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λ −STRONGLY SUMMABLE AND λ −STATISTICALLY CONVERGENT FUNCTIONS*

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Abstract – In this study, by using the notion of (V, λ)-summability, we introduce and study the concepts of λ-strongly summable and λ-statistiacally convergent functions.

Keywords – Statistical convergence, strongly summable function

1. INTRODUCTION

The idea of statistical convergence which is closely related to the concept of natural density or asymptotic density of a subset of the set of natural numbers ℕ, was first introduced by Fast [1]. The concept of statistical convergence plays an important role in the summability theory and functional analysis. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg [2]. In [3], Borwein introduced and studied strongly summable functions.

Strongly summable number sequences and statistically convergent number sequences were studied in [4] and [5], respectively. In [6], λ −statistically convergent number sequences was defined. In this paper, by taking real valued functions $x(t)$ measurable (in the Lebesque sense) in the interval $(1,\infty)$ instead of sequences we will introduce λ–strongly summable and λ -statistically convergent functions and give some inclusion relations.

Let $\lambda=(\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$. Λ denote the set of all such sequences. For a sequence $x=(x_k)$ the generalized de la Vallee Poussin mean is defined by

$$t_n(x):=\frac{1}{n}\sum_{k\in I_n}x_k,$$

where $I_n=[n-\lambda_n+1,n]$.
A sequence $x=(x_k)$ is said to be (V,λ)- summable to a number l if $t_n(x)\to l$ as $n\to\infty$. If $\lambda_n=n$, then (V,λ) −summability reduces to $(C,1)$ summability.

2 λ-STRONGLY SUMMABLE FUNCTIONS

A real valued function $x(t)$, measurable(in the Lebesque sense) in the interval $(1,\infty)$, is said to be strongly summable to l if

$$\lim_{n\to\infty}\frac{1}{n}\int_1^n|x(t)-l|dt=0.$$

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$[W]$ will denote the space of all strongly summable functions.
Also, W will denote the space of $x(t)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\int_1^n x(t)dt = l.$$

In this section we will introduce λ-strongly summable function.

Definition 2.1 Let $\lambda \in \Lambda$ and $x(t)$ be a real valued function which is measurable (in the Lebesque sense) in the interval $(1,\infty)$, if

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\int_{n-\lambda_n+1}^n |x(t)-l|dt = 0.$$

Then we say that the function $x(t)$ is λ–strongly summable to l. In this case we write $[W,\lambda] - lim\, x(t) = l$ and

$$[W,\lambda] := \{x(t):\ \exists\ l = l_x, [W,\lambda] - lim\, x(t) = l\}.$$

If $\lambda_n = n$, then $[W,\lambda]$ is the same as $[W]$.

3 λ −STATISTICALLY CONVERGENT FUNCTIONS

$x(t)$ is a real valued function which is measurable(in the Lebesque sense) in the interval $(1,\infty)$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{t \le n:\ \ |x(t)-l| \ge \varepsilon\}| = 0$$

then we say that the function $x(t)$ is statistically convergent to l. Where the vertical bars indicate the Lebesque measure of the enclosed set. In this case we write $S -\ lim\, x(t) = l$ and

$$S := \{x(t):\ \exists\ l = l_x, S -\ lim\, x(t) = l\ \}.$$

Definition 3.1. Let $\lambda \in \Lambda$ and $x(t)$be a real valued function which is measurable (in the Lebesque sense) in the interval $(1,\infty)$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{t \in I_n:\ \ |x(t)-l| \ge \varepsilon\}| = 0$$

then we say that the function $x(t)$ is λ-statistically convergent to l. In this case we write $S_\lambda -\ lim\, x(t) = l$ and

$$(S,\lambda) := \{x(t):\ \exists\ l = l_x, S_\lambda\text{-}\ lim\, x(t) = l\}.$$

If $\lambda_n = n$, then (S,λ) is the same as S, the set of statistically convergent functions.

Theorem 3.1. Let $\lambda \in \Lambda$ and $x(t)$ be a real valued function which is measurable (in the Lebesque sense) in the interval $(1,\infty)$, then
(i) $[W,\lambda] \subset (S,\lambda)$ and the inclusion is proper.
(ii) If $x(t)$ is bounded and S_λ-$x(t)$ =l then $[W,\lambda] - lim\, x(t) = l$ and hence $W - lim\, x(t) = l$ provided $x(t)$ is not eventually constant.
(iii) If $x(t)$ is bounded then $(S,\lambda) = [W,\lambda]$.

Proof: (i) Let $\varepsilon > 0$ and $[W, \lambda] - lim\, x(t) = l$. We write

$$\int_{t\in I_n} |x(t)-l|\, dt \geq \int_{\substack{t\in I_n \\ |x(t)-l|\geq\varepsilon}} |x(t)-l|\, dt \geq \epsilon |\{t \in I_n: \ |x(t)-l| \geq \varepsilon\}|.$$

Therefore $[W, \lambda] - lim\, x(t) = l$ implies S_λ-$lim\, x(t) = l$.
Define a function $x(t)$ by

$$x(t) = \begin{cases} t, & n-(\lambda_n)^{\frac{1}{2}}+1 \leq t \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Then $x(t)$ is not a bounded function and for every $\varepsilon (0 < \varepsilon \leq 1)$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{t \in I_n: \ |x(t)-0| \geq \varepsilon\}| = \lim_{n\to\infty} \frac{(\lambda_n)^{\frac{1}{2}}}{\lambda_n} = 0,$$

i.e., S_λ-$lim\, x(t) = 0$. But

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \int_{n-\lambda_n+1}^{n} |x(t)-0| dt = \infty,$$

i.e., $x(t) \notin [W, \lambda]$. Therefore, the inclusion is proper.
(ii) Suppose that S_λ-$lim\, x(t) = l$ and $x(t)$ be a bounded function, say $|x(t)-l| \leq M$ for all t. Given $\varepsilon > 0$, we have that

$$\frac{1}{\lambda_n} \int_{t\in I_n} |x(t)-l|\, dt = \frac{1}{\lambda_n} \int_{\substack{t\in I_n \\ |x(t)-l|\geq\varepsilon}} |x(t)-l|\, dt + \frac{1}{\lambda_n} \int_{\substack{t\in I_n \\ |x(t)-l|<\varepsilon}} |x(t)-l|\, dt$$
$$\leq \frac{M}{\lambda_n} |\{t \in I_n: \ |x(t)-l| \geq \varepsilon\}| + \varepsilon$$

which implies that $[W, \lambda] - lim\, x(t) = l$.
Also, we have, since $\frac{\lambda_n}{n} \leq 1$ for all n,

$$\frac{1}{n} \int_1^n (x(t)-l)\, dt = \frac{1}{n} \int_1^{n-\lambda_n} (x(t)-l)\, dt + \frac{1}{n} \int_{t\in I_n} (x(t)-l)\, dt$$
$$\leq \frac{1}{n} \int_1^{n-\lambda_n} |x(t)-l|\, dt + \frac{1}{n} \int_{t\in I_n} |x(t)-l|\, dt$$
$$\leq \frac{2}{\lambda_n} \int_{t\in I_n} |x(t)-l|\, dt.$$

Hence $[W] - lim\, x(t) = l$, since $[W, \lambda] - lim\, x(t) = l$.
(iii) This follows from (i) and (ii).
It is easy to see that $(S, \lambda) \subset S$ for all λ, since $\frac{\lambda_n}{n} \leq 1$. Now we prove the following inclusion.

Theorem 3.2. $S \subset (S, \lambda)$ if and only if

$$\liminf_{n\to\infty}\frac{\lambda_n}{n} > 0. \tag{*}$$

Proof: For given $\varepsilon > 0$, we have

$$\{t \le n:\ \ |x(t) - l| \ge \varepsilon\} \supset \{t \in I_n:\ \ |x(t) - l| \ge \varepsilon\}.$$

Therefore,

$$\begin{aligned}\frac{1}{n}|\{t \le n:\ \ |x(t) - l| \ge \varepsilon\}| &\ge \frac{1}{n}|\{t \in I_n:\ \ |x(t) - l| \ge \varepsilon\}| \\ &\ge \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{t \in I_n:\ \ |x(t) - l| \ge \varepsilon\}|.\end{aligned}$$

Hence by using (*) and taking the limit as $n \to \infty$, we get $x(t) \to l(S)$ implies $x(t) \to l(S,\lambda)$.
Conversely, suppose that $\liminf_{n\to\infty}\frac{\lambda_n}{n} = 0$. We can choose a subsequence (n_j) such that $\frac{\lambda_{n_j}}{n_j} < \frac{1}{j}$. Define a function $x(t)$ by $x(t) = 1$ if $t \in I_{n_j}$, $j = 1,2,\ldots$ and $x(t) = 0$ otherwise. Then $x(t) \in [W]$ and hence $x(t) \in S$. But $x(t) \notin [W,\lambda]$ and Theorem 3.1(ii) implies that $x(t) \notin (S,\lambda)$. Hence (*) is necessary.

Finally, we conclude this paper with a definition which generalizes Definition 2.1 of Section 2 and two theorems related to this definition.

Definition 3.2. Let $\lambda \in \Lambda$, p is a real number and $x(t)$ be a real valued function which is measurable (in the Lebesque sense) in the interval $(1,\infty)$, if

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\int_{n-\lambda_n+1}^{n}|x(t) - l|^p dt = 0.$$

Then we say that the function $x(t)$ is λp–strongly summable to *l*. In this case we write $[W_p,\lambda] - lim\, x(t) = l$ and

$$[W_p,\lambda] \coloneqq \{x(t)\colon\ \exists\ l = l_x, [W_p,\lambda]\text{-}\ lim\, x(t) = l\}.$$

If $\lambda_n = n$, then $[W_p,\lambda\,]$ is the same as $[W_p\,]$, the set of strongly p-Cesaro summable functions.

Theorem 3.3. Let $1 \le p < \infty$. If a function $x(t)$ is λp–strongly summable to *l*, then it is λ–statistically convergent to *l*.
The proof of the theorem is similar to that of Theorem 3.1.(i). So it was omitted.

Theorem 3.4. Let $1 \le p < \infty$. If a bounded function $x(t)$ is λ–statistically convergent to *l*, then it is λp–strongly summable to *l*.
The proof of the theorem is similar to that of Theorem 3.1.(ii). So it was omitted.

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