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TWO-DIRECTION POLY-SCALE REFINEMENT EQUATIONS WITH NONNEGATIVE COEFFICIENTS*

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Abstract – In this paper, we study L^1-solutions of the following two-direction poly-scale refinement equation

$$f(x)=\sum_{m=1}^{M-1}\sum_{n=-N}^{N}[c_{m,n}^{1} f(\lambda^m x-n)+c_{m,n}^{-1} f(-\lambda^m x-n)]$$

We prove that the vector space of all L^1-solutions of the above equation is at most one-dimensional and consists of compactly supported functions of constant sign. We also show that any-L^1 solution of the above equation is either positive or negative on its support under a special assumption. With regard to the L^1-solutions of the equation, some simple sufficient conditions for the existence of nontrivial L^1-solutions and the nonexistence of such solutions are given.

Keywords – Two-direction poly-scale refinement equation, L^1-solutions, iterated function systems

1. INTRODUCTION

Two-scale refinement equation $f(x)=\sum_{k\in\mathbb{Z}} c_k f(2x-k)$ plays a basic role in the construction and application of wavelets [1-6]. Recently, poly-scale refinement equation

$$f(x)=\sum_{m=1}^{M-1}\sum_{n\in\mathbf{Z}} c_{m,n} f(\lambda^m x-n) \tag{1}$$

has been studied [7-9]. It is the extension of the two-scale refinement equation, and poly-scale refinable functions have more good properties than two-scale refinable functions [7]. In addition, two-direction refinement equation

$$f(x)=\sum_{n} c_n^{1} f(\lambda x-n)+\sum_{n} c_n^{-1} f(-\lambda x-n) \tag{2}$$

has been studied and a system theory was built [10-12]. In [13] and [14], Kapica and Morawiec focus on L^1-solutions of (2) with nonnegative coefficients, and give necessary and sufficient conditions for the existence of non-trivial L^1-solutions in several special cases. In [15], the existence of solutions of two-direction poly-scale refinement equation

$$f(x)=\sum_{m=1}^{M-1}\sum_{n\in\mathbb{Z}}[c_{m,n}^{1} f(\lambda^m x-n)+c_{m,n}^{-1} f(-\lambda^m x-n)] \tag{3}$$

is investigated, and conditions for the existence of a compactly supported distributional solution of (3) are derived,

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together with an explicit form of its Fourier transform.

In this paper, based on [13] and [14], we study L^1-solutions of

$$f(x)=\sum_{m=1}^{M-1}\sum_{n=-N}^{N}[c_{m,n}^{1}f(\lambda^m x-n)+c_{m,n}^{-1}f(-\lambda^m x-n)] \tag{4}$$

where the coefficients are nonnegative and integers $\lambda \ge 2, N \ge 1, M \ge 2$. Define two matrices

$$\mathbf{C}^{+}=\begin{bmatrix} c_{1,-N}^{1} & c_{1,-N+1}^{1} & \cdots & c_{1,N}^{1} \\ c_{2,-N}^{1} & c_{2,-N+1}^{1} & \cdots & c_{2,N}^{1} \\ \vdots & \vdots & & \vdots \\ c_{M-1,-N}^{1} & c_{M-1,-N+1}^{1} & \cdots & c_{M-1,N}^{1} \end{bmatrix},$$

$$\mathbf{C}^{-}=\begin{bmatrix} c_{1,-N}^{-1} & c_{1,-N+1}^{-1} & \cdots & c_{1,N}^{-1} \\ c_{2,-N}^{-1} & c_{2,-N+1}^{-1} & \cdots & c_{2,N}^{-1} \\ \vdots & \vdots & & \vdots \\ c_{M-1,-N}^{-1} & c_{M-1,-N+1}^{-1} & \cdots & c_{M-1,N}^{-1} \end{bmatrix}.$$

Throughout this paper, we always assume that

$$\sum_{(m,n,\varepsilon)\in\mathbf{S}}\lambda^{-m}c_{m,n}^{\varepsilon}=1, \tag{5}$$

where $\mathbf{S}=\{(m,n,\varepsilon): m\in\{1,...,M-1\}, n\in\{-N,...,N\}, \varepsilon\in\{1,-1\}, c_{m,n}^{\varepsilon}>0\}$. Then we can rewrite (4) into

$$f(x)=\sum_{(m,n,\varepsilon)\in\mathbf{S}}c_{m,n}^{\varepsilon}f(\varepsilon\lambda^m x-n). \tag{6}$$

If $M=2$, then (6) takes the form of (2). If $C^{-}=\mathbf{0}$, then (6) takes the form of (1).

This paper is arranged as follows. In the next section we show that the vector space of all L^1-solutions of (6) is at most one-dimensional. Moreover, any non-trivial L^1-solution of (6) is compactly supported and it is either positive or negative on it is support almost everywhere. In section 3, sufficient conditions for the existence of non-trivial L^1-solutions of (6) as well as for the nonexistence of such solutions are given. At last, we finish this paper with two remarks and two examples.

2. BASIC PROPERTIES OF L^1-SOLUTIONS OF (6)

Obviously the set of all L^1-solutions of (6) is a real vector space. Denote by $\mathbf{V_c}$ this space.
$\dim\mathbf{V_c}$ is the dimension of $\mathbf{V_c}$. For $l\in\mathbf{N}$, put

$$\mathbf{S}^{l}=\{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l):(m_j,n_j,\varepsilon_j)\in\mathbf{S}, j\in\{1,2,...,l\}\}$$

and $\mathbf{S}^{\mathbf{N}}=\lim_{l\to+\infty}\mathbf{S}^{l}$. First, we get the Fourier transform of $f(x)$ for $\forall f\in\mathbf{V_c}$.

Theorem 2.1. Let $f\in\mathbf{V_c}$ and define a sequence of function

$$f_l(t)=\sum_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in\mathbf{S}^l}\frac{c_{m_1,n_1}^{\varepsilon_1}\cdots c_{m_l,n_l}^{\varepsilon_l}}{\lambda^{m_1+\cdots+m_l}}e^{-it\sum_{j=1}^{l}\frac{\varepsilon_1\cdots\varepsilon_j}{\lambda^{m_1+\cdots+m_j}}n_j}, \; l\in\mathbf{N}.$$

Then $\{f_l : l \in \mathbf{N}\}$ converges uniformly on any compact subset of $\mathbf{R}$,

$$\hat{f}(t) = \hat{f}(0)\lim_{l\to\infty} f_l(t), \quad t \in \mathbf{R}.$$

Proof: Fix $l \in \mathbf{N}$ and $t \in \mathbf{R}$. Then by (5), we have

$$\begin{aligned}
|f_l(t) - f_{l-1}(t)| &\le \sum_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in \mathbf{S}^l} \frac{c_{m_1,n_1}^{\varepsilon_1}\cdots c_{m_l,n_l}^{\varepsilon_l}}{\lambda^{m_1+\cdots+m_l}}\left| e^{-it\frac{\varepsilon_1\cdots\varepsilon_l}{\lambda^{m_1+\cdots+m_l}}n_l} - 1\right| \\
&\le \sum_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in \mathbf{S}^l} \frac{c_{m_1,n_1}^{\varepsilon_1}\cdots c_{m_l,n_l}^{\varepsilon_l}}{\lambda^{m_1+\cdots+m_l}}\left| t\frac{\varepsilon_1\cdots\varepsilon_l}{\lambda^{m_1+\cdots+m_l}}n_l\right| \\
&\le \frac{N}{\lambda^l}|t|.
\end{aligned}$$

That is, $\{f_l : l \in \mathbf{N}\}$ satisfies the uniform Cauchy condition, and thus converges uniformly on every compact subset of $\mathbf{R}$.

Next, we take the Fourier transform of $\hat{f}$, by (6} we get

$$\hat{f}(t) = \int_{\mathbf{R}} e^{-itx} f(x)dx = \sum_{(m,n,\varepsilon)\in \mathbf{S}} \frac{c_{m,n}^{\varepsilon}}{\lambda^m} e^{-it\frac{\varepsilon}{\lambda^m}n}\hat{f}(\frac{\varepsilon}{\lambda^m}t).$$

By iterating,

$$\hat{f}(t) = \sum_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in \mathbf{S}^l} \frac{c_{m_1,n_1}^{\varepsilon_1}\cdots c_{m_l,n_l}^{\varepsilon_l}}{\lambda^{m_1+\cdots+m_l}} e^{-it\sum_{j=1}^{l}\frac{\varepsilon_1\cdots\varepsilon_j}{\lambda^{m_1+\cdots+m_j}}n_j}\hat{f}(\frac{\varepsilon_1\cdots\varepsilon_l}{\lambda^{m_1+\cdots+m_l}}t)$$

for $l \in \mathbf{N}$ and $t \in \mathbf{R}$. When $l \to +\infty$, we get our assertion.

For $M = 2$, by Theorem 2.1, we have the following corollary. This result is the same as in [14].

Corollary 3.1. In Theorem 2.1, assume $M = 2$. Let $f \in \mathbf{V_c}$ and define a sequence of function $f_l(t) = \sum_{(1,n_1,\varepsilon_1,\cdots,1,n_l,\varepsilon_l)\in \mathbf{S}^l} \frac{c_{1,n_1}^{\varepsilon_1}\cdots c_{1,n_l}^{\varepsilon_l}}{\lambda^l} e^{-it\sum_{j=1}^{l}\frac{\varepsilon_1\cdots\varepsilon_j}{\lambda^j}n_j}$, $l \in \mathbf{N}$. Then $\{f_l : l \in \mathbf{N}\}$ converges uniformly on any compact subset of $\mathbf{R}$, and $\hat{f}(t) = \hat{f}(0)\lim_{l\to\infty} f_l(t), \quad t \in \mathbf{R}.$

Let $N_j = \max\{|n| : c_{j,n}^{\varepsilon} > 0\}$ for $j \in \{1,...,M-1\}$ and

$$\Upsilon = \left\{ k : \frac{N_k}{\lambda^k} = \max_{j\in\{1,...,M-1\}} \frac{N_j}{\lambda^j}, k \in \{1,...,M-1\}\right\}.$$

Fix $K \in \Upsilon$. For every $(m,n,\varepsilon) \in \mathbf{S}$, we can define a map $T_{m,n,\varepsilon} : \mathbf{R} \to \mathbf{R}$ by

$$T_{m,n,\varepsilon}(x) = \frac{x+n}{\varepsilon\lambda^m}.$$

Clearly these mappings consist of an iterated function system (IFS) by independently choosing the map $T_{m,n,\varepsilon}$ with probability $\frac{c_{m,n}^{\varepsilon}}{\lambda^m}$. Put

$$T_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l} = T_{m_1,n_1,\varepsilon_1}\circ\cdots\circ T_{m_l,n_l,\varepsilon_l}$$

for $(m_1, n_1, \varepsilon_1, \cdots, m_l, n_l, \varepsilon_l) \in \mathbf{S}^l$, $l \in \mathbf{N}$. According to [16], there exists a unique nonempty compact set $\mathbf{J_c}$ satisfying

$$\bigcup_{(m,n,\varepsilon)\in\mathbf{S}} T_{m,n,\varepsilon}(\mathbf{J_c}) = \mathbf{J_c}. \tag{7}$$

Furthermore, for arbitrary closed bounded $A \in \mathbf{R}$, let

$$T(A) = \bigcup_{(m,n,\varepsilon)\in\mathbf{S}} T_{m,n,\varepsilon}(A), \quad T^{p+1}(A) = T(T^p(A)) \quad \text{for} \quad p \in \mathbf{N}.$$

Then $T^p(A) \to \mathbf{J_c}$. As a result,

$$\begin{aligned}
\mathbf{J_c} &= \left\{ \lim_{l\to\infty} T_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}(0) : \{(m_l, n_l, \varepsilon_l) : l \in \mathbf{N}\} \in \mathbf{S}^{\mathbf{N}} \right\} \\
&= \left\{ \sum_{l\in\mathbf{N}} \frac{\varepsilon_1 \cdots \varepsilon_l}{\lambda^{m_1+\cdots+m_l}} n_l : \{(m_l, n_l, \varepsilon_l) : l \in \mathbf{N}\} \in \mathbf{S}^{\mathbf{N}} \right\} \\
&\subset \left[-\frac{N_K}{\lambda^K - 1}, \frac{N_K}{\lambda^K - 1} \right] = \mathbf{J}
\end{aligned}$$

and

$$\mathbf{J_c} = \bigcap_{l\in\mathbf{N}} \left(\bigcup_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in\mathbf{S}^l} T_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}(\mathbf{J}) \right). \tag{8}$$

Given probability weights $\{\frac{c^{\varepsilon}_{m,n}}{\lambda^m} : (m, n, \varepsilon) \in \mathbf{S}\}$, there exists a unique Borel regular measure of total mass 1 such that

$$\mu(\bullet) = \sum_{(m,n,\varepsilon)\in\mathbf{S}} \frac{c^{\varepsilon}_{m,n}}{\lambda^m} \mu \circ T^{-1}{}_{m,n,\varepsilon}(\bullet). \tag{9}$$

Moreover, μ is supported on $\mathbf{J_c}$ ([16]).

Let's give some remarks, which can be easily derived from (7) and (8).

Remark 2.1. Assume

$$\mathbf{S} \cap \{(m,n,1),(m,-n,-1)\} \neq \varnothing \Leftrightarrow \mathbf{S} \cap \{(m,-n,1),(m,n,-1)\} \neq \varnothing \tag{10}$$

for $m \in \{1,\ldots,M-1\}$ and $n \in \{1,\ldots,N_m\}$. Then $\mathbf{J_c} = -\mathbf{J_c}$.

Remark 2.2. Denote

$$(\mathbf{J_c})_j = \left\{ \lim_{l\to\infty} T_{j,n_1,\varepsilon_1,\cdots,j,n_l,\varepsilon_l}(0) : \{(j, n_l, \varepsilon_l) : l \in \mathbf{N}\} \in \mathbf{S}^{\mathbf{N}} \right\}$$

for $j \in 1,\ldots,M-1$. Then

$$\bigcup_{j=1}^{M-1} (\mathbf{J_c})_j \subset \mathbf{J_c}. \tag{11}$$

Remark 2.3. Assume

$$\lambda^K \le 2N_K + 1 \tag{12}$$

and

$$\mathbf{S} \cap \{(K,n,1),(K,-n,-1)\} \ne \varnothing \quad \text{for} \quad n \in \{-N_K,\dots,N_K\}. \tag{13}$$

Then $\mathbf{J_c} = \mathbf{J}$.

Remark 2.3. Let

$$\tilde{\mathbf{C}}^+ = \begin{bmatrix} \tilde{c}^1_{1,-N} & \tilde{c}^1_{1,-N+1} & \cdots & \tilde{c}^1_{1,N} \\ \tilde{c}^1_{2,-N} & \tilde{c}^1_{2,-N+1} & \cdots & \tilde{c}^1_{2,N} \\ \vdots & \vdots & & \vdots \\ \tilde{c}^1_{M-1,-N} & \tilde{c}^1_{M-1,-N+1} & \cdots & \tilde{c}^1_{M-1,N} \end{bmatrix}$$

and

$$\tilde{\mathbf{C}}^- = \begin{bmatrix} \tilde{c}^{-1}_{1,-N} & \tilde{c}^{-1}_{1,-N+1} & \cdots & \tilde{c}^{-1}_{1,N} \\ \tilde{c}^{-1}_{2,-N} & \tilde{c}^{-1}_{2,-N+1} & \cdots & \tilde{c}^{-1}_{2,N} \\ \vdots & \vdots & & \vdots \\ \tilde{c}^{-1}_{M-1,-N} & \tilde{c}^{-1}_{M-1,-N+1} & \cdots & \tilde{c}^{-1}_{M-1,N} \end{bmatrix}$$

be two matrices of nonnegative reals satisfying $\mathbf{S} \subset \tilde{\mathbf{S}} = \{(m,n,\varepsilon) : \tilde{c}^{\varepsilon}_{m,n} > 0\}$ and $\sum_{(m,n,\varepsilon)\in\tilde{\mathbf{S}}} \lambda^{-m}\tilde{c}^{\varepsilon}_{m,n} = 1$. Then $\mathbf{J_c} \subset \mathbf{J_{\tilde{c}}}$.

In the next section, we will see that (12) is necessary for $\dim \mathbf{V_c} = 1$.

In [14], Morawiec shows that every nontrivial L^1-solution f of (6) is compactly supported with $\operatorname{supp} f = \mathbf{J_c}$ in the case of $M = 2$. In [15] Yang and Xue discuss the support of any compactly supported solution of (6). For both of the generalizations, we give the next theorem of the precise information about the support of $f \in \mathbf{V_c}$.

Theorem 2.2. Suppose $f \in \mathbf{V_c} \setminus \{0\}$. Then f is of constant sign with $\operatorname{supp} f = \mathbf{J_c}$.

Proof: By (6) and (5),

$$\| f \|_1 = \int_{\mathbf{R}} |f(x)|\,dx \le \sum_{(m,n,\varepsilon)\in\mathbf{S}} c^{\varepsilon}_{m,n} \int_{\mathbf{R}} |f(\varepsilon\lambda^m x - n)|\,dx = \| f \|_1.$$

Hence $|f| \in \mathbf{V_c}$. Jointly with Theorem 2.1, we get that f is either nonnegative or nonpositive.

Without loss of generality we can assume that f is nonnegative with $\| f \|_1 = 1$. Let

$$\mu(B) = \int_B f(x)\,dx$$

for any Borel set $B \in \mathbf{R}$. Then $\mu(B)$ defines a probability Borel measure on $\mathbf{R}$. Notice that

$$f(x) = \sum_{(m,n,\varepsilon)\in\mathbf{S}} c^{\varepsilon}_{m,n} f(T^{-1}{}_{m,n,\varepsilon}(x)), \tag{14}$$

thus we get (9). As a result, $\mu(B)$ is the unique probability Borel measure satisfying (9). $\operatorname{supp} f = \mathbf{J_c}$ since $\mu(B)$ is supported on $\mathbf{J_c}$.

For $M = 2$, by Theorem2.2, we have the following corollary. This result has been derived by Morawiec in [14].

Corollary 2.2. In Theorem 2.2, suppose $M = 2$. Let $f \in \mathbf{V_c} \setminus \{0\}$. Then f is of constant sign with $\operatorname{supp} f = \mathbf{J_c}$.

From Theorem 2.1, we see that $\mathbf{V_c}$ is one-dimensional or zero-dimensional. By Theorem 2.2 we get that if $\dim \mathbf{V_c} = 1$, then $l_1(\mathbf{J_c}) > 0$, where l_1 is the Lebesgue measure on the real line. Condition $l_1(\mathbf{J_c}) > 0$ is necessary, but not sufficient, to guarantee $\dim \mathbf{V_c} = 1$. From the proof of Theorem 2.2 we see that if $\dim \mathbf{V_c} = 1$, then μ is absolutely continuous. On the other hand, if μ is absolutely continuous, then its density (i.e. the Radon-Nikodym derivative) belongs to $\mathbf{V_c}$, which jointly with Theorem 2.1 leads to $\dim \mathbf{V_c} = 1$. Thus, we conclude that $\dim \mathbf{V_c} = 1$ if and only if the unique probability Borel measure μ satisfying (9) is absolutely continuous with respect to l_1. But it is not easy to test the absolute continuity of μ.

In general, the absolute continuity of a Borel measure μ with respect to l_1 does not guarantee its density to be positive on its support. But, in our case, we show that the density of the unique probability Borel measure μ satisfying (9) is positive on its support.

Let's give some denotes and two lemmas before the next theorem.

For all $(m_1, n_1, \varepsilon_1, \cdots, m_l, n_l, \varepsilon_l) \in \mathbf{S}^l$ with $l \in \mathbf{N}$, let

$$J_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l} = T_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}(\mathbf{J_c}).$$

According to (7), we get

$$\bigcup_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in\mathbf{S}^l} J_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l} = \mathbf{J_c} \quad \text{for} \quad \forall l \in \mathbf{N}. \tag{15}$$

For a given measurable set $A \subseteq \mathbf{J_c}$, put

$$A_0 = A, \quad A_{-(l+1)} = \bigcup_{(m,n,\varepsilon)\in\mathbf{S}} T^{-1}{}_{m,n,\varepsilon}(A_{-l} \cap J_{m,n,\varepsilon}) \text{for} l \in \mathbf{N}_0, \quad \Lambda_- = \bigcup_{l\in\mathbf{N}_0} A_{-l},$$

where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. It is easy to check that $\Lambda_- \subset \mathbf{J_c}$ by (7).

Lemma 2.1. Suppose $\mathbf{J_c}$ is a non-degenerated interval. Let $A \subseteq \mathbf{J_c}$ be a measurable set. Then either $l_1(\Lambda_-) = 0$ or $l_1(\Lambda_-) = l_1(\mathbf{J_c})$.

Proof: Put $\Lambda = \mathbf{J_c} \setminus \Lambda_-$. For the case of $l_1(\Lambda) = 0$, we get $l_1(\Lambda_-) = l_1(\mathbf{J_c})$. For the case of $l_1(\Lambda) > 0$, we prove that $l_1(\Lambda_-) = 0$.

Assume that $l_1(\Lambda) > 0$. Fix $x \in \Lambda$ and suppose that $T_{m,n,\varepsilon}(x) \notin \Lambda$ for some $(m,n,\varepsilon) \in \mathbf{S}$. Then $T_{m,n,\varepsilon}(x) \in \Lambda_-$ since $\Lambda_- \subset \mathbf{J_c}$. So there is a positive integer l satisfying $T_{m,n,\varepsilon}(x) \in A_{-l} \cap J_{m,n,\varepsilon}$, and therefore $x \in A_{-(l+1)} \subset \Lambda_-$, which is a contradiction. Hence $T_{m,n,\varepsilon}(\Lambda) \subset \Lambda$ for $(m,n,\varepsilon) \in \mathbf{S}$. By induction we get

$$T_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}(\Lambda) \subset \Lambda \cap J_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}$$

for $(m_1, n_1, \varepsilon_1, \cdots, m_l, n_l, \varepsilon_l) \in \mathbf{S}^l$, $l \in \mathbf{N}$. Thus

$$\frac{l_1(\Lambda)}{\lambda^{m_1+\cdots m_l}} = l_1(T_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}(\Lambda)) \le l_1(\Lambda \cap J_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}) \tag{16}$$

for $(m_1, n_1, \varepsilon_1, \cdots, m_l, n_l, \varepsilon_l) \in \mathbf{S}^l$.

Fix an interval $I \subset \mathbf{J_c}$. Taking into account (15), we get that there exists a nonnegative integer l and a set

$$P_l = \{(m_1, n_1, \varepsilon_1, \cdots, m_l, n_l, \varepsilon_l) \in \mathbf{S}^l : J_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l} \subset I\}$$

satisfying

$$\begin{gathered} J_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l} \bigcap J_{m_{1'},n_{1'},\varepsilon_{1'},\cdots,m_{l'},n_{l'},\varepsilon_{l'}} = \varnothing \\ \text{for } (m_1, n_1, \varepsilon_1, \cdots, m_l, n_l, \varepsilon_l), (m_{1'}, n_{1'}, \varepsilon_{1'}, \cdots, m_{l'}, n_{l'}, \varepsilon_{l'}) \in P_l \ \text{ with} \\ (m_1, n_1, \varepsilon_1, \cdots, m_l, n_l, \varepsilon_l) \neq (m_{1'}, n_{1'}, \varepsilon_{1'}, \cdots, m_{l'}, n_{l'}, \varepsilon_{l'}), \ \textit{and} \\ \sum_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in P_l} \frac{l_1(\mathbf{J_c})}{\lambda^{m_1+\cdots m_l}} = \sum_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in P_l} l_1(J_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}) \\ \geq \frac{1}{4} l_1(I). \end{gathered} \tag{17}$$

Therefore, by (16) and (17),

$$\begin{aligned} l_1(\Lambda \cap I) &\geq \sum_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in P_l} l_1(\Lambda \cap J_{m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l}) \\ &\geq \sum_{(m_1,n_1,\varepsilon_1,\cdots,m_l,n_l,\varepsilon_l)\in P_l} \frac{l_1(\Lambda)}{\lambda^{m_1+\cdots m_l}} \\ &\geq \alpha l_1(I), \end{aligned} \tag{18}$$

where $\alpha = \dfrac{l_1(\Lambda)}{4 l_1(\mathbf{J_c})} > 0$.

Now fix $\varepsilon > 0$ and choose a collection of non-interaction intervals $\{I_l : l \in \mathbf{N}\}$ such that

$$\Lambda_- \subset \bigcup_{l\in\mathbf{N}} I_l \quad \text{and} \quad \sum_{l\in\mathbf{N}} l_1(I_l) \leq l_1(\Lambda_-) + \varepsilon.$$

Then, by (18), we have

$$\varepsilon \geq \sum_{l\in\mathbf{N}} l_1(I_l \cap \Lambda) \geq \sum_{l\in\mathbf{N}} \alpha l_1(I_l) \geq \alpha l_1(\Lambda_-),$$

which leads to $l_1(\Lambda_-) = 0$.

Lemma 2.2. Suppose $\mathbf{J_c}$ is a non-degenerated interval. Let $A \subseteq \mathbf{J_c}$ be a measurable set and $f \in \mathbf{V_c}$. If f vanishes on A, then f vanishes on the set Λ_-.

Proof: Without the restriction of generality we can assume that f is nonnegative. We use the mathematical induction to prove that f vanishes on A_{-l} for every $l \in \mathbf{N}_0$. Thus f vanishes on the set Λ_-. Obviously, f vanishes on A_0. Fix $l \in \mathbf{N}_0$ and suppose that f vanishes on A_{-l}. Then by (14) we get that f vanishes on $T^{-1}_{m,n,\varepsilon}(A_{-l})$ for every $(m, n, \varepsilon) \in \mathbf{S}$. Thus f vanishes on $A_{-(l+1)}$.

According to Theorem 2.2 and Lemma 2.1 - Lemma 2.2 we can easily derive the next theorem.

Theorem 2.3. Suppose $\mathbf{J_c}$ is a non-degenerated interval. Let $f \in \mathbf{V_c} \setminus \{0\}$. Then f is either positive or negative on $\mathbf{J_c}$ almost everywhere.

A direct application of Theorem 2.3 and Remark 2.3 is the next assertion.

Corollary 2.3. Suppose (12) and (13). Let $f \in \mathbf{V_c} \setminus \{0\}$. Then f is either positive or negative on $\mathbf{J}$ almost everywhere.

We get the following corollary by Theorem 2.3 when $M=2$. This result has already shown in [14].

Corollary 2.4. In Theorem 2.3, suppose $M=2$. Assume that $\mathbf{J_c}$ is a non-degenerated interval. Let $f\in\mathbf{V_c}\setminus\{0\}$. Then f is either positive or negative on $\mathbf{J_c}$ almost everywhere.

3. THE SPACE $\mathbf{V_c}$

In this section we will determine the dimension of $\mathbf{V_c}$ for some special cases. Even in the case of (2), known conditions characterizing the dimension of $\mathbf{V_c}$ are rather difficult to check [13]. We will not give conditions characterizing the dimension of $\mathbf{V_c}$. But we present simple sufficient conditions for $\dim\mathbf{V_c}=0$ and $\dim\mathbf{V_c}=1$. Note that some of the conditions are the same

as in [14], for the case of $M=2$.

Fix $m\in\{1,...,M-1\}$. Put $\mathbf{S}_m=\{(m,n,\varepsilon)\in\mathbf{S}\}$. Let $\mathrm{card}\mathbf{S}_m$ be the number of the elements of $\mathbf{S}_m$. For a set $S\subset\mathbf{R}$, denote by $\mathrm{int}S$ the interior of S, by $\mathrm{cl}S$ the closure of S.

Remark 3.1. Suppose $\sum_{m=1}^{M-1}\frac{\mathrm{card}\mathbf{S}_m}{\lambda^m}<1$. Then $\mathbf{V_c}=\{0\}$.

Proof: By (7), we get

$$l_1(\mathbf{J_c})\le\sum_{(m,n,\varepsilon)\in\mathbf{S}}l_1(T_{m,n,\varepsilon}(\mathbf{J_c}))=\sum_{(m,n,\varepsilon)\in\mathbf{S}}\frac{l_1(\mathbf{J_c})}{\lambda^m}=(\sum_{m=1}^{M-1}\frac{\mathrm{card}\mathbf{S}_m}{\lambda^m})l_1(\mathbf{J_c})$$

Thus $l_1(\mathbf{J_c})=0$ since $\sum_{m=1}^{M-1}\frac{\mathrm{card}\mathbf{S}_m}{\lambda^m}<1$. Finally, according to Theorem 2.2, $\mathbf{V_c}=\{0\}$.

Remark 3.2. Suppose $\sum_{m=1}^{M-1}\frac{2N_m+1}{\lambda^m}<1$. Then $\mathbf{V_c}=\{0\}$.

Proof: We will show that $l_1(\mathbf{J_c})=0$. According to Remark 2.4 we can assume

$$\mathbf{S}\cap\{(m,n,1),(m,-n,-1)\}\neq\varnothing\quad\text{for}\quad m\in\{1,...,M-1\},n\in\{-N_m,...,N_m\}.\tag{19}$$

Hence, by Remark 2.1, we get $\mathbf{J_c}=-\mathbf{J_c}$. Using (7) and (19), we get

$$\begin{aligned}l_1(\mathbf{J_c})&=l_1(\bigcup_{(m,n,1)\in\mathbf{S}}T_{m,n,1}(\mathbf{J_c})+\bigcup_{(m,-n,-1)\in\mathbf{S}}T_{m,-n,-1}(-\mathbf{J_c}))\\&\le\sum_{m=1}^{M-1}\sum_{n=-N_m}^{N_m}l_1(T_{m,n,1}(\mathbf{J_c}))=\sum_{m=1}^{M-1}\frac{2N_m+1}{\lambda^m}l_1(\mathbf{J_c}).\end{aligned}$$

Since $\sum_{m=1}^{M-1}\frac{2N_m+1}{\lambda^m}<1$, we get $l_1(\mathbf{J_c})=0$. Finally, according to Theorem 2.2, $\mathbf{V_c}=\{0\}$.

According to Theorem 1.1 in [17], the following two results can be easily derived.

Theorem 3.1. Suppose $\sum_{(m,n,\varepsilon)\in\mathbf{S}}\lambda^{-m}c^{\varepsilon}_{m,n}\log c^{\varepsilon}_{m,n}>0$. Then $\mathbf{V_c}=\{0\}$.

Theorem 3.2. Suppose $\sum_{(m,n,\varepsilon)\in\mathbf{S}}\lambda^{-m}c^{\varepsilon}_{m,n}\log c^{\varepsilon}_{m,n}=0$ but $c^{\varepsilon}_{m,n}\neq1$ for some $(m,n,\varepsilon)\in\mathbf{S}$. Then $\mathbf{V_c}=\{0\}$.

In Theorem 3.2, it requires $c^{\varepsilon}_{m,n}\neq1$ for some $(m,n,\varepsilon)\in\mathbf{S}$. If not, we have the following result.

Theorem 3.3. Suppose $c^{\varepsilon}_{m,n}=1$ for all $(m,n,\varepsilon)\in\mathbf{S}$. Then:

(i) $\mathbf{V_c} = \{\alpha \mathcal{X}_{\mathbf{J_c}}\}$, where $\mathcal{X}_{\mathbf{J_c}}$ denotes the characteristic function of the set $\mathbf{J_c}$;
(ii) $\dim \mathbf{V_c} = 1$ if and only if $l_1(\mathbf{J_c}) > 0$.

Proof: By (5), we get $\sum_{m=1}^{M-1} \frac{\mathrm{card}\mathbf{S}_m}{\lambda^m} = 1$. This jointly with (7) gives $l_1(T_{m',n',\varepsilon'}(\mathbf{J_c}) \cap T_{m,n,\varepsilon}(\mathbf{J_c})) = 0$ for $(m',n',\varepsilon'),(m,n,\varepsilon) \in \mathbf{S}$ with $(m',n',\varepsilon') \neq (m,n,\varepsilon)$. A simple calculation shows that $\mathcal{X}_{\mathbf{J_c}} \in \mathbf{V_c}$. Thus by Theorem 2.1 and Theorem 2.2 we conclude the assertions (i) and (ii).

For $M = 2$, by Theorem 3.1 - Theorem 3.3, we have the following corollaries. These results are the same as in [14].

Corollary 3.1. In Theorem 3.1, assume $M = 2$. Suppose $\sum_{(1,n,\varepsilon)\in\mathbf{S}} c_{1,n}^{\varepsilon} \log c_{1,n}^{\varepsilon} > 0$. Then $\mathbf{V_c} = \{0\}$.

Corollary 3.2. In Theorem 3.2, assume $M = 2$. Suppose $\sum_{(1,n,\varepsilon)\in\mathbf{S}} c_{1,n}^{\varepsilon} \log c_{1,n}^{\varepsilon} = 0$ but $c_{1,n}^{\varepsilon} \neq 1$ for some $(1,n,\varepsilon) \in \mathbf{S}$. Then $\mathbf{V_c} = \{0\}$.

Corollary 3.3. In Theorem3.3, assume $M = 2$. Suppose $c_{1,n}^{\varepsilon} = 1$ for all $(1,n,\varepsilon) \in \mathbf{S}$. Then:
(i) $\mathbf{V_c} = \{\alpha \mathcal{X}_{\mathbf{J_c}}\}$, where $\mathcal{X}_{\mathbf{J_c}}$ denotes the characteristic function of the set $\mathbf{J_c}$;
(ii) $\dim \mathbf{V_c} = 1$ if and only if $l_1(\mathbf{J_c}) > 0$.

From Theorem 1.1 in [17] we can see that the condition $l_1(\mathbf{J_c}) > 0$ of assertion (ii) in Theorem 3.3 can be replaced by the open-set condition, that is, there exists an open set $U \subset \mathbf{J}$ such that $T_{m,n,\varepsilon}(U) \subset U$ and $T_{m,n,\varepsilon}(U) \cap T_{m',n',\varepsilon'}(U) = \varnothing$ for $(m,n,\varepsilon),(m',n',\varepsilon') \in \mathbf{S}$ with $(m,n,\varepsilon) \neq (m',n',\varepsilon')$. From [18] it follows that if $l_1(\mathbf{J_c}) > 0$, then $\mathrm{int}\mathbf{J_c} \neq \varnothing$ and $\mathrm{cl}(\mathrm{int}\mathbf{J_c}) = \mathbf{J_c}$. As a result, $l_1(\mathbf{J_c}) > 0$ if and only if the set $\mathrm{int}\mathbf{J_c}$ is nonempty and satisfies the open set condition. This, as well as (8), may sometimes be helpful for determining $\mathbf{J_c}$ (see [14] in the case of $M = 2$). If $\mathbf{J_c}$ is not easy to determine, we can use the algorithm proposed in [18] to calculate $l_1(\mathbf{J_c})$.

In the case of $\sum_{(m,n,\varepsilon)\in\mathbf{S}} \lambda^{-m} c_{m,n}^{\varepsilon} \log c_{m,n}^{\varepsilon} \geq 0$, we have Theorem 3.1–Theorem 3.3 to determine $\dim \mathbf{V_c}$. We do not know any applicable counterpart of those theorems for iterated function systems if $\sum_{(m,n,\varepsilon)\in\mathbf{S}} \lambda^{-m} c_{m,n}^{\varepsilon} \log c_{m,n}^{\varepsilon} < 0$. It turns out that in the latter case it is very difficult to find any formula for $f \in \mathbf{V_c} \setminus \{0\}$. Therefore, we are limited to determine only $\dim \mathbf{V_c}$.

We give some denotes and four lemmas which will be used to prove the next theorem.

From now on, we make the following assumption.

$$(H)\begin{cases} c_{m,n}^{\varepsilon} = 0 \quad \text{for } \varepsilon \in \{-1,1\}, m \in \{1,\dots,M-1\}, n \in \mathbf{R} \setminus \{-N,\dots,N\}, \\ \sum_{m=1}^{M-1} \lambda^{-m} (c_{m,k(\lambda^m-1)}^{1} + c_{m,-k(\lambda^m+1)}^{-1}) < 1 \quad \forall k \in \mathbf{R}, \\ p_{m,n}^{\varepsilon} = \lambda^{-m} c_{m,n}^{\varepsilon} \text{for all } \varepsilon \in \{-1,1\}, m \in \{1,\dots,M-1\}, n \in \mathbf{R}. \end{cases} \tag{20}$$

Next we show the connection between the L^1-solutions of (6) and the Grincevičjus series. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let random variables $L, U : \Omega \to \mathbf{R}$ satisfy

$$L \neq 0 \quad a.e., \quad 0 < \int_{\Omega} \log |L(\omega)| \, dP(\omega) < +\infty, \tag{21}$$

$$\int_{\Omega} logmax \left\{ \frac{|U(\omega)|}{|L(\omega)|}, 1 \right\} dP(\omega) < +\infty, \tag{22}$$

$$P(Lc-U=c)<1 \quad \text{for all } c\in\mathbf{R}.$$
(23)

Fix a sequence $\{(\xi_n,\eta_n):n\in\mathbf{N}\}$ of independent identically distributed vectors of random variables distributed as $(\frac{1}{L},\frac{U}{L})$. It's shown that in [19] (also [13]) that the Grincevi c jus series

$$\sum_{n=1}^{\infty}\eta_n\prod_{m=1}^{n-1}\xi_m \tag{24}$$

converges almost surely and its probability distribution function F is either absolutely continuous or purely singular, and satisfies the functional-integral equation

$$F(x)=\int_{L>0}F(L(\omega)x-U(\omega))dP(\omega)+\int_{L<0}[1-F(L(\omega)x-U(\omega))dP(\omega)]. \tag{25}$$

All the integrals in (21)-(25) are Lebesgue-Stieltjes integrals.

Assume $L(\Omega)\subset\{\varepsilon\lambda^m:\varepsilon\in\{-1,1\},m\in\{1,...,M-1\}\}$ and $U(\Omega)\subset\{-N,...,N\}$. Suppose $p_{m,n}^{\varepsilon}=P(L=\varepsilon\lambda^m,U=n)$ and $c_{m,n}^{\varepsilon}=\lambda^m p_{m,n}^{\varepsilon}$ for all $\varepsilon\in\{-1,1\}$, $m\in\{1,...,M-1\}$, $n\in\mathbf{R}$. We get $\sum_{m=1}^{M-1}\sum_{n=-N}^{N}(p_{m,n}^{1}+p_{m,n}^{-1})=1$ since P is a probability measure, it follows that (5) is satisfied. Obviously, (21) and (22) also hold. Condition (23) can be rewritten as (20). (25) now takes the form

$$F(x)=\sum_{m=1}^{M-1}\sum_{n=-N}^{N}p_{m,n}^{1}F(\lambda^m x-n)+\sum_{m=1}^{M-1}\sum_{n=-N}^{N}p_{m,n}^{-1}[1-F(-\lambda^m x-n)]. \tag{26}$$

The next Lemma is easily derived from [20].

Lemma 3.1. Suppose (H). Then (\ref{e38}) has exactly one solution in the class of all continuous probability distribution functions. Moreover, this solution is either absolutely continuous or purely singular.

Assume (H), we denote by F_* the unique probability distribution solution of (26). Just like the case of M=2 in [13], we have the following result.

Lemma 3.2. Assume (H). $\dim\mathbf{V_c}=1$ if and only if F_* is absolutely continuous.

Proof: If F_* is absolutely continuous, then it's density is a non-trivial L^1-solution of (6). On the other hand, if $\dim\mathbf{V_c}=1$, fix $f(x)\in\mathbf{V_c}\setminus\{0\}$. By Theorem 2.3, without loss of generality we can assume that $f(x)$ is nonnegative. Let $a=\int_{\mathbf{R}}f(x)dx$. It is clear that $a\neq 0$. By a simple calculation we get $F_*(x)=\frac{1}{a}\int_{-\infty}^{x}f(t)dt$. Hence, $F_*(x)$ is absolutely continuous.

Based on [21] (also [13]), we get the following lemma in a similar way.

Lemma 3.3. Let F be a continuous probability distribution function. If $\hat{F}(2n\pi)=0$ for $n\in\mathbf{Z}\setminus\{0\}$, then F is a contraction; in particular, F is absolutely continuous.

Lemma 3.4. Assume (H) and suppose that there exists nonnegative constants $C_m^{\varepsilon},m=1,...,M-1,\varepsilon=\pm 1$ such that

$$\sum_{n\in\mathbf{Z}}p_{m,\lambda^m n+\gamma}^{\varepsilon}=C_m^{\varepsilon}\quad\forall\quad\varepsilon\in\{-1,1\},\gamma\in\{0,...,\lambda^m-1\}, \tag{27}$$

$$\sum_{m=1}^{M-1} \lambda^m (C_m^1 + C_m^{-1}) = 1. \tag{28}$$

Then F_* is absolutely continuous.

Proof: First, we show that under assumptions (5) and (27), (28) is necessary. Actually, by (27) and (5), we get

$$\sum_{m=1}^{M-1} \lambda^m (C_m^1 + C_m^{-1}) = \sum_{m=1}^{M-1} \sum_{\varepsilon=\pm 1} \sum_{\gamma=1}^{\lambda^m -1} \sum_{n\in\mathbf{Z}} p_{m,\lambda^m n+\gamma}^{\varepsilon} = \sum_{(m,n,\varepsilon)\in\mathbf{S}} p_{m,n}^{\varepsilon} = 1.$$

According to Lemma 3.3, it's enough to prove that $\hat{F}_*(2k\pi) = 0$ for every $k \in \mathbf{Z} \setminus \{0\}$.

Taking the Fourier-Stieltjes transform of F_* in (26), we get

$$\hat{F}_*(t) = \sum_{(m,n,\varepsilon)\in\mathbf{S}} p_{m,n}^{\varepsilon} e^{\frac{-i\varepsilon nt}{\lambda^m}} \hat{F}_*(\frac{\varepsilon t}{\lambda^m}). \tag{29}$$

Fix $k \in \mathbf{Z} \setminus \{0\}$. Using (29) and (27), we get

$$\begin{aligned} \hat{F}_*(2k\pi) &= \sum_{(m,n,\varepsilon)\in\mathbf{S}} p_{m,n}^{\varepsilon} e^{\frac{-i\varepsilon n 2k\pi}{\lambda^m}} \hat{F}_*(\frac{\varepsilon 2k\pi}{\lambda^m}) \\ &= \sum_{m=1}^{M-1} \sum_{\varepsilon=\pm 1} \sum_{\gamma=0}^{\lambda^m -1} \sum_{n\in\mathbf{Z}} p_{m,\lambda^m n+\gamma}^{\varepsilon} e^{\frac{-i\varepsilon 2k\gamma\pi}{\lambda^m}} \hat{F}_*(\frac{\varepsilon 2k\pi}{\lambda^m}) \\ &= \sum_{m=1}^{M-1} \sum_{\varepsilon=\pm 1} C_m^{\varepsilon} \hat{F}_*(\frac{\varepsilon 2k\pi}{\lambda^m}) \sum_{\gamma=0}^{\lambda^m -1} e^{\frac{-i\varepsilon 2k\gamma\pi}{\lambda^m}}. \end{aligned} \tag{30}$$

Since $\sum_{\gamma=0}^{\lambda^m -1} e^{\frac{-i\varepsilon 2\gamma\pi}{\lambda^m}} = 0,$ we get

$$\hat{F}_*(2\pi) = \sum_{m=1}^{M-1} \sum_{\varepsilon=\pm 1} C_m^{\varepsilon} \hat{F}_*(\frac{\varepsilon 2\pi}{\lambda^m}) \sum_{\gamma=0}^{\lambda^m -1} e^{\frac{-i\varepsilon 2\gamma\pi}{\lambda^m}} = 0.$$

And so does $\hat{F}_*(-2\pi) = 0$. Fix now $l \in \mathbf{Z}^+$ and suppose that $\hat{F}_*(2k\pi) = 0$ for any $k \in \mathbf{Z}$ such that $0 < |k| < l$. By (30), we obtain

$$\hat{F}_*(2l\pi) = \sum_{m=1}^{M-1} \sum_{\varepsilon=\pm 1} C_m^{\varepsilon} \hat{F}_*(\frac{\varepsilon 2l\pi}{\lambda^m}) \sum_{\gamma=0}^{\lambda^m -1} e^{\frac{-i\varepsilon 2l\gamma\pi}{\lambda^m}}.$$

Fix $m \in \{1,\dots,M-1\}$. If $\lambda^m | \ l$, then we obtain $\hat{F}_*(\frac{\varepsilon 2l\pi}{\lambda^m}) = 0$. If $\lambda^m \nmid l$, then we get $\sum_{\gamma=0}^{\lambda^m -1} e^{\frac{-i\varepsilon 2l\gamma\pi}{\lambda^m}} = 0.$ Hence, we get $\hat{F}_*(2l\pi) = 0$. By induction, we get $\hat{F}_*(2k\pi) = 0$ for every $k \in \mathbf{Z} \setminus \{0\}$.

From Lemma 3.1 - Lemma 3.4, we can easily derive the following result.

Theorem 3.4. Assume (H) and suppose that there exists nonnegative constants $D_m^{\varepsilon}, m = 1,\dots,M-1, \varepsilon = \pm 1$ such that

$$\sum_{n\in\mathbf{Z}} c_{m,\lambda^m n+\gamma}^{\varepsilon} = D_m^{\varepsilon} \quad \forall \varepsilon \in \{-1,1\}, \gamma \in \{0,\dots,\lambda^m -1\}, \tag{31}$$

$$\sum_{m=1}^{M-1} (D_m^1 + D_m^{-1}) = 1. \tag{32}$$

Then $\dim \mathbf{V_c} = 1$.

We derive the following corollary by Theorem 3.4 for $M=2$. This result has been proved in [13].

Corollary 3.4. In Theorem 3.4, assume $M=2$. Suppose (H) and assume that there exists nonnegative constant D_1^1, D_1^{-1} such that

$$\sum_{n\in\mathbf{Z}} c_{1,\lambda n+\gamma}^{\varepsilon} = D_1^{\varepsilon} \quad \text{for all} \quad \varepsilon\in\{-1,1\}, \gamma\in\{0,1\}, \tag{33}$$

$$D_1^1 + D_1^{-1} = 1. \tag{34}$$

Then $\dim \mathbf{V_c} = 1$.

4. REMARKS AND EXAMPLES

We finish this paper with two remarks and two examples.

Remark 4.1. All the results mentioned in this paper, except Lemma 3.4 and Theorem 3.4, also hold if we replace the integer $\lambda \geq 2$ with arbitrary real $\lambda > 1$.

Remark 4.2. Lemma 3.1 - Lemma 3.4 and Theorem 3.4 also hold when $N \to +\infty$. In this case, we should add the following condition:

$$\sum_{m=1}^{M-1} \sum_{|n|>\lambda^m} \lambda^{-m} log\,|\,n\,|\,(c_{m,n}^1 + c_{m,n}^{-1}) < +\infty.$$

Example 1: Let $M=2, \lambda=2, N=5.$ Then (6) takes the form

$$f(x) = \sum_{n=-5}^{5} \left[c_{1,n}^1 f(2x-n) + c_{1,n}^{-1} f(-2x-n) \right].$$

If the nonnegative coefficients $c_{m,n}^{\varepsilon}$ satisfy

$$\begin{aligned} & c_{1,k}^1 + c_{1,-3k}^{-1} < 2 \quad \forall k\in\mathbf{R}, \\ & \sum_{n\in\mathbf{Z}} c_{1,2n+\gamma}^{\varepsilon} = D_1^{\varepsilon} \quad \text{for all} \quad \varepsilon\in\{-1,1\}, \gamma\in\{0,1\}, \\ & D_1^1 + D_1^{-1} = 1, \end{aligned}$$

then by Corollary 3.4, $\dim \mathbf{V_c} = 1.$ Particularly, we can choose $D_1^1 = 0.4082$, $D_1^{-1} = 0.5918$. Select $c_{1,2}^1 = c_{1,5}^1 = 0.102, c_{1,3}^1 = c_{1,4}^1 = 0.3062, c_{1,0}^{-1} = 0.0458, c_{1,1}^{-1} = 0.3418, c_{1,2}^{-1} = 0.546, c_{1,3}^{-1} = 0.25$ and $c_{1,n}^{\varepsilon} = 0$ for the rest of the coefficients. It is clear that the given coefficients satisfy (H)
and (33), (34). Thus, according to Corollary 3.4, $\dim \mathbf{V_c} = 1.$ Moreover, for $f \in \mathbf{V_c} \setminus \{0\}$, according to Theorem 2.3, f is either positive or negative on $\mathbf{J_c}$ almost everywhere. This example has already been shown in [10].

Example 2. Let $M=3, \lambda=2, N=2.$ Then (6) takes the form

$$f(x) = \sum_{n=-2}^{2} \left[c_{1,n}^1 f(2x-n) + c_{1,n}^{-1} f(-2x-n) \right] + \sum_{n=-2}^{2} \left[c_{2,n}^1 f(4x-n) + c_{2,n}^{-1} f(-4x-n) \right].$$

If the nonnegative coefficients $c_{m,n}^{\varepsilon}$ satisfy

$$\frac{1}{2}(c_{1,k}^{1}+c_{1,-3k}^{-1})+\frac{1}{4}(c_{2,3k}^{1}+c_{2,-5k}^{-1})<1 \quad \forall k\in\mathbf{R},$$
$$(D_1^1+D_1^{-1}+D_2^1+D_2^{-1})=1,$$
$$\sum_{n\in\mathbf{Z}} c_{m,2^m n+\gamma}^{\varepsilon}=D_m^{\varepsilon} \quad \forall m\in\{1,2\},\varepsilon\in\{-1,1\},\gamma\in\{0,\dots,2^m-1\},$$

then by Theorem 3.4, $\dim\mathbf{V_c}=1.$ Especially, we can choose $D_1^1=D_1^{-1}=D_2^{-1}=D_2^1=\frac{1}{4}.$ Select $c_{1,-2}^{1}=c_{1,0}^{1}=c_{1,2}^{1}=\frac{1}{12},c_{1,-1}^{1}=c_{1,1}^{1}=c_{1,-1}^{-1}=c_{1,1}^{-1}=\frac{1}{8},c_{1,0}^{-1}=c_{2,0}^{1}=c_{2,1}^{1}=c_{2,0}^{-1}=c_{2,-1}^{-1}=\frac{1}{4}$ and $c_{\varepsilon}^{m,n}=0$ for the rest of the coefficients. We can easily get (H) and (31), (32) in this case. Thus,

according to Theorem 3.4, we get $\dim\mathbf{V_c}=1.$ Moreover, by Corollary 2.3, f is either positive or negative on $\mathbf{J}=[-2,2]$ for $f\in\mathbf{V_c}\setminus\{0\}.$

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