STRONGLY SUMMABLE AND STATISTICALLY CONVERGENT FUNCTIONS*

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Abstract – In this study, by using the notion of \((V, \lambda)\)-summability, we introduce and study the concepts of \(\lambda\)-strongly summable and \(\lambda\)-statistically convergent functions.

Keywords – Statistical convergence, strongly summable function

1. INTRODUCTION

The idea of statistical convergence which is closely related to the concept of natural density or asymptotic density of a subset of the set of natural numbers \(\mathbb{N}\), was first introduced by Fast [1]. The concept of statistical convergence plays an important role in the summability theory and functional analysis. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg [2]. In [3], Borwein introduced and studied strongly summable functions.

Strongly summable number sequences and statistically convergent number sequences were studied in [4] and [5], respectively. In [6], \(\lambda\) –statistically convergent number sequences was defined. In this paper, by taking real valued functions \(x(t)\) measurable (in the Lebesque sense) in the interval \((1, \infty)\) instead of sequences we will introduce \(\lambda\)–strongly summable and \(\lambda\) -statistically convergent functions and give some inclusion relations.

Let \(\lambda = (\lambda_n)\) be a nondecreasing sequence of positive numbers tending to \(\infty\) such that \(\lambda_{n+1} \leq \lambda_n + 1\), \(\lambda_1 = 1\). \(\Lambda\) denote the set of all such sequences. For a sequence \(x = (x_k)\) the generalized de la Vallee Poussin mean is defined by

\[
t_n(x) := \frac{1}{n} \sum_{k \in I_n} x_k,
\]

where \(I_n = [n - \lambda_n + 1, n]\).

A sequence \(x = (x_k)\) is said to be \((V, \lambda)\)-summable to a number \(l\) if \(t_n(x) \to l\) as \(n \to \infty\). If \(\lambda_n = n\), then \((V, \lambda)\) –summability reduces to \((C, 1)\) summability.

2 \(\lambda\)-STRONGLY SUMMABLE FUNCTIONS

A real valued function \(x(t)\), measurable(in the Lebesque sense) in the interval \((1, \infty)\), is said to be strongly summable to \(l\) if

\[
\lim_{n \to \infty} \frac{1}{n} \int_1^n |x(t) - l| dt = 0.
\]

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[W] will denote the space of all strongly summable functions. Also, \( W \) will denote the space of \( x(t) \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \int_1^n x(t) \, dt = l.
\]

In this section we will introduce \( \lambda \)-strongly summable function.

**Definition 2.1** Let \( \lambda \in \Lambda \) and \( x(t) \) be a real valued function which is measurable (in the Lebesque sense) in the interval \((1, \infty)\), if

\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \int_{n-\lambda_n+1}^n |x(t) - l| \, dt = 0.
\]

Then we say that the function \( x(t) \) is \( \lambda \)-strongly summable to \( l \). In this case we write \( [W, \lambda] - \lim x(t) = l \).

If \( \lambda_n = n \), then \( [W, \lambda] \) is the same as \( W \).

### 3.2 - Statistically Convergent Functions

\( x(t) \) is a real valued function which is measurable (in the Lebesque sense) in the interval \((1, \infty)\), if for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{t \leq n: |x(t) - l| \geq \varepsilon} 1 = 0
\]

then we say that the function \( x(t) \) is statistically convergent to \( l \). Where the vertical bars indicate the Lebesque measure of the enclosed set. In this case we write \( S - \lim x(t) = l \) and

\[
S := \{ x(t): \exists \ l = l_x, S - \lim x(t) = l \}.
\]

**Definition 3.1.** Let \( \lambda \in \Lambda \) and \( x(t) \) be a real valued function which is measurable (in the Lebesque sense) in the interval \((1, \infty)\), if for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{t \in I_n: |x(t) - l| \geq \varepsilon} 1 = 0
\]

then we say that the function \( x(t) \) is \( \lambda \)-statistically convergent to \( l \). In this case we write \( S_\lambda - \lim x(t) = l \) and

\[
(S, \lambda) := \{ x(t): \exists \ l = l_x, S_\lambda - \lim x(t) = l \}.
\]

If \( \lambda_n = n \), then \( (S, \lambda) \) is the same as \( S \), the set of statistically convergent functions.

**Theorem 3.1.** Let \( \lambda \in \Lambda \) and \( x(t) \) be a real valued function which is measurable (in the Lebesque sense) in the interval \((1, \infty)\), then

(i) \([W, \lambda] \subseteq (S, \lambda) \) and the inclusion is proper.

(ii) If \( x(t) \) is bounded and \( S_\lambda - x(t) = l \) then \([W, \lambda] - \lim x(t) = l \) and hence \( W - \lim x(t) = l \) provided \( x(t) \) is not eventually constant.

(iii) If \( x(t) \) is bounded then \( (S, \lambda) = [W, \lambda] \).
Proof: (i) Let \( \varepsilon > 0 \) and \( [W, \lambda] - \lim x(t) = l \). We write

\[
\int_{t \in I_n} |x(t) - l| \, dt \geq \int_{t \in I_n} |x(t) - l| \, dt \geq \varepsilon |\{ t \in I_n : \ |x(t) - l| \geq \varepsilon \}|.
\]

Then \( [W, \lambda] - \lim x(t) = l \) implies \( S_\lambda - \lim x(t) = l \).

Define a function \( x(t) \) by

\[
x(t) = \begin{cases} 
\{ t, \ n - (\lambda_n)^2 + 1 \leq t \leq n \\
0, \quad \text{otherwise.}
\end{cases}
\]

Then \( x(t) \) is not a bounded function and for every \( \varepsilon (0 < \varepsilon \leq 1) \),

\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \left| \{ t \in I_n : \ |x(t) - 0| \geq \varepsilon \} \right| = \lim_{n \to \infty} \frac{\varepsilon^2}{\lambda_n} = 0,
\]

i.e., \( S_\lambda - \lim x(t) = 0 \). But

\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \int_{n - \lambda_n + 1}^{n} |x(t) - 0| \, dt = \infty,
\]

i.e., \( x(t) \notin [W, \lambda] \). Therefore, the inclusion is proper.

(ii) Suppose that \( S_\lambda - \lim x(t) = l \) and \( x(t) \) be a bounded function, say \( |x(t) - l| \leq M \) for all \( t \). Given \( \varepsilon > 0 \), we have that

\[
\frac{1}{\lambda_n} \int_{t \in I_n} |x(t) - l| \, dt = \frac{1}{\lambda_n} \int_{t \in I_n} |x(t) - l| \, dt + \frac{1}{\lambda_n} \int_{t \in I_n} |x(t) - l| \, dt
\]

\[
\leq \frac{M}{\lambda_n} |\{ t \in I_n : \ |x(t) - l| \geq \varepsilon \}| + \varepsilon
\]

which implies that \( [W, \lambda] - \lim x(t) = l \).

Also, we have, since \( \frac{\lambda_n}{n} \leq 1 \) for all \( n, \)

\[
\frac{1}{n} \int_{1}^{n} (x(t) - l) \, dt = \frac{1}{n} \int_{1}^{n-\lambda_n} (x(t) - l) \, dt + \frac{1}{n} \int_{t \in I_n} (x(t) - l) \, dt
\]

\[
\leq \frac{1}{n} \int_{1}^{n-\lambda_n} |x(t) - l| \, dt + \frac{1}{n} \int_{t \in I_n} |x(t) - l| \, dt
\]

\[
\leq \frac{2}{\lambda_n} \int_{t \in I_n} |x(t) - l| \, dt.
\]

Hence \( [W] - \lim x(t) = l \), since \( [W, \lambda] - \lim x(t) = l \).

(iii) This follows from (i) and (ii).

It is easy to see that \( (S, \lambda) \subseteq S \) for all \( \lambda \), since \( \frac{\lambda_n}{n} \leq 1 \). Now we prove the following inclusion.

**Theorem 3.2.** \( S \subseteq (S, \lambda) \) if and only if
\[
\liminf_{n \to \infty} \frac{\lambda_n}{n} > 0. 
\]

(**)

**Proof:** For given \( \varepsilon > 0 \), we have

\[
\{ t \leq n : |x(t) - l| \geq \varepsilon \} \supset \{ t \in I_n : |x(t) - l| \geq \varepsilon \}.
\]

Therefore,

\[
\frac{1}{n} \sum_{t \leq n} |x(t) - l| \geq \frac{1}{n} \sum_{t \in I_n} |x(t) - l| \geq \frac{\lambda_n}{n} \frac{1}{\lambda_n} \sum_{t \in I_n} |x(t) - l| \geq \varepsilon.
\]

Hence by using (**), and taking the limit as \( n \to \infty \), we get \( x(t) \to l \) implies \( x(t) \to l(S, \lambda) \).

Conversely, suppose that \( \liminf_{n \to \infty} \frac{\lambda_n}{n} = 0 \). We can choose a subsequence \( (n_j) \) such that \( \frac{\lambda_{n_j}}{n_j} < \frac{1}{j} \). Define a function \( x(t) \) by \( x(t) = 1 \) if \( t \in I_{n_j}, j = 1, 2, \ldots \) and \( x(t) = 0 \) otherwise. Then \( x(t) \in [W] \) and hence \( x(t) \in S \). But \( x(t) \notin [W, \lambda] \) and Theorem 3.1(ii) implies that \( x(t) \notin (S, \lambda) \). Hence (***) is necessary.

Finally, we conclude this paper with a definition which generalizes Definition 2.1 of Section 2 and two theorems related to this definition.

**Definition 3.2.** Let \( \lambda \in \Lambda \), \( p \) is a real number and \( x(t) \) be a real valued function which is measurable (in the Lebesque sense) in the interval \((1, \infty)\), if

\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \int_{n - \lambda_n + 1}^{n} |x(t) - l|^p dt = 0.
\]

Then we say that the function \( x(t) \) is \( \lambda^p \)-strongly summable to \( l \). In this case we write \([W, \lambda]^p \)-\( \lim x(t) = l \) and

\[
[W, \lambda] = \{ x(t) : \exists l = l_x, [W, \lambda], \lim x(t) = l \}.
\]

If \( \lambda_n = n \), then \([W, \lambda] \) is the same as \([W_p, \lambda] \), the set of strongly \( p \)-Cesaro summable functions.

**Theorem 3.3.** Let \( 1 \leq p < \infty \). If a function \( x(t) \) is \( \lambda^p \)-strongly summable to \( l \), then it is \( \lambda \)-statistically convergent to \( l \).

The proof of the theorem is similar to that of Theorem 3.1(i). So it was omitted.

**Theorem 3.4.** Let \( 1 \leq p < \infty \). If a bounded function \( x(t) \) is \( \lambda \)-statistically convergent to \( l \), then it is \( \lambda^p \)-strongly summable to \( l \).

The proof of the theorem is similar to that of Theorem 3.1(ii). So it was omitted.

**REFERENCES**