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DISCRIMINANT ANALYSIS IN AR(p) PLUS DIFFERENT NOISES PROCESSES*

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Abstract – The problem of discrimination between two stationary AR(p) plus noise processes is considered when the noise process are different in two models. The discrimination rule leads to a quadratic form with cumbersome matrices. An approximate and analytic form is given to distribution of the discriminant. The simulation study has been used to show the performance of discrimination rule. The cumulants of discriminant function are obtained and show them to be very close to the true values given in literature.

Keywords – Log-likelihood discrimination, AR(p) plus noise process, band matrix, pearson-curves

1. INTRODUCTION

Discrimination of time series data is an important area with applications in various disciplines. In cardiology, where electrocardiographics (ECGs) signals taken from different patients are classified to a particular type of patient. In seismology, the general problem of interest is in distinguishing the underground nuclear explosions from natural earthquakes ([1-4]).

Detecting a signal embedded in a noise series is also an important technique in statistical pattern recognition. Some other applications are in biology and developmental psychology (see [5, 6] and therein references for some more applications).

The majority of works in time series discrimination, however, is devoted to considering ARMA processes which can be expressed as a linear combination of white noise processes [7]. Recently, much attention has been paid to other processes. However, these approaches usually lead to numerical methods instead of analytic methods [8-11].

The log-likelihood ratio is usually considered as an appropriate criterion to discriminate between the two models. For Gaussian models, the discrimnant function is expressed in terms of a linear combination of independent chi-square random variables, each with one degree of freedom. The coefficients are the eigenvalues of a matrix based on the covariance matrices for the two models. The eigenvluaes are calculated numerically. Chan et al. [12] gave an approximate analytic solution for the coefficients in ARMA processes. This was followed by Chinipardaz [13] for an autoregressive model of order one, AR(1), with an extra noise and again extended for autoregressive model of order p, AR(p), with an extra noise in [14]. In the two latter papers it is assumed that the extra noises are the same for the two competition models.

In this article the discrimination has been considered for AR(p) models with these different extra noises for the competition models. To give a motivation example, assume that a missile fired from a submarine is tracked using the observations taken from the satellite observations, y_t, which include an extra noise with the actual position of the missile. The problem of interest is to allocate the actual position,

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x_t, of the missile to one of two competition AR(p) models. However, the obtained observations are collected by different radars.

This article is organized as follows: In the next section we present our method as well as some primary results required in the next section. In section three the discriminant rule based on log-likelihood ratio is given. In the forth section a simulation study has been done to show the performance of the criterion given in section three. The distribution of the discriminant function is studied in section five. Finally, in the last section the first four cumulants of the discriminant function are obtained and compared with those given in numerical methods in classical approaches.

2. DISCRIMINATION BETWEEN TWO AR(p) PLUS NOISES MODELS

Consider an observed time series vector, $\mathbf{y} = (y_1, y_2, ..., y_T)'$ to be classified, to one of two models $H_i (i = 1,2)$ where

$$H_1: y_t = x_t + \varepsilon_{1t}$$
$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \cdots + \alpha_p x_{t-p} + \eta_{1t}$$

and

$$H_2: y_t = x_t + \varepsilon_{2t}$$
$$x_t = \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_p x_{t-p} + \eta_{2t} \tag{1}$$

such that

$$\mathbf{y} \big| H_1 \sim N(0, \mathbf{\Sigma}_1)$$
$$\mathbf{y} \big| H_2 \sim N(0, \mathbf{\Sigma}_2)$$

and

$$\mathbf{\Sigma}_1 = \sigma^2_{\eta_1} \mathbf{\Sigma}_\alpha + \sigma^2_{\varepsilon_1} \mathbf{I}, \qquad \mathbf{\Sigma}_2 = \sigma^2_{\eta_2} \mathbf{\Sigma}_\beta + \sigma^2_{\varepsilon_2} \mathbf{I} \tag{2}$$

Where $\mathbf{\Sigma}_\alpha$ and $\mathbf{\Sigma}_\beta$ are covariance matrices of AR(p) with parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_p)$ $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_p)$, respectively, and $\mathbf{I}$ is identity matrix of dimension T. We assume that ε_{it} and η_{it} are uncorrelated white noise disturbances with mean zero and variances $\sigma^2_{\varepsilon_i}$ and $\sigma^2_{\eta_i}$ respectively for $i = 1,2$, and $\text{cov}(\varepsilon_{it}, \eta_{it}) = 0$, for all t and i.
The probability density function $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_T)$ under hypothesis H_1 is:

$$p\left(\boldsymbol{\eta} | \sigma_{\eta_1}{}^2\right) = \left(2\pi\sigma^2_{\eta_1}\right)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2\sigma^2_{\eta_1}} \sum_{t=1}^{T} \eta_t^2\right\}$$

Transforming from $\left\{\eta_{t_i}\right\}$ to $\left\{x_t\right\}$, with Jacobin=1, the conditional probability density of $\mathbf{x}$ conditional on $\mathbf{x}_0 = \left(x_{1-p}, \ldots, x_{-1}, x_0\right)$ under hypothesis H_1 is:

$$p\left(\mathbf{x} \mid \sigma^2_{\eta_1}, \mathbf{x}_0\right) = \left(2\pi\sigma^2_{\eta_1}\right)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2\sigma^2_{\eta_1}} \sum_{t=1}^{T} \left(x_t - \alpha_1 x_{t-1} - \cdots - \alpha_p x_{t-p}\right)^2\right\}.$$

With some manipulations

$$p\left(\mathbf{x} \mid \sigma_{\eta_1}^2, \mathbf{x}_0\right) = \left(2\pi\sigma_{\eta_1}^2\right)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2\sigma_{\eta_1}^2}\sum_{i=0}^{p}\sum_{j=0}^{p}\left(-1\right)^{i+j}\alpha_i\alpha_j\sum_{t=1}^{T}x_{t-i}x_{t-j}\right\}$$

$$= \left(2\pi\sigma_{\eta_1}^2\right)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2\sigma_{\eta_1}^2}\left[\sum_{i=\circ}^{p}\alpha_i^2\sum_{t=i+1}^{T+i}x_{t-i}^2 + 2\sum_{i=\circ}^{p-1}\sum_{j=i+1}^{p}(-1)^{i+j}\alpha_i\alpha_j\sum_{t=j+1}^{T+i}x_{t-i}x_{t-j}\right]\right\}$$

$$= \left(2\pi\sigma_{\eta 1}^2\right)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2\sigma_{\eta_1}^2}\left[\mathbf{x}'\mathbf{B}_{2p+1,\alpha}\mathbf{x}\right]\right\} \tag{3}$$

$\mathbf{B}_{2p+1,\alpha}$ is the band matrix of dimension *T* of band width (2p+1) defined by

$$\mathbf{B}_{2p+1,\alpha} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_p & 0 & \cdots & 0 \\ a_1 & a_0 & a_1 & \cdots & a_{p-1} & a_p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_p & a_{p-1} & \cdots & a_0 \end{pmatrix}. \tag{4}$$

Therefore, $\mathbf{y} \mid H_1 \sim N(0,\ \sigma_{\eta_1}^2\mathbf{B}_{2p+1,\alpha}^{-1} + \sigma_{\varepsilon_1}^2\mathbf{I})$ and $\mathbf{y} \mid H_2 \sim N(0,\ \sigma_{\eta_2}^2\mathbf{B}_{2p+1,\beta}^{-1} + \sigma_{\varepsilon_2}^2\mathbf{I})$.
The properties of this band matrix have been studied in [15].
The band matrix of order 2p+1 can be approximated with a polynomial of order 3, $\mathbf{B}_3$. i. e.

$$\mathbf{B}_{2p+1,\alpha} = \sum_{j=0}^{p} c_j\mathbf{B}_3^j, \tag{5}$$

Where $[\mathbf{B}_3]_{ij} = \begin{cases} -1 & \mid i-j \mid = 1 \\ 0 & \text{otherwise} \end{cases}$ and the coefficients, $c_j \quad (j = 0,1,\ldots,p)$, have to be obtained from the structure of the covariance matrix. The *r*th eigenvalue of band matrix $\mathbf{B}_3$ is

$$\lambda_r = -2\cos\left(\frac{r\pi}{T+1}\right)$$

and the *r*th normalized eigenvector associated with λ_r, denoted by $\boldsymbol{\xi}_r$, is given by

$$\boldsymbol{\xi}_r' = \sqrt{\frac{2}{T+1}}\left\{\sin\frac{r\pi}{T+1}, \sin\frac{2r\pi}{T+1}, \ldots, \sin\frac{Tr\pi}{T+1}\right\} \quad \left(r = 1,\ldots,T\right) \text{ [13].}$$

The $T \times T$ symmetric matrix of eigenvectors is $\mathbf{L} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_T)$.

3. DISCRIMINATION BASED ON LIKELIHOOD RATIO

The log-likelihood ratio discriminant function, is

$$LLR = -\frac{1}{2}\ln\frac{\left|\boldsymbol{\Sigma}_1\right|}{\left|\boldsymbol{\Sigma}_2\right|} + \frac{1}{2}\mathbf{y}'\left(\boldsymbol{\Sigma}_2^{-1} - \boldsymbol{\Sigma}_1^{-1}\right)\mathbf{y}$$
$$= -\frac{1}{2}\ln\frac{\left|\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\mathbf{B}_{2p+1,\alpha}^{-1}\right|}{\left|\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\mathbf{B}_{2p+1,\beta}^{-1}\right|} \tag{6}$$
$$+ \frac{1}{2}\mathbf{y}'\left[\left(\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\mathbf{B}_{2p+1,\beta}^{-1}\right)^{-1} - \left(\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\mathbf{B}_{2p+1,\alpha}^{-1}\right)^{-1}\right]\mathbf{y}.$$

We assign $\mathbf{y}$ to H_1 if $LLR \geq 0$, i.e.

$$\mathbf{y}'\left[\left(\sigma^2_{\varepsilon_2}\mathbf{I}+\sigma^2_{\eta_2}\mathbf{B}^{-1}_{2p+1,\beta}\right)^{-1}-\left(\sigma^2_{\varepsilon_1}\mathbf{I}+\sigma^2_{\eta_1}\mathbf{B}^{-1}_{2p+1,\alpha}\right)^{-1}\right]\mathbf{y}\geq \ln\frac{\left|\sigma^2_{\varepsilon_1}\mathbf{I}+\sigma^2_{\eta_1}\mathbf{B}^{-1}_{2p+1,\alpha}\right|}{\left|\sigma^2_{\varepsilon_2}\mathbf{I}+\sigma^2_{\eta_2}\mathbf{B}^{-1}_{2p+1,\beta}\right|} \tag{7}$$

and to H_2 otherwise. Now, we have this theorem:

Theorem: The log-likelihood ratio is approximated by

$$LLR=-\frac{1}{2}\ln\left(\frac{\prod_{r=1}^{T}V_{1r}}{\prod_{r=1}^{T}V_{2r}}\right)+\frac{1}{2}\sum_{r=1}^{T}\left[\frac{1}{V_{2r}}-\frac{1}{V_{1r}}\right]z_r^2 \tag{8}$$

Where

$$V_{1r}=\sigma^2_{\varepsilon_1}+\sigma^2_{\eta_1}\left[\sum_{j=0}^{p}c_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1} \tag{9}$$

$$V_{2r}=\sigma^2_{\varepsilon_2}+\sigma^2_{\eta_2}\left[\sum_{j=0}^{p}d_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1} \tag{10}$$

and

$$Z_r=\boldsymbol{\xi}_r'\mathbf{Y}=\sqrt{\frac{2}{T+1}}\sum_{t=1}^{T}Y_t\sin\left(\frac{tr\pi}{T+1}\right)$$

where c_j and d_j are obtained from the structure of $\mathbf{B}_{2p+1,\alpha}$ and $\mathbf{B}_{2p+1,\beta}$, respectively.

Proof: Appendix

Therefore, by discrimination rule we assign **y** to H_1 if

$$\sum_{r=1}^{T}\left[\frac{1}{V_{2r}}-\frac{1}{V_{1r}}\right]z_r^2\geq\ln\left(\frac{\prod_{r=1}^{T}V_{1r}}{\prod_{r=1}^{T}V_{2r}}\right). \tag{11}$$

3.1. Some special cases and generalizations

AR(1) process without observed noise and $\sigma^2_{\eta_2}=\sigma^2_{\eta_1}=\sigma^2_{\eta}$.

In this case we have: $c_\circ=1+\alpha^2, c_1=\alpha$ and $d_\circ=1+\beta^2, d_1=\beta$

$$V_{1r}=\sigma^2_\eta\left[1+\alpha^2-2\alpha\cos\frac{r\pi}{T+1}\right]^{-1}$$

$$V_{2r}=\sigma^2_\eta\left[1+\beta^2-2\beta\cos\frac{r\pi}{T+1}\right]^{-1}$$

Therefore, the discrimination rule leads to assign **y** to H_1 if

$$\frac{\beta-\alpha}{\sigma_\eta^2}\sum_{r=1}^{T}\left(\beta+\alpha-2\cos\frac{r\pi}{T+1}\right)z_r^2 \geq \ln\frac{\prod_{r=1}^{T}\left[1+\beta^2-2\beta\cos\left(\frac{r\pi}{T+1}\right)\right]}{\prod_{r=1}^{T}\left[1+\alpha^2-2\alpha\cos\left(\frac{r\pi}{T+1}\right)\right]}$$

and to H_2, otherwise. This conclusion has been obtained by Chan et al. [12].

AR(1) process with equal observed noise.

Consider the discrimination between two AR(1) processes with the same observation noise or equal variance noises. We have $c_0 = 1+\alpha^2, c_1 = \alpha$ and $d_0 = 1+\beta^2, d_1 = \beta$ so

$$V_{1r} = \sigma_\varepsilon^2 + \sigma_\eta^2\left[1+\alpha^2-2\alpha\cos\frac{r\pi}{T+1}\right]^{-1}$$

and

$$V_{2r} = \sigma_\varepsilon^2 + \sigma_\eta^2\left[1+\beta^2-2\beta\cos\frac{r\pi}{T+1}\right]^{-1}$$

Discrimination rule will be obtained by putting this quantity in (11). This has been obtained in [13].

Our approach may be generalized for non stationary *ARI(p, d)* models. Let

$$H_1 : x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \cdots + \alpha_p x_{t-p} + \eta_{1t}$$
$$H_2 : x_t = \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_p x_{t-p} + \eta_{2t}$$

and $y_t = (1-B)^d x_t$, where B is the backward shift, we have

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = \begin{pmatrix} 1 & \circ & \circ & \cdots & \circ \\ -1 & 1 & \circ & \cdots & \circ \\ \circ & -1 & 1 & \cdots & \circ \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \circ & \circ & \circ & -1 & 1 \end{pmatrix}^d \begin{pmatrix} x_1 \\ \vdots \\ x_T \end{pmatrix} = \begin{pmatrix} \binom{d}{\circ} & \circ & \circ & \circ & \cdots & \circ \\ -\binom{d}{1} & \binom{d}{\circ} & \circ & \circ & \cdots & \circ \\ \binom{d}{2} & -\binom{d}{1} & \binom{d}{\circ} & \circ & \cdots & \circ \\ -\binom{d}{3} & \binom{d}{2} & -\binom{d}{1} & \binom{d}{\circ} & \cdots & \circ \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \circ & \circ & \cdots & -\binom{d}{d-1} & \cdots & \binom{d}{\circ} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_T \end{pmatrix}$$

In fact, $\mathbf{y} = \mathbf{W}\,\mathbf{x}$ where the (i,j) th element of the matrix for difference is

$$\left[\mathbf{W}\right]_{i,j} = \begin{cases} \binom{d}{i-j}(-1)^{i-j} & i = j, j+1, \ldots, j+d \\ \circ & \text{otherwise.} \end{cases}$$

It can be shown that inverse matrix is given as

$$\left[\mathbf{W}^{-1}\right]_{i,j} = \begin{cases} \binom{i-j+d-1}{d-1} & i \geq j \\ \circ & \text{otherwise } [16]. \end{cases}$$

$$\mathbf{y} \sim N\left(\circ, \mathbf{W}\,\boldsymbol{\Sigma}\,\mathbf{W}'\right)$$

Under this transformation whit Jacobin=1, the log-likelihood is

$$\begin{aligned} LLR &= -\frac{1}{2}\ln\frac{\left|\mathbf{W}\,\boldsymbol{\Sigma}_1\mathbf{W}'\right|}{\left|\mathbf{W}\,\boldsymbol{\Sigma}_2\mathbf{W}'\right|} + \frac{1}{2}\mathbf{y}'\left((\mathbf{W}\,\boldsymbol{\Sigma}_2\mathbf{W}')^{-1} - (\mathbf{W}\,\boldsymbol{\Sigma}_1\mathbf{W}')^{-1}\right)\mathbf{y} \\ &= -\frac{1}{2}\ln\frac{\left|\sigma^2_{\eta_1}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\alpha}\mathbf{W}'\right|}{\left|\sigma^2_{\eta_2}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\beta}\mathbf{W}'\right|} \\ &+ \frac{1}{2}\mathbf{y}'\left[\left(\sigma^2_{\eta_2}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\beta}\mathbf{W}'\right)^{-1} - \left(\sigma^2_{\eta_1}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\alpha}\mathbf{W}'\right)^{-1}\right]\mathbf{y}. \end{aligned}$$

and assign $\mathbf{y}$ to H_1 if

$$\mathbf{y}'\left[\left(\sigma^2_{\eta_2}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\beta}\mathbf{W}'\right)^{-1} - \left(\sigma^2_{\eta_1}\,\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\alpha}\mathbf{W}'\right)^{-1}\right]\mathbf{y} \geq \ln\frac{\left|\sigma^2_{\eta_1}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\alpha}\mathbf{W}'\right|}{\left|\sigma^2_{\eta_2}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\beta}\mathbf{W}'\right|}$$

$$\mathbf{y}'\left[\frac{1}{\sigma^2_{\eta_2}}(\mathbf{W}')^{-1}\mathbf{B}_{2p+1,\beta}\mathbf{W}^{-1} - \frac{1}{\sigma^2_{\eta_1}}(\mathbf{W}')^{-1}\mathbf{B}_{2p+1,\alpha}\mathbf{W}^{-1}\right]\mathbf{y} \geq \ln\frac{\left|\sigma^2_{\eta_1}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\alpha}\mathbf{W}'\right|}{\left|\sigma^2_{\eta_2}\mathbf{W}\,\mathbf{B}^{-1}_{2p+1,\beta}\mathbf{W}'\right|}$$

$$\mathbf{y}'\left[\frac{1}{\sigma^2_{\eta_2}}(\mathbf{W}')^{-1}\sum_{j=1}^{p}\mathbf{B}^{j}_{3,\beta}\mathbf{W}^{-1} - \frac{1}{\sigma^2_{\eta_1}}(\mathbf{W}')^{-1}\sum_{j=1}^{p}\mathbf{B}^{j}_{3,\alpha}\mathbf{W}^{-1}\right]\mathbf{y} \geq \ln\frac{\sigma^2_{\eta_1}\left|\sum_{j=1}^{p}\mathbf{B}^{j}_{3,\alpha}\right|}{\sigma^2_{\eta_2}\left|\sum_{j=1}^{p}\mathbf{B}^{j}_{3,\beta}\right|}$$

and to H_2 otherwise.

4. NUMERICAL STUDY

The performance of the discrimination function can be studied by the numerical methods. For models AR(1) and AR(2) the misclassification rate was investigated by simulation.

Two hundred time series each of length one hundred were simulated from H_1. Then each time series was allocated to H_1 or H_2 according to (11). The number of misclassified observations was calculated. The results are given in Tables 1 and 2 for AR(1) plus noise and AR(2) plus noise processes, respectively. The model and observation noises also allowed to take different values. The results show that the method works well.

Table 1. The number of misclassifications for the discrimination between two AR(1) processes plus noise for some different values of α and β and various variances

$(\alpha,\beta)\rightarrow$ $(\sigma^2_{\eta_1},\sigma^2_{\eta_2},\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2})\downarrow$	(-0.2,-0.6)	(0.3,0.5)	(0.2,-0.2)	(-0.1,-0.8)	(0.2,0.9)
(1,1,0,0)	5	26	7	0	0
(2,2,0,0)	2	31	7	0	0
(1,2,0,0)	0	2	0	0	0
(1,1,1,1)	0	0	35	0	0
(1,1,2,2)	0	0	44	0	0
(1,1,1,2)	0	0	10	0	0
(1,1,3,1)	0	0	3	0	0
(1,2,1,2)	0	0	1	0	0
(1,10,1,1)	0	0	0	0	0
(1,1,1,10)	0	0	0	0	87

Table 2. The number of misclassifications for the discrimination between two AR(2) processes plus noise for some different values of α_1, α_2 and β_1, β_2 and various variances

$(\alpha_1, \alpha_2, \beta_1, \beta_2) \rightarrow$ $(\sigma^2_{\eta_1}, \sigma^2_{\eta_2}, \sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}) \downarrow$	(-.1,-.2,-.2,-.2)	(-.2,-.1,-.2,-.2)	(-.2,-.1,.2,-.1)	(-.3,-.2,.3,.2)	(.3,-.3,-.2,-.4)
(1,1,0,0)	17	2	8	20	0
(2,2,0,0)	29	0	2	17	0
(3,3,0,0)	17	1	4	19	0
(1,2,0,0)	22	3	1	11	25
(2,1,0,0)	0	1	0	0	0
(1,1,1,1)	25	15	34	81	20
(1,1,2,2)	74	58	68	98	58
(1,1,1,2)	32	20	17	37	61
(1,1,2,1)	0	0	0	0	0
(1,2,1,2)	3	2	2	2	3
(2,1,2,1)	0	0	0	0	0
(3,1,3,1)	0	0	0	0	0
(1,5,1,1)	0	0	0	0	0
(1,1,1,5)	0	0	0	0	0

As can be seen from Table 1 and 2 the results may be worse in the case of the parameters in the two models are close. Also, the misclassification rate is large if the variances of observation errors, $\sigma^2_{\varepsilon_1}$ and $\sigma^2_{\varepsilon_2}$ are large, especially if the second population is large. In the case of $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = 0$, the results are similar with those given in [12] as was expected.

5. DISCRIMINATION OF DISCRIMINANT FUNCTION

Suppose that $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_T)'$ is a vector time series from H_1, then

$$\mathbf{Z} = \mathbf{LY} = \sqrt{\frac{2}{T+1}} \begin{pmatrix} \sum_{t=1}^{T} Y_t \sin\left(\frac{t\pi}{T+1}\right) \\ \sum_{t=1}^{T} Y_t \sin\left(\frac{2t\pi}{T+1}\right) \\ \vdots \\ \sum_{t=1}^{T} Y_t \sin\left(\frac{Tt\pi}{T+1}\right) \end{pmatrix}$$

using the normality of $\mathbf{Y}$ and the independence of elements of $\mathbf{Z}$, $\mathbf{Z} = (Z_1, \ldots, Z_T)'$ has a multivariate normal distribution with zero mean vector and diagonal covariance matrix with (r, r) th element

$$V_{1r} = \sigma^2_{\varepsilon_1} + \sigma^2_{\eta_1} \left[\sum_{j=0}^{p} c_j \left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j \right]^{-1}.$$

Therefore,

$$\frac{Z_r^2}{V_{1r}} \sim \chi^2_{1,r}$$

Where $\chi^2_{1,r}$ is the chi-square random variable with one degree of freedom.

From (8), ignoring the constant term, the discriminate function is

$$d_Q(\mathbf{z}) = \sum_{r=1}^{T}\left[\frac{1}{V_{2r}} - \frac{1}{V_{1r}}\right]z_r^2 = \sum_{r=1}^{T}\left[\frac{V_{1r}}{V_{2r}} - 1\right]\chi_{1,r}^2.$$

Hence under H_1 the discriminant function has a linear combination of chi-square random variable with one degree of freedom with *r*th coefficient

$$\left[\frac{V_{1r}}{V_{2r}} - 1\right] = \frac{\sigma_{\varepsilon_1}^2 + \sigma_{\eta_1}^2\left[\sum_{j=0}^{p} c_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1}}{\sigma_{\varepsilon_2}^2 + \sigma_{\eta_2}^2\left[\sum_{j=0}^{p} d_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1}} - 1 \tag{12}$$

by a similar manner, if **Y** belongs to H_2 then

$$d_Q(\mathbf{z}) = \sum_{r=1}^{T}\left[1 - \frac{V_{2r}}{V_{1r}}\right]\chi_{1,r}^2 = \sum_{r=1}^{T}\left[1 - \frac{\sigma_{\varepsilon_2}^2 + \sigma_{\eta_2}^2\left[\sum_{j=0}^{p} d_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1}}{\sigma_{\varepsilon_1}^2 + \sigma_{\eta_1}^2\left[\sum_{j=0}^{p} c_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1}}\right]\chi_{1,r}^2 \tag{13}$$

6. COMPARING THE CUMULANTS OF DISCRIMINANT FUNCTION: ANALYTICAL VERSUS NUMERICAL METHODS

As was shown, the quadratic distribution of the discriminant function leads to

$$d_Q(\mathbf{x}) = \sum_{r=1}^{T}\lambda_r\chi_{1,r}^2$$

where λ_j is given in (12) or (13) based on the observations taken from H_1 *or* H_2, respectively. Many authors have tried to tabulate this distribution function [17, 18]. They consider $j>0,\ T=5,$ which is not the case of time series where *j* can be positive or negative. An alternative method is based on Pearson curves given in [19] and [20]. In this method the true density function is approximated by equating the first four cumulants. Solomon and Stephens [20] showed that the *s*th cumulant is given by

$$\kappa_s\left(d\left(\mathbf{x}\right)\right) = 2^{s-1}(s-1)!\sum_{r=1}^{T}\lambda_r^s$$

These are derived using the classical method, i.e.

$$\lambda_j = \text{eigenvalues } \frac{1}{T}\boldsymbol{\Sigma}_j(\boldsymbol{\Sigma}_2^{-1} - \boldsymbol{\Sigma}_1^{-1}), \qquad j = 1,2.$$

where $\boldsymbol{\Sigma}_j$ is the covariance matrix of *j*th model [13]. A numerical comparison between the classical methods and the method given in this paper has been considered for the first four cumulants. AR(1) plus noise model AR(2) plus noise model are given in Table 3 and Table 4, respectively. The various values of parameters and variances are considered for T=100. It should be mentioned that results given in [13] and [14] may be included in our results, considering the same observation errors.

The numerical comparison shows that our method agrees well with the classical methods being close, especially when the parameter values are large with different signs.

Table 3. Comparison of the first four cumulants of $d_Q(\mathbf{x})$ approximated by different methods for AR(1) plus noise processes $(\sigma^2_{\eta_1},\sigma^2_{\eta_2},\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2})$

$(\alpha,\beta)\downarrow$		Method	(1,1,0,0)	(5,5,0,0)	(5,1,0,0)	(1,1,1,1)	(1,1,5,5)	(1,1,2,1)	(2,1,2,1)
(0.2,0.4)	κ_1	I	3.999×10^{-2}	3.999×10^{-2}	4.199	1.79×10^{-2}	1.675×10^{-2}	2.162×10^{-2}	3.008×10^{-1}
		II	4.246×10^{-2}	4.246×10^{-2}	4.212	1.656×10^{-2}	1.622×10^{-2}	4.854×10^{-1}	9.668×10^{-1}
	κ_2	I	1.552×10^{-3}	1.552×10^{-3}	3.907×10^{-1}	4.584×10^{-4}	8.236×10^{-5}	2.486×10^{-4}	1.990×10^{-3}
		II	1.548×10^{-3}	1.548×10^{-3}	3.926×10^{-1}	4.546×10^{-4}	8.143×10^{-5}	6.344×10^{-3}	2.049×10^{-2}
	κ_3	I	2.812×10^{-7}	2.812×10^{-7}	7.753×10^{-2}	3.176×10^{-6}	3.618×10^{-7}	1.608×10^{-6}	2.739×10^{-5}
		II	5.266×10^{-7}	5.270×10^{-7}	7.800×10^{-2}	3.117×10^{-6}	3.563×10^{-7}	1.747×10^{-4}	9.137×10^{-4}
	κ_4	I	4.389×10^{-7}	4.389×10^{-7}	2.399×10^{-2}	6.464×10^{-8}	1.064×10^{-6}	2.314×10^{-8}	5.762×10^{-7}
		II	4.369×10^{-7}	4.369×10^{-7}	2.415×10^{-2}	6.373×10^{-8}	9.995×10^{-6}	7.441×10^{-6}	6.286×10^{-5}
(−0.5,0.5)	κ_1	I	1.320	1.320	10.600	3.117×10^{-1}	4.192×10^{-2}	1.480×10^{-1}	6.448×10^{-1}
		II	1.311	1.311	10.555	3.089×10^{-1}	4.146×10^{-2}	8.133×10^{-1}	1.618
	κ_2	I	1.806×10^{-1}	1.806×10^{-1}	5.890	1.987×10^{-2}	1.822×10^{-3}	7.644×10^{-3}	3.104×10^{-2}
		II	1.791×10^{-1}	1.791×10^{-1}	5.845	1.966×10^{-2}	1.801×10^{-3}	3.690×10^{-2}	1.234×10^{-1}
	κ_3	I	4.420×10^{-2}	4.420×10^{-2}	7.997	1.299×10^{-3}	1.857×10^{-5}	2.538×10^{-4}	2.862×10^{-3}
		II	4.375×10^{-2}	4.376×10^{-2}	7.821	1.282×10^{-3}	1.831×10^{-5}	3.441×10^{-3}	2.198×10^{-2}
	κ_4	I	1.800×10^{-2}	1.800×10^{-2}	17.691	1.709×10^{-4}	9.953×10^{-7}	2.164×10^{-5}	4.469×10^{-4}
		II	1.780×10^{-2}	1.780×10^{-2}	17.504	1.686×10^{-4}	9.807×10^{-7}	5.401×10^{-4}	6.459×10^{-3}
(−0.3,0.8)	κ_1	I	2.659×10^{-1}	2.659×10^{-1}	5.330	2.263×10^{-2}	7.101×10^{-2}	5.897×10^{-2}	2.361×10^{-1}
		II	2.775×10^{-1}	2.775×10^{-1}	5.387	1.540×10^{-2}	6.631×10^{-2}	5.096×10^{-1}	9.692×10^{-1}
	κ_2	I	9.119×10^{-3}	9.119×10^{-3}	7.607×10^{-1}	2.491×10^{-3}	1.149×10^{-3}	1.833×10^{-3}	3.635×10^{-3}
		II	9.097×10^{-3}	9.097×10^{-3}	7.694×10^{-1}	2.410×10^{-3}	1.904×10^{-3}	1.194×10^{-2}	2.841×10^{-2}
	κ_3	I	1.377×10^{-4}	1.377×10^{-4}	2.289×10^{-1}	5.277×10^{-5}	2.711×10^{-5}	4.485×10^{-5}	1.929×10^{-5}
		II	1.441×10^{-4}	1.441×10^{-4}	2.316×10^{-1}	4.984×10^{-5}	2.559×10^{-5}	3.404×10^{-4}	1.484×10^{-3}
	κ_4	I	1.417×10^{-5}	1.417×10^{-5}	1.065×10^{-1}	2.677×10^{-6}	1.064×10^{-6}	2.015×10^{-6}	2.170×10^{-6}
		II	1.412×10^{-5}	1.412×10^{-5}	1.078×10^{-1}	2.536×10^{-6}	9.995×10^{-6}	2.236×10^{-5}	1.258×10^{-4}

Table 4. Comparison of the first four cumulants of $d_Q(\mathbf{x})$ approximated by different methods for AR(2) plus noise processes $(\sigma^2_{\eta_1},\sigma^2_{\eta_2},\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2})$

$(\alpha_1,\alpha_2,\beta_1,\beta_2)$		Method	(1,1,0,0)	(5,5,0,0)	(5,1,0,0)	(1,1,1,1)	(1,1,5,5)	(1,1,2,1)	(2,1,2,1)
(−0.1,−0.2,−0.2,−0.2)	κ_1	I	1.005×10^{-2}	1.005×10^{-2}	4.050	5.659×10^{-3}	5.746×10^{-3}	4.506	8.943
		II	1.084×10^{-2}	1.084×10^{-2}	4.054	5.103×10^{-3}	5.484×10^{-3}	4.521	8.949
	κ_2	I	4.107×10^{-4}	4.107×10^{-4}	3.383×10^{-1}	1.405×10^{-4}	2.397×10^{-5}	4.254×10^{-1}	1.614
		II	4.039×10^{-4}	4.039×10^{-4}	3.388×10^{-1}	1.376×10^{-4}	2.343×10^{-5}	4.279×10^{-1}	1.615
	κ_3	I	1.058×10^{-7}	1.058×10^{-7}	5.804×10^{-2}	3.996×10^{-7}	5.061×10^{-8}	8.275×10^{-2}	5.870×10^{-1}
		II	7.278×10^{-8}	7.274×10^{-8}	5.812×10^{-2}	3.851×10^{-7}	4.906×10^{-8}	8.342×10^{-2}	5.878×10^{-1}
	κ_4	I	4.148×10^{-8}	4.148×10^{-8}	1.528×10^{-2}	6.745×10^{-9}	2.846×10^{-10}	2.462×10^{-2}	3.226×10^{-1}
		II	4.056×10^{-8}	4.056×10^{-8}	1.530×10^{-2}	6.558×10^{-9}	2.763×10^{-10}	2.487×10^{-2}	3.231×10^{-1}
(−0.2,−0.5,0.2,0.1)	κ_1	I	2.151×10^{-1}	2.151×10^{-1}	5.076	4.861×10^{-2}	5.665×10^{-3}	4.545	9.486
		II	2.120×10^{-1}	2.120×10^{-1}	5.060	4.701×10^{-2}	5.102×10^{-3}	4.541	9.470
	κ_2	I	1.412×10^{-2}	1.412×10^{-2}	8.452×10^{-1}	2.379×10^{-3}	2.258×10^{-4}	4.336×10^{-1}	2.033
		II	1.408×10^{-2}	1.408×10^{-2}	8.416×10^{-1}	2.374×10^{-3}	2.257×10^{-4}	4.330×10^{-1}	2.027
	κ_3	I	7.902×10^{-4}	7.902×10^{-4}	3.607×10^{-1}	4.055×10^{-5}	8.510×10^{-7}	8.682×10^{-2}	9.751×10^{-1}
		II	7.862×10^{-4}	7.862×10^{-4}	3.592×10^{-1}	4.023×10^{-5}	8.409×10^{-7}	8.666×10^{-2}	9.712×10^{-1}
	κ_4	I	8.529×10^{-5}	8.529×10^{-5}	2.550×10^{-1}	1.819×10^{-6}	1.317×10^{-8}	2.726×10^{-2}	7.662×10^{-1}
		II	8.489×10^{-5}	8.490×10^{-5}	2.538×10^{-1}	1.809×10^{-6}	1.311×10^{-8}	2.720×10^{-2}	7.627×10^{-1}
(0.3,−0.3,−0.2,−0.4)	κ_1	I	2.923×10^{-1}	2.923×10^{-1}	5.461	1.238×10^{-1}	2.616×10^{-2}	4.666	10.238
		II	2.844×10^{-1}	2.844×10^{-1}	5.422	1.202×10^{-1}	2.552×10^{-2}	4.693	10.202
	κ_2	I	1.788×10^{-2}	1.788×10^{-2}	1.001	6.417×10^{-3}	1.106×10^{-3}	4.851×10^{-1}	2.707
		II	1.713×10^{-2}	1.713×10^{-2}	9.757×10^{-1}	6.149×10^{-3}	1.060×10^{-3}	4.895×10^{-1}	2.668
	κ_3	I	1.324×10^{-3}	1.324×10^{-3}	4.873×10^{-1}	2.565×10^{-4}	1.038×10^{-5}	1.047×10^{-1}	1.773
		II	1.254×10^{-3}	1.254×10^{-3}	4.681×10^{-1}	2.435×10^{-4}	9.873×10^{-6}	1.059×10^{-1}	1.724
	κ_4	I	2.060×10^{-4}	2.060×10^{-4}	4.208×10^{-1}	2.686×10^{-5}	6.908×10^{-7}	3.454×10^{-2}	2.059
		II	1.944×10^{-4}	1.944×10^{-4}	4.004×10^{-1}	2.545×10^{-5}	6.558×10^{-7}	3.498×10^{-2}	1.981

7. APPENDIX: PROOF FOR THEOREM

Applying the transformation $\mathbf{z} = \mathbf{L}'\mathbf{y}$ to (6) leads us to

$$LLR = -\frac{1}{2}\ln\frac{\left|\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\mathbf{L}\mathbf{B}_{2p+1,\alpha}^{-1}\mathbf{L}'\right|}{\left|\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\mathbf{L}\mathbf{B}_{2p+1,\beta}^{-1}\mathbf{L}'\right|} + \frac{1}{2}\mathbf{z}'\left[\left(\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\mathbf{L}\mathbf{B}_{2p+1,\beta}^{-1}\mathbf{L}'\right)^{-1} - \left(\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\mathbf{L}\mathbf{B}_{2p+1,\alpha}^{-1}\mathbf{L}'\right)^{-1}\right]\mathbf{z}$$

Note that $\mathbf{L}$ is an orthogonal matrix. By considering the other properties of $\mathbf{L}$ which are given in [13], we have:

$$\begin{aligned}
&\mathbf{z}'\left[\left(\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\mathbf{L}\mathbf{B}_{2p+1,\beta}^{-1}\mathbf{L}'\right)^{-1} - \left(\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\mathbf{L}\mathbf{B}_{2p+1,\alpha}^{-1}\mathbf{L}'\right)^{-1}\right]\mathbf{z}\\
&= \mathbf{z}'\left[\left(\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\mathbf{L}\left(\sum_{j=0}^{p} d_j\mathbf{B}_3^j\right)^{-1}\mathbf{L}'\right)^{-1} - \left(\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\mathbf{L}\left(\sum_{j=0}^{p} c_j\mathbf{B}_3^j\right)^{-1}\mathbf{L}'\right)^{-1}\right]\mathbf{z}\\
&= \mathbf{z}'\left[\left(\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\left(\sum_{j=0}^{p} d_j\left(\mathbf{L}\mathbf{B}_3^j\mathbf{L}'\right)\right)^{-1}\right)^{-1} - \left(\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\left(\sum_{j=0}^{p} c_j\left(\mathbf{L}\mathbf{B}_3^j\mathbf{L}'\right)\right)^{-1}\right)^{-1}\right]\mathbf{z}\\
&= \mathbf{z}'\left[\left(\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\left(\sum_{j=0}^{p} d_j\mathbf{\Lambda}_3^j\right)^{-1}\right)^{-1} - \left(\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\left(\sum_{j=0}^{p} c_j\mathbf{\Lambda}_3^j\right)^{-1}\right)^{-1}\right]\mathbf{z}
\end{aligned}$$

where $\mathbf{\Lambda}_3$ is a diagonal matrix with rth diagonal element

$$\lambda_r = [\mathbf{\Lambda}_3]_{r,r} = -2\cos\left(\frac{r\pi}{T+1}\right)$$

hence

$$\mathbf{z}'\left[\left(\sigma_{\varepsilon_2}^2\mathbf{I} + \sigma_{\eta_2}^2\left(\sum_{j=0}^{p} d_j\mathbf{\Lambda}_3^j\right)^{-1}\right)^{-1} - \left(\sigma_{\varepsilon_1}^2\mathbf{I} + \sigma_{\eta_1}^2\left(\sum_{j=0}^{p} c_j\mathbf{\Lambda}_3^j\right)^{-1}\right)^{-1}\right]\mathbf{z}$$

$$= \mathbf{z}\left\{\begin{bmatrix} \sigma_{\varepsilon_2}^2 + \sigma_{\eta_2}^2\left(\sum_{j=\circ}^{p} d_j\lambda_1^j\right)^{-1} & \circ & \cdots & \circ \\ \circ & \sigma_{\varepsilon_2}^2 + \sigma_{\eta_2}^2\left(\sum_{j=\circ}^{p} d_j\lambda_2^j\right)^{-1} & \cdots & \circ \\ \vdots & \vdots & \ddots & \vdots \\ \circ & \circ & \cdots & \sigma_{\varepsilon_2}^2 + \sigma_{\eta_2}^2\left(\sum_{j=\circ}^{p} d_j\lambda_T^j\right)^{-1} \end{bmatrix}^{-1}\right.$$

$$-\left[\begin{matrix}\sigma_{\varepsilon_1}^2+\sigma_{\eta_1}^2\left(\sum_{j=\circ}^{p}c_j\lambda_1^j\right)^{-1} & \circ & \cdots & \circ\\ \circ & \sigma_{\varepsilon_1}^2+\sigma_{\eta_1}^2\left(\sum_{j=\circ}^{p}c_j\lambda_1^j\right)^{-1} & \cdots & \circ\\ \vdots & \vdots & \ddots & \vdots\\ \circ & \circ & \cdots & \sigma_{\varepsilon_1}^2+\sigma_{\eta_1}^2\left(\sum_{j=\circ}^{p}c_j\lambda_T^j\right)^{-1}\end{matrix}\right]^{-1}\Bigg\}\mathbf{z}$$

$$=\sum_{r=1}^{T}\left\{\left[\frac{1}{\sigma_{\varepsilon_2}^2+\sigma_{\eta_2}^2\left(\sum_{j=0}^{p}d_j\lambda_r^j\right)^{-1}}\right]-\left[\frac{1}{\sigma_{\varepsilon_1}^2+\sigma_{\eta_1}^2\left(\sum_{j=0}^{p}c_j\lambda_r^j\right)^{-1}}\right]\right\}z_r^2$$

considering $\lambda_r=-2\cos\left(\frac{r\pi}{T+1}\right)$ we have:

$$=\sum_{r=1}^{T}\left\{\left[\frac{1}{\sigma_{\varepsilon_2}^2+\sigma_{\eta_2}^2\left[\sum_{j=0}^{p}d_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1}}\right]-\left[\frac{1}{\sigma_{\varepsilon_1}^2+\sigma_{\eta_1}^2\left[\sum_{j=0}^{p}c_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1}}\right]\right\}z_r^2$$

In order to abbreviate, let

$$V_{1r}=\sigma_{\varepsilon_1}^2+\sigma_{\eta_1}^2\left[\sum_{j=0}^{p}c_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1}\qquad V_{2r}=\sigma_{\varepsilon_2}^2+\sigma_{\eta_2}^2\left[\sum_{j=0}^{p}d_j\left(-2\cos\left(\frac{r\pi}{T+1}\right)\right)^j\right]^{-1}$$

so we have

$$\mathbf{z}'\left[\left(\sigma_{\varepsilon_2}^2\mathbf{I}+\sigma_{\eta_2}^2\mathbf{L}\mathbf{B}_{2p+1,\beta}^{-1}\mathbf{L}'\right)^{-1}-\left(\sigma_{\varepsilon_1}^2\mathbf{I}+\sigma_{\eta_1}^2\mathbf{L}\mathbf{B}_{2p+1,\alpha}^{-1}\mathbf{L}'\right)^{-1}\right]\mathbf{z}=\sum_{r=1}^{T}\left[\frac{1}{V_{2r}}-\frac{1}{V_{1r}}\right]z_r^2$$

Similarly, we can write

$$\left|\left(\sigma_{\varepsilon_1}^2\mathbf{I}+\sigma_{\eta_1}^2\mathbf{L}\mathbf{B}_{2p+1,\alpha}^{-1}\mathbf{L}'\right)\right|=\prod_{r=1}^{T}V_{1r}$$

and

$$\left|\left(\sigma_{\varepsilon_2}^2\mathbf{I}+\sigma_{\eta_2}^2\mathbf{L}\mathbf{B}_{2p+1,\beta}^{-1}\mathbf{L}'\right)\right|=\prod_{r=1}^{T}V_{2r}$$

$$\Rightarrow \ln \frac{\left| \sigma_{\varepsilon_1}^2 \mathbf{I} + \sigma_{\eta_1}^2 \mathbf{L} \mathbf{B}_{2p+1,\alpha}^{-1} \mathbf{L}' \right|}{\left| \sigma_{\varepsilon_2}^2 \mathbf{I} + \sigma_{\eta_2}^2 \mathbf{L} \mathbf{B}_{2p+1,\beta}^{-1} \mathbf{L}' \right|} = \ln \frac{\prod_{r=1}^{T} V_{1r}}{\prod_{r=1}^{T} V_{2r}}$$

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