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REDUCED DIFFERENTIAL TRANSFORM METHOD FOR SOLVING LINEAR AND NONLINEAR WAVE EQUATIONS*

Y. KESKIN** AND G. OTURANC

Department of Mathematics, Science Faculty, Selcuk University, Konya 42003, Turkey
Email: yildiraykeskin@yahoo.com

Abstract – Reduced differential transform method (RDTM) is applied to various wave equations. To assess the accuracy of the solutions, we compare the results with the exact solutions and variational iteration method. The results reveal that the RDTM is very effective, convenient and quite accurate to systems of nonlinear equations.

Keywords – Reduced differential transform method, wave equation

1. INTRODUCTION

There are many wave equations, which are quite useful and applicable in engineering and physics such as the well-known linear and nonlinear wave equations, the wave equation in an unbounded domain, kdv equation and so on. This problem includes a vibrating string, vibrating membrane, longitudinal vibrations of an elastic rod or beam, shallow water waves, acoustic problems for the velocity potential for a fluid flow through which sound can be transmitted, transmission of electric signals along a cable, shock waves, chemical exchange processes in chromatography, sediment transport in rivers and waves in plasmas, and both electric and magnetic fields in the absence of charge and dielectric [1]. Since solving these equations requires some nonphysical assumptions, some various approximate methods have recently been developed to solve linear and nonlinear differential equations [2-9]

In this paper, some wave equations are solved by the reduced differential transform method which is presented to overcome the demerit of complex calculation of differential transform method. This method, like the differential transform method, was first introduced by Zhou in 1986 [10-11] and the variational iteration method introduced by He [12] has been used by many mathematicians and engineers to solve various functional equations. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms. Analytical solutions enable researchers to study the effect of different variables or parameters on the function under study easily. Its small size of computation in comparison with the computational size required in other numerical methods, and its rapid convergence show that the method is reliable and introduces a significant improvement in solving wave equations over existing methods. The solution procedure of the RDTM is simpler than that of traditional DTM, and the amount of computation required in RDTM is much less than that in traditional DTM. The solution obtained by the reduced differential transform method is an infinite power series for initial value problems, which can be, in turn, expressed in a closed form, the exact solution.

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**Corresponding author

2. REDUCED DIFFERENTIAL TRANSFORM METHOD

Consider a function of two variables $u(x,t)$ and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,t) = f(x)g(t)$. Based on the properties of differential transform, function $u(x,t)$ can be represented as

$$u(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{k=0}^{\infty} U_k(x)t^k$$

where $U_k(x)$ is called t-dimensional spectrum function of $u(x,t)$.

The basic definitions and operations of reduced differential transform method [13-17] are introduced as follows:

Definition 2.1. If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \tag{1}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function, while the uppercase $U_k(x)$ stand for the transformed function.

The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k . \tag{2}$$

Then combining equation (1) and (2) we write

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k . \tag{3}$$

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion. For example,

Consider $u(x,t) = e^{x+t}$, this function can be written as

$$u(x,t) = e^{x+t} = \underbrace{\left(1 + x + \frac{x^2}{2} + \ldots\right)}_{e^x}\underbrace{\left(1 + t + \frac{t^2}{2} + \ldots\right)}_{e^t} = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j$$

otherwise,

$$u(x,t)=e^{x+t}=e^{x}\underbrace{\left(1+t+\frac{t^2}{2}+...\right)}_{e^t}=e^{x}+e^{x}t+\frac{e^{x}t^2}{2}+...=\sum_{k=0}^{\infty}\underbrace{\frac{e^x}{k!}}_{U_k(x)}t^k=\sum_{k=0}^{\infty}U_k(x)t^k.$$

To illustrate the basic concepts of RDTM, consider the following general wave equation [17]:

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t),$$

With initial condition

$$u(x,0)=f(x),$$

where $L=\frac{\partial}{\partial t}$, $R=\frac{\partial^2}{\partial x^2}$ is a linear operator, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term.

According to the RDTM and Table 1, we can construct the following iteration formula:

$$(k+1)U_{k+1}(x)=G_k(x)-RU_k(x)-NU_k(x) \quad (4)$$

where $U_k(x), RU_k(x), NU_k(x)$ and $G_k(x)$ are the transformations of the functions $Lu(x,t)$, $Ru(x,t)$, $Nu(x,t)$ and $g(x,t)$ respectively.
From the initial condition, we write

$$U_0(x)=f(x). \quad (5)$$

Substituting (5) into (4) and by straightforward iterative calculations, we get the following $U_k(x)$ values. Then the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives the approximation solution as,

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n}U_k(x)t^k$$

where n is order of approximation solution.

Therefore, the exact solution of the problem is given by

$$u(x,t)=\lim_{n\to\infty}\tilde{u}_n(x,t). \quad (6)$$

Table 1. Reduced differential transformation

Functional Form	**Transformed Form**
$u(x,t)$	$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t)=u(x,t)\pm v(x,t)$	$W_k(x)=U_k(x)\pm V_k(x)$
$w(x,t)=\alpha u(x,t)$	$W_k(x)=\alpha U_k(x)$ (α is a constant)

Table 1. (Continued)

$w(x,y)=x^m t^n$	$W_k(x)=x^m\delta(k-n)$
$w(x,y)=x^m t^n u(x,t)$	$W_k(x)=x^m U_{k-n}(x)$
$w(x,t)=u(x,t)v(x,t)$	$W_k(x)=\sum_{r=0}^{k}V_r(x)U_{k-r}(x)=\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$
$w(x,t)=\frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x)=(k+1)...(k+r)U_{k+1}(x)=\frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t)=\frac{\partial}{\partial x}u(x,t)$	$W_k(x)=\frac{\partial}{\partial x}U_k(x)$
$Nu(x,t)$	**Maple Code for** Nonlinear Function restart; NF:=Nu(x,t):#Nonlinear Function m:=5: # Order u[t]:=sum(u[b]*t^b,b=0..m): NF[t]:=subs(Nu(x,t)=u[t],NF): s:=expand(NF[t],t): dt:=unapply(s,t): for i from 0 to m do n[i]:=((D@@i)(dt)(0)/i!): print(N[i],n[i]); #Transform Function od:

3. APPLICATIONS

In order to assess the advantages and accuracy of RDTM for solving wave equations, the following examples were considered.

3.1. Example

We first consider the homogeneous wave equation [18]

$$u_{tt}=u_{xx}-3u,\ \ 0<x<\pi, t>0 \tag{7}$$

with the boundary and initial conditions:

$$\begin{aligned}&B.C.\ u(0,t)=\sin(2t),\ \ u(\pi,t)=-\sin(2t)\\&I.C.\ u(x,0)=0,\ \ u_t(x,0)=2\cos(x)\end{aligned} \tag{8}$$

where $u=u(x,t)$ is a function of the variables x and t.

Then, by using the basic properties and theorems of the reduced differential transformation, we can find the transformed form of equation (7) as

$$\frac{(k+2)!}{k!}U_{k+2}(x)=\frac{\partial^2}{\partial x^2}U_k(x)-3U_k(x)\,. \tag{9}$$

Using the initial condition (8), we have

$$U_0(x)=0,\ U_1(x)=2\cos(x) \tag{10}$$

Now, substituting (10) into (9), we obtain the following $U_k\left(x\right)$ values successively

$$U_2(x)=0$$

$$U_3(x)=-\frac{4}{3}\cos(x), U_4(x)=0,\ U_5(x)=\frac{4}{155}\cos(x), U_6(x)=0, U_7(x)=-\frac{8}{315}\cos(x),\cdots$$

$$U_k(x)=\begin{cases}\dfrac{(-1)^{\frac{k-1}{2}}}{k!}2^k\cos(x), & \text{for k is odd}\\ 0, & \text{for k is even}\end{cases}$$

Finally, the differential inverse transform of $U_k(x)$ gives

$$u\left(x,t\right)=\sum_{k=0}^{\infty}U_k(x)t^k=\cos(x)\sum_{k=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{k-1}{2}}}{k!}2^k t^k\,. \tag{11}$$

Hence the closed form of (11) is

$$u\left(x,t\right)=\cos(x)\sin(2t)$$

which is the exact solutions of (7)-(8).

3.2. Example

We next consider the inhomogeneous nonlinear wave equation [18]

$$u_{tt}=u_{xx}+u+u^2-xt-x^2t^2,\ 0<x<\pi, t>0 \tag{12}$$

with the boundary and initial conditions:

$$\begin{aligned}&B.C.\ u(0,t)=0,\ u(\pi,t)=\pi t\\ &I.C.\ u(x,0)=0,\ u_t(x,0)=x\end{aligned} \tag{13}$$

Taking the differential transform of (12) and the initial condition (13) respectively, we obtain

$$\frac{(k+2)!}{k!}U_{k+2}(x)=\frac{\partial^2}{\partial x^2}U_k(x)+U_k(x)+\sum_{r=0}^{k}U_r(x)U_{k-r}(x)-x\delta(k-1)-x^2\delta(k-2) \tag{14}$$

and the transformed initial condition

$$U_0(x)=0, U_1(x)=x\,. \tag{15}$$

Then, substituting (15) into (14) we have

$$U_2(x)=0,\ U_3(x)=0$$

and

$$U_k(x)=0,\ k=4,5,6,\ldots.$$

Finally, the differential inverse transform of $U_k(x)$ gives

$$u(x,t)=\sum_{k=0}^{\infty} U_k(x)t^k = xt$$

which is the exact solution.

3.3. Example

One of the most attractive and surprising wave phenomenum is the creation of solitary waves or solitons. For this reason, consider the KdV equation which takes the form [19]:

$$u_t - 6uu_x + u_{xxx} = 0, x\in R, \tag{16}$$

and initial conditions

$$u(x,0) = -\frac{k^2}{2} sech^2\left[\frac{k}{2}x\right] \tag{17}$$

where $u=u(x,t)$ is a function of the variables x and t.
Taking the differential transform of (16), the following can be obtained

$$(k+1)U_{k+1}(x) = 6\sum_{r=0}^{k} U_{k-r}(x)\frac{\partial}{\partial x}U_r(x) - \frac{\partial^3}{\partial x^3}U_k(x) \tag{18}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function.

From the initial condition (17) we write

$$U_0(x) = -\frac{k^2}{2} sech^2\left[\frac{k}{2}x\right] \tag{19}$$

Substituting (19) into (18), we obtain the following $U_k(x)$ values successively

$$U_1(x) = -\frac{1}{2}\frac{\sinh\left(\frac{kx}{2}\right)k^5}{\cosh\left(\frac{kx}{2}\right)^3},\quad U_2(x) = -\frac{1}{8}\frac{k^8\left(-3+2\cosh\left(\frac{kx}{2}\right)^2\right)}{\cosh\left(\frac{kx}{2}\right)^4}$$

$$U_3(x) = -\frac{1}{12}\frac{k^{11}\sinh\left(\frac{kx}{2}\right)\left(\cosh\left(\frac{kx}{2}\right)^2-3\right)}{\cosh\left(\frac{kx}{2}\right)^5},$$

$$U_4(x) = -\frac{1}{96}\frac{k^{14}\left(2\cosh\left(\frac{kx}{2}\right)^4+15-15\cosh\left(\frac{kx}{2}\right)^2\right)}{\cosh\left(\frac{kx}{2}\right)^6}$$

and so on.

Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{4}$ gives four term (Order 4) approximation solution as

$$\tilde{u}_4(x,t) = \sum_{k=0}^{4} U_k(x)t^k = -\frac{1}{96}\frac{k^2\left(15k^{12}-15k^{12}\cosh\left(\frac{kx}{2}\right)^2+2k^{12}\cosh\left(\frac{kx}{2}\right)^4\right)}{\cosh\left(\frac{kx}{2}\right)^6}t^4$$

$$-\frac{1}{96}\frac{k^2\left(8k^9\sinh\left(\frac{kx}{2}\right)\cosh\left(\frac{kx}{2}\right)^3-24k^9\sinh\left(\frac{kx}{2}\right)\cosh\left(\frac{kx}{2}\right)\right)}{\cosh\left(\frac{kx}{2}\right)^6}t^3$$

$$-\frac{1}{96}\frac{k^2\left(-36k^6\cosh\left(\frac{kx}{2}\right)^2-24k^6\cosh\left(\frac{kx}{2}\right)^4\right)}{\cosh\left(\frac{kx}{2}\right)^6}t^2-\frac{1}{2}\frac{k^5\sinh\left(\frac{kx}{2}\right)}{\cosh\left(\frac{kx}{2}\right)^3}t-\frac{1}{2}\frac{k^2}{\cosh\left(\frac{kx}{2}\right)^2}$$

Therefore, the exact solution of the problem is given by

$$u(x,y) = \lim_{n\to\infty}\tilde{u}_n(x,y).$$

This solution is convergent to the exact solution [19] and the same as the approximate solution of the variational iteration method [20]. (Fig. 1)

$$u(x,t) = -\frac{k^2}{2}\sec h\left(\frac{k}{2}\left(x-k^2t\right)\right).$$

which is the exact solutions of (16)-(17)

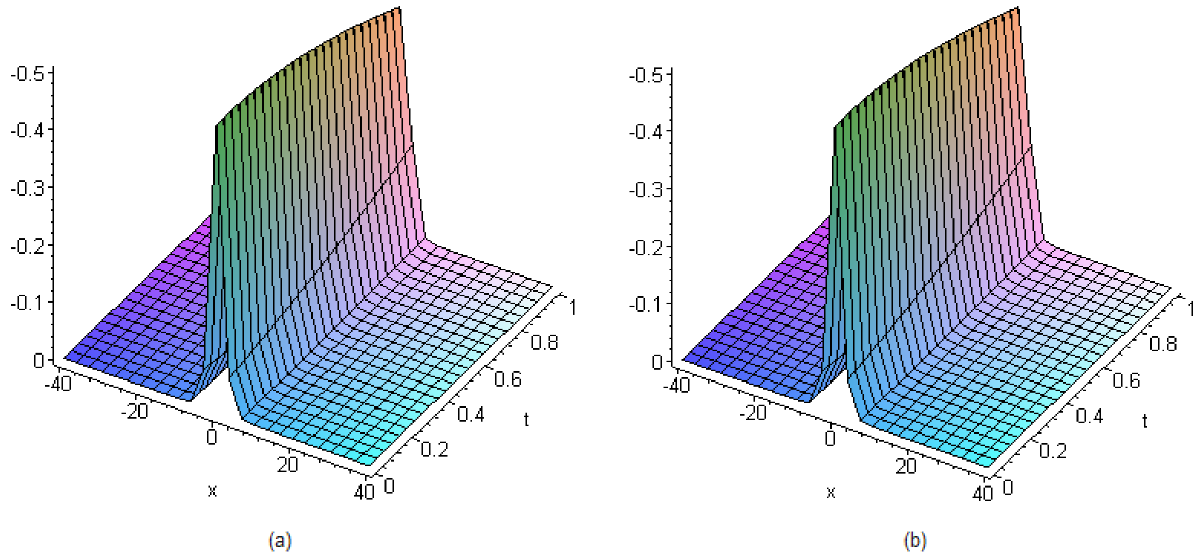

Fig. 1. The numerical results for $u_4(x,y)$ when $k=1$: (a) in comparison with the analytical solutions $u(x,t)=-\frac{k^2}{2}\sec h\left(\frac{k}{2}\left(x-k^2t\right)\right)$, (b) for the solitary wave solution with the initial condition of 3.3 Example

3.4. Example

We finally study the wave equation in an unbounded domain [18]

$$u_{tt}=u_{xx},\ -\infty<x<\infty, t>0 \tag{20}$$

with initial conditions:

$$I.C.\ u(x,0)=\sin(x),\ \ u_t(x,0)=0 \tag{21}$$

which we know the exact solution $u(x,t)=\sin(x)\cos(t)$.

Taking the differential transform of (20) and the initial condition (21) respectively, we obtain

$$\frac{(k+2)!}{k!}U_{k+2}(x)=\frac{\partial^2}{\partial x^2}U_k(x) \tag{22}$$

and the transformed initial condition

$$U_0(x)=\sin(x), U_1(x)=0 \tag{23}$$

Then substituting (23) into (22) we have

$$U_2(x)=-\frac{1}{2}\sin(x),\ U_3(x)=0, U_4(x)=\frac{1}{24}\sin(x), U_5(x)=0, U_6(x)=-\frac{1}{7202}\sin(x),\ldots$$

and so on, we can calculated $U_k(x)$. Substituting all $U_k(x)$ into (2.2) we obtain

$$u(x,t)\cong\sum_{k=0}^{6}U_k(x)t^k=\sin(x)-\frac{1}{2}\sin(x)t^2+\frac{1}{24}\sin(x)t^4-\frac{1}{720}\sin(x)t^6$$

It is obvious that a higher number of iterations makes $u(x,t)$ converge to the exact solution, $\sin(x)\cos(t)$.

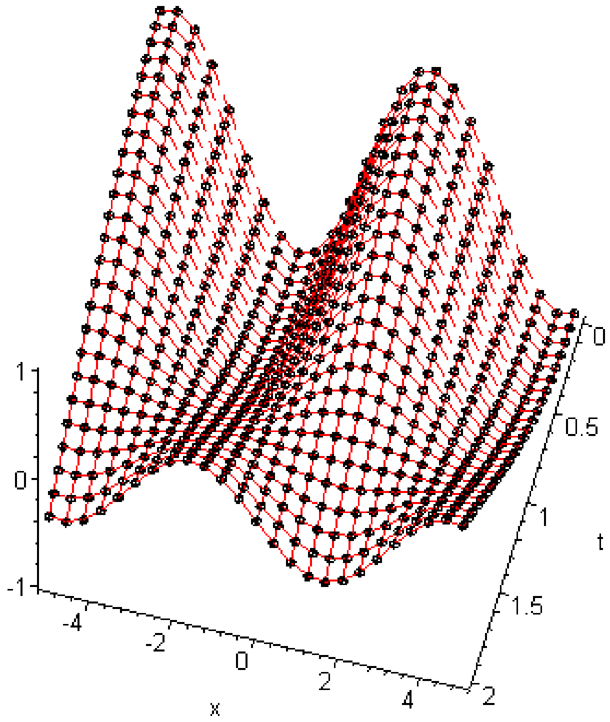

Fig. 2. The comparison of the RDTM approximation and the exact solution

Figure 2 shows the comparison of the RDTM approximation solution of order eight and the exact solution $u(x,t) = \sin(x)\cos(t)$, the solid line represents the solution by the reduced differential transform method, while the circle represents the exact solution. From Fig. 2, it is clearly seen that the RDTM approximation and the exact solution are in good agreement.

4. CONCLUSIONS

In this work, we present the analytical approximation to a solution for wave equations in four different cases. We have achieved this goal by applying reduced differential transform method. Using the RDTM, it is possible to find the exact solution or a good approximate solution of the equation. It can be concluded that RDTM is a very powerful and efficient technique for finding exact solutions for wide classes of problems. Computations are performed using the Maple package.

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