$, where $e_{m+s}$, $1\le s\le k-m$, is a timelike vector in the central space $Z_{k-m}(t)$ and the hyperbolic angles between timelike unit vector $e$ and base vectors $e_1,e_2,\dots,e_{m+s},\dots,e_k$ are $\theta_1,\theta_2,\dots,\theta_{m+s},\dots,\theta_k$, respectively, so that

$$e=\sum_{\substack{\nu=1\\ \nu\neq m+s}}^{k}\sinh\theta_\nu e_\nu+\cosh\theta_{m+s}e_{m+s}\quad and \quad \sum_{\substack{\nu=1\\ \nu\neq m+s}}^{k}\sinh^2\theta_\nu-\cosh^2\theta_{m+s}=-1.$$

**Proof:** Let the coordinates of the timelike unit vector $e$ and the spacelike normal tangent vector $n$ in $\Re_1^n$ be $(\beta_0,\beta_1,\dots,\beta_k)$ and $(\gamma_0,\gamma_1,\dots,\gamma_k)$, respectively, taking $e_{m+s}$, $1\le s\le k-m$, to be a timelike vector within the central space $Z_{k-m}(t)$ and the hyperbolic angles between timelike unit vector $e$ and base vectors $e_1,e_2,\dots,e_{m+s},\dots,e_k$ to be $\theta_1,\theta_2,\dots,\theta_{m+s},\dots,\theta_k$, respectively. From the equations (3), (4), (21), (23) and (26) one can easily see that equation (37) holds between the timelike sectional curvature $(e,n)$ and the principal sectional curvatures of $M$ at the point $\zeta\in\Omega$. The equation (37) is called **IV. type Lorentzian Beltrami-Euler formula** for the timelike sectional curvature of generalized timelike ruled surface $M$ with timelike central ruled surface at the central point $\zeta\in\Omega$.

**Example 4.1.** Let $\Re_1^5$ be a five-dimensional Minkowski space, given by Lorentz metric

$$\left\langle \vec{X},\vec{Y}\right\rangle = x_1y_1+x_2y_2+x_3y_3+x_4y_4-x_5y_5$$

where $\vec{X}=(x_1,x_2,x_3,x_4,x_5),\vec{Y}=(y_1,y_2,y_3,y_4,y_5)\in\Re^5$. Taking $\kappa$, $\tau$ and $\varepsilon=\sqrt{\kappa^2+\tau^2}$ to be arbitrary constants, a spacelike curve $\alpha:I\to\Re_1^5$ is given as follows;

$$\alpha(t)=\frac{1}{\varepsilon}\Big(2\tau\varepsilon t,\sqrt{3}\kappa\cosh\varepsilon t+\kappa\sinh\varepsilon t+\varepsilon\tau t,2\kappa\varepsilon t,\sqrt{3}\tau\cosh\varepsilon t+\tau\sinh\varepsilon t-\kappa\varepsilon t,\sqrt{3}\varepsilon\sinh\varepsilon t+\varepsilon\cosh\varepsilon t\Big).$$

In this case, the orthonormal vector field system $\{e_1(t),e_2(t)\}$ defined at every point of curve $\alpha$ is expressed as

$$e_1(t)=\frac{1}{\varepsilon}\Big(\kappa+\tau,\ \sqrt{3}\kappa\sinh\varepsilon t,\ \kappa-\tau,\ \sqrt{3}\tau\sinh\varepsilon t,\ \sqrt{3}\varepsilon\cosh\varepsilon t\Big),$$
$$e_2(t)=\frac{1}{\sqrt{3}\varepsilon}\big(\tau-\kappa,\ \kappa\cosh\varepsilon t,\ \kappa+\tau,\ \tau\cosh\varepsilon t,\ \varepsilon\sinh\varepsilon t\big).$$

This system spans a $2-$dimensional subspace of tangent space at the point $\alpha(t)\in\Re_1^5$ and this subspace is denoted by $E_2(t)$. It is clear that $E_2(t)$ is timelike subspace. The parametrization

$$\varphi(t,u_1,u_2)=\alpha(t)+\sum_{v=1}^{2}u_v e_v(t)$$

defines a $3-$dimensional timelike ruled surface denoted by $M$, with the spacelike base curve and the timelike generating space. Therefore, taking $\{e_1(t),e_2(t)\}$ to be the principal frame of the generating space $E_2(t)$ and $\{e_1(t),e_2(t),\dot{e}_1(t),\dot{e}_2(t),\dot{\alpha}(t)\}$ to be the base of the tangential bundle of $M$ and by using the Gramm-Schmidth method, we reach the orthonormal base vectors as follows;

$$\begin{aligned}
a_3(t)&=\frac{1}{\sqrt{6}\varepsilon}\left(\kappa-\tau,\ 2\kappa\cosh\varepsilon t,\ -\kappa-\tau,\ 2\tau\cosh\varepsilon t,\ 2\varepsilon\sinh\varepsilon t\right),\\
a_4(t)&=-\frac{1}{\sqrt{2}\varepsilon}\left(\sqrt{3}(\kappa+\tau),\ 2\kappa\sinh\varepsilon t,\ \sqrt{3}(\kappa-\tau),\ 2\tau\sinh\varepsilon t,\ 2\varepsilon\cosh\varepsilon t\right),\\
a_5(t)&=\frac{1}{\varepsilon}(0,\ \tau,\ 0,\ -\kappa,\ 0).
\end{aligned}$$

That is, there exists an orthonormal base $\{e_1(t),e_2(t),a_3(t),a_4(t),a_5(t)\}$ for the tangential bundle of $M$. The differential equation of the principal frame of the generating space of $M$ and the velocity vector of the base curve $\alpha$ are given by the following equations;

$$\begin{aligned}
\dot{e}_1(t)&=\varepsilon e_2(t)+\sqrt{2}\varepsilon a_3(t),\\
\dot{e}_2(t)&=\varepsilon e_1(t)+\sqrt{\frac{2}{3}}\varepsilon a_4(t),\\
\dot{\alpha}&=-\varepsilon e_1+\sqrt{3}\varepsilon e_2+\varepsilon a_5,
\end{aligned}$$

respectively. Thus the first fundamental form's regular matrix of $3-$dimensional timelike ruled surface $M$ with timelike generating space becomes

$$\left[g_{ij}\right]_{3\times 3}=\begin{bmatrix}
2\sqrt{3}\varepsilon^2u_1-2\varepsilon^2u_2+3\varepsilon^2u_1^2-\frac{\varepsilon^2}{3}u_2^2+3\varepsilon^2 & -\varepsilon-\varepsilon u_2 & \sqrt{3}\varepsilon+\varepsilon u_1\\
-\varepsilon-\varepsilon u_2 & -1 & 0\\
\sqrt{3}\varepsilon+\varepsilon u_1 & 0 & -1
\end{bmatrix}$$

so that, the determinant of the first fundamental form of $M$ is

$$g=\det\left[g_{ij}\right]=-2\varepsilon^2u_1^2-\frac{2}{3}\varepsilon^2u_2^2-\varepsilon^2. \tag{38}$$

The normal tangent vector of the timelike ruled surface $M$, which is orthogonal to the generating space $E_2(t)$, is obtained at the point $\forall\xi\in M$ to be

$$n=\sqrt{2}\varepsilon u_1a_3+\sqrt{\frac{2}{3}}\varepsilon u_2a_4+\varepsilon a_5.$$

Here, it is quite obvious that the normal tangent vector $n$ is a spacelike vector. In this example, the first principal tangential section $(e_1, n)$ is a timelike plane while the second principal tangential section $(e_2, n)$ becomes a spacelike plane. First and second principal sectional curvatures can be expressed as

$$K_\xi(e_1, n) = \frac{1}{2g}\frac{\partial^2 g}{\partial u_1^{\ 2}} - \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_1}\right)^2$$

and

$$K_\xi(e_2, n) = -\frac{1}{2g}\frac{\partial^2 g}{\partial u_2^{\ 2}} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_2}\right)^2.$$

If we substitute the first and the second order partial differentials of equation (38) into these last two equations we find curvatures of the first principal timelike tangential section $(e_1, n)$ and of the second principal spacelike tangential section $(e_2, n)$ as

$$K_\xi(e_1, n) = \frac{12u_2^2 + 18}{\left(6u_1^2 + 2u_2^2 + 3\right)^2}$$

and

$$K_\xi(e_2, n) = -\frac{12u_1^2 + 6}{\left(6u_1^2 + 2u_2^2 + 3\right)^2}$$

respectively. Furthermore, since $u_1 = u_2 = 0$ at the central point $\forall \zeta \in \Omega$, $K_\zeta(e_1, n)$ and $K_\zeta(e_2, n)$ becomes

$$K_\zeta(e_1, n) = 2 \quad \text{and} \quad K_\zeta(e_2, n) = -\frac{2}{3}.$$

In addition to these, since the first and second principal distribution parameter of $M$ at the point $\forall \zeta \in \Omega$

$$P_1 = \frac{\varepsilon}{\sqrt{2}\varepsilon} = \frac{\sqrt{2}}{2} \quad \text{and} \quad P_2 = \frac{\varepsilon}{\sqrt{\frac{2}{3}}\varepsilon} = \frac{\sqrt{6}}{2}$$

we reach

$$K_\zeta(e_1, n) = \frac{1}{P_1^2} = 2 \quad \text{and} \quad K_\zeta(e_2, n) = -\frac{1}{P_2^2} = -\frac{2}{3}.$$

Which is the same result as before. Let $e$ be an arbitrary unit vector within the timelike subspace $E_2(t)$. It is clear that unit vector $e$ is either a spacelike or a timelike vector.

First, we suppose that $e$ is a spacelike unit vector in $E_2(t)$. Taking angles between spacelike unit vector $e$ and base vectors $e_1$, $e_2$ to be $\theta_1$, $\theta_2$, we write the following relations

$$e = \sinh\theta_1 e_1 + \cosh\theta_2 e_2 \quad , \quad -\sinh^2\theta_1 + \cosh^2\theta_2 = 1.$$

In this case, from the I. type Lorentzian Beltrami-Euler formula, the curvature of spacelike tangential section $(e,n)$ of $M$ at the central point is found to be

$$K_\zeta(e,n) = -2\sinh^2\theta_1 - \frac{2}{3}\cosh^2\theta_2.$$

Secondly, we suppose that the unit vector $e^*$ is a timelike unit vector in $E_2(t)$ and the angles between timelike unit vector $e^*$ and base vectors $e_1$, $e_2$ are the hyperbolic angles $\theta_1^*$, $\theta_2^*$, respectively. In this case timelike unit vector $e^*$ is written in the form

$$e^* = \cosh\theta_1^* e_1 + \sinh\theta_2^* e_2 \quad , \quad -\cosh^2\theta_1^* + \sinh^2\theta_2^* = -1.$$

From the last equations and II. type Lorentzian Beltrami-Euler formula, the curvature of timelike tangential section $(e^*,n)$ of $M$ at the central point is found to be

$$K_\zeta(e^*,n) = 2\cosh^2\theta_1^* + \frac{2}{3}\sinh^2\theta_2^*.$$

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