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ON THE CASES OF EXPLICIT SOLVABILITY OF A THIRD ORDER PARTIAL DIFFERENTIAL EQUATION*

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Abstract – In this paper, the Goursat problem of a third order equation on cases of explicit solvability is investigated, with the help of the Riemann function. Some results and one theorem are given concerning the existence and uniqueness for the solution of the suggested problem.

Keywords – Third order partial differential equation, the Goursat problem, the Riemann function

1. INTRODUCTION

In the domain $D = \{(x,y); \quad x_0 < x < x_1, \quad y_0 < y < y_1\}$, we consider the equation

$$L(u) \equiv u_{xxy} + au_{xx} + bu_{xy} + cu_x + du_y + eu = f \tag{1}$$

where

$$a\ , b,\ c, d, e,\ f \in C^{1+2}(D),$$

The special cases of the equation (1) are encountered during the investigation of processes of moisture absorption by plants [1], where the class C^{k+l} means the existence and continuity for all derivatives

$$\partial^{r+s} / \partial x^r \partial y^s \qquad (r = 0,...k; s = 0,...,l).$$

We will call the solution of the class a regular.

The solution of the Goursat problem for the equation (1), with the help of the Riemann function $R(x, y;\zeta,\eta)$, introduced as a solution of the following integral equation

$$\begin{gathered} V(x,y) - \int_\eta^y a(x,\tau)V(x,\tau)\,d\tau - \int_\varsigma^x [b(t,y) - (x-t)\,d(t,y)]V(t,y)\,dt + \\ + \int_\zeta^x \int_\eta^y [-c(t,\tau) - (x-t)\,e(t,\tau)]V(t,\tau)d\tau\,dt = 1 \end{gathered} \tag{2}$$

was obtained in [2] and [3] (see also, [4-5]).

Also, $V(x, y)$ remains the solution of the following equation adjoint to the equation (1)

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$$L^*(V) = V_{xxy} - (aV)_{xx} - (bV)_{xy} + (cV)_x + (dV)_y - eV = 0. \quad (3)$$

In this paper, some cases of constructing the above function in quadratures are considered, which provides for the solution of the original problem explicitly.

2. MAIN RESULTS

1^0) Let $b \equiv c \equiv d \equiv e \equiv 0$ and $a \neq 0$. Then the solution (2) will obviously be the following function:

$$V = R(x, y; \eta) = \exp \int_{\eta}^{y} a(x, \tau)\, d\tau.$$

2^0) A similar variant appears when $b \neq 0$ and $a \equiv c \equiv d \equiv e \equiv 0$. Here

$$V = R(x, y; \zeta) = \exp \int_{\eta}^{y} b(t, y)\, dt.$$

3^0) The following case is analogous to the previous two ($a \equiv c \equiv e \equiv b + xd \equiv 0$ and $d \neq 0$). Hence, the equation (2) takes the form:

$$V(x, y) + x \int_{\zeta}^{x} d(t, y) V(t, y)\, dt = 1.$$

Put

$$\lambda(x, y) = \int_{\zeta}^{x} d(t, y) V(t, y)\, dt,$$

then, by differentiating it with respect to *x* we get:

$$\lambda_x(x, y) + x\lambda(x, y)\, d(x, y) = d(x, y).$$

With the additional condition $\lambda(\zeta, y) = 0$. Solving this linear equation, we have

$$\exp\left[\int_{\zeta}^{x} t\, d(t, y)\, dt\right] \lambda(x, y) = \int_{\zeta}^{x} d(t, y) \exp\left[-\int_{t}^{\zeta} s\, d(s, y)\, ds\right] dt,$$

or

$$\lambda(x, y) = \int_{\zeta}^{x} d(t, y) \exp\left[\int_{x}^{t} s\, d(s, y)\, ds\right] dt.$$

As a result, with the help of the formula

$$V(x, y) = \frac{\lambda_x(x, y)}{d(x, y)},$$

it is clear that:

$$V = R(x,y;\zeta) = 1 - x\int_{\zeta}^{x} d(t,y)\left[\exp\int_{x}^{t} s\,d(s,y)\,ds\right]dt,$$

Let us consider some less obvious cases.

4^0) $a \equiv b \equiv d \equiv e \equiv 0$, and $c = m(x)n(y)$.

Here the result of the paper [6] may be used and we can prove that

$$R(x,y;\zeta,\eta) = J_0\{2[\int_{\zeta}^{x} m(t)dt\int_{\eta}^{y} n(\tau)d\tau]^{\frac{1}{2}}\}, \tag{4}$$

where J_0 is the Bessel function of the first kind and order zero

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k^2}}{(k!)^2}\left(\frac{z}{2}\right)^{2k}.$$

5^0) $a \equiv b \equiv c \equiv e \equiv 0$, and $d(x,y) = \pm\gamma^2(y) \neq 0.$

It is obvious that:

$$R = \cos\left[(x-\zeta\gamma(y))\right], \quad (d \geq 0);$$
$$R = \cosh\left[(x-\zeta\gamma(y))\right], \quad (d \leq 0).$$

6^0) $a \equiv b \equiv c \equiv d \equiv 0$, and $e(x,y) = \pm e(y) \neq 0.$

Using the standard method of finding the solution of the integral equation (2) in the form of the Neumann's series, the following equation can be obtained

$$R(x,y;\zeta,\eta) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{\eta}^{y} e(\tau)d\tau\right)^{k} \frac{(x-\zeta)^{2k}}{(2k)!}.$$

7^0) $a \equiv b \equiv c \equiv d \equiv 0$, and $c(x,y) \equiv xe(x,y) \equiv m(x)n(y).$

Using the notation

$$\theta(x,y) = \int_{\zeta}^{x}\int_{\eta}^{y} t\,e(t,\tau)V(t,\tau)\,d\tau\,dt,$$

The equation (2) can be written in the following form

$$\theta_{xy} + m(x)n(y)\theta = xe(x,y), \qquad \theta\big|_{x=\zeta} = \theta\big|_{y=\eta} = 0. \tag{5}$$

The Riemann function R^* for the equation (5) coincides with the right part (4).

Using ([7]; p. 448), we have the following:

$$\omega(\zeta,\eta) = R(\zeta_0,\eta_0;\zeta,\eta)\omega(\zeta_0,\eta_0) + \int_{\zeta_0}^{\zeta} d\zeta_1 \int_{\eta_0}^{\eta} R(\zeta_1,\eta_1;\zeta,\eta)\, L[\omega(\zeta_1,\eta_1)] d\eta_1 +$$

$$+ \int_{\zeta_0}^{\zeta} R(\zeta_0,\eta_0;\zeta,\eta)\left[\frac{\partial\omega(\zeta_1,\eta_0)}{\partial\zeta_1} + b(\zeta_1,\eta_0)\omega(\zeta_1,\eta_0)\right] d\zeta_1 +$$

$$+ \int_{\eta_0}^{\eta} R(\zeta_0,\eta_1;\zeta,\eta)\left[\frac{\partial\omega(\zeta_0,\eta_1)}{\partial\eta_1} + a(\zeta_0,\eta_1)\omega(\zeta_0,\eta_1)\right] d\eta_1 ,$$

$$\theta = R^*(\zeta,\eta;x,y) + \int_{\zeta}^{x}\int_{\eta}^{y} R^*(t,\tau;x,y)\, t\, e(t,\tau)\, d\tau\, dt.$$

Using the notation for θ and (5), it is clear that $V = 1-\theta$, because

$$R(x,y;\zeta,\eta) = 1 - J_0\left\{2\left[\int_{\zeta}^{x} m(t)dt \int_{\eta}^{y} n(\tau)d\tau\right]^{\frac{1}{2}}\right\} -$$

$$-\int_{\zeta}^{x}\int_{\eta}^{y} J_0\left\{2\left[\int_{t}^{x} m(s)ds \int_{\tau}^{y} n(\sigma)d\sigma\right]^{\frac{1}{2}}\right\} m(t)n(\tau)d\tau dt.$$

8^0) $d \equiv e \equiv 0.$

In this case, the equation (2) transforms into the equation of Riemann function, which responds with the following

$$u_{xy} + au_x + bu_y + cu = 0.$$

if

$$a_x + ab - c = 0,$$

or

$$b_y + ab - c = 0,$$

respectively (see [8]):

$$R = \exp\left[\int_{y}^{\eta} a(x,\tau)d\tau + \int_{x}^{\zeta} b(t,\tau)dt\right],$$

$$R = \exp\left[\int_{y}^{\eta} a(x,y)d\tau + \int_{x}^{\zeta} b(t,\tau)dt\right].$$

if

$$a=a_1(y)+\lambda x, \qquad b=b_1(x)+\lambda y, \qquad c-ab-\lambda=m(x)n(y), \quad \lambda=const,$$

then (see again, [8] and [9])

$$R(x,y;\zeta,\eta)=J_0\left\{2\left[\int_\zeta^x m(t)dt\int_\eta^y n(\tau)d\tau\right]^{\frac{1}{2}}\right\}\exp\left[\lambda(xy-\zeta\eta)+\int_\zeta^x b_1(t)dt+\int_\eta^y a_1(\tau)d\tau\right],$$

9^0) $b+xd\equiv c+xe\equiv 0.$

Put the new unknown function

$$V=xW,$$

Then the equation (2) takes the form

$$W(x,y)-\int_\eta^y a(x,\tau)W(x,\tau)d\tau+\int_\zeta^x td(t,y)W(t,y)dt-$$

$$-\int_\zeta^x\int_\eta^y te(t,\tau)W(t,\tau)d\tau dt=\frac{1}{x},$$

which is equivalent to the Goursat problem (taken into consideration that: $\zeta=x_*$, $\eta=y_*$):

$$W_{xy}-aW_x+xdW_y(xd_y-a_x-xe)W=0,$$

$$W(x_*,y)=\frac{1}{x_*}\exp\int_{y_*}^y a(x_*,\tau)d\tau, \tag{6}$$

$$W(x\ ,y_*)=\frac{1}{x_*}\exp\int_{x_*}^x td(t,y_*)dt-\int_{x_*}^x\left[\exp\int_x^t sd(s,y_*)ds\right]\frac{dt}{t^2}.$$

To justify the conducted operations it is required that $x_*\neq 0$, i.e. the domain D should be strictly found either to the right or to the left from the Y-axis.

The equation (6) has the same form as in 8^0). From the addition to 9^0) of any identity, we have

$$d_y+ad-e\equiv 0, \qquad a_x-x(ad-e)\equiv 0,$$

or the representations:

$$a=a_1(y)+\lambda x, \qquad xd=b_1(x)+\lambda y,$$
$$xd_y+ad-e-2\lambda=m(x)n(y), \quad \lambda=const,$$

leads to the possibility of constructing $W(x,y;x_*,y_*)$ explicitly, after which

$$R(x,y;\zeta,\eta)=xW(x,y;\zeta,\eta).$$

Let us also consider cases when the problem of finding the Riemann function is solved in quadratures through the direct integration of the adjoint differential equation (3).

Therefore, the operation in its left part is split, for example
10^0)

$$(\frac{\partial}{\partial x}-\alpha)(\frac{\partial}{\partial y}-\gamma)(\frac{\partial V}{\partial x}-\beta V)=0.$$

So,

$$\frac{\partial V}{\partial x}-\beta V=V_1\,; \quad \frac{\partial V_1}{\partial y}-\gamma V_1=V_2\,; \quad \frac{\partial V_2}{\partial x}-\alpha V_2=0.$$

Then to find V, it is necessary to find V_2 and V_1. Since

$$V_2=C(y)\exp\left[\int_{x_*}^{x}\alpha(t,y)dt\right],$$

Therefore

$$\frac{\partial V_1}{\partial y}-\gamma V_1=C(y)\exp\left[\int_{x_*}^{x}\alpha(t,y)dt\right], \tag{7}$$

and hence,

$$\begin{aligned} V_1(x,y)=\{C_1(x)+\int_{y_0}^{y}C(\eta)\exp\left[\int_{x_*}^{x}\alpha(t,y)dt\right]\bullet \\ \bullet\exp\left[-\int_{y_*}^{y}\gamma(x,\eta_1)d\eta_1\right]d\eta\}\exp\left[\int_{y_*}^{y}\gamma(x,s)ds\right], \end{aligned} \tag{8}$$

Then,

$$\begin{aligned} V(x,y)=\exp\left[\int_{x_*}^{x}\beta(\zeta,y)d\zeta\right]\bullet \\ \bullet\left(C_2(y)+\int_{x_0}^{x}V_1(\zeta,y)\exp\left[-\int_{x_*}^{\zeta}\beta(\zeta_1,y)d\zeta_1\right]d\zeta\right). \end{aligned} \tag{9}$$

In order to find C_2, we put $\eta=y_0$, $\zeta=x$ and then $x=x_0$ in (2). This leads to the equation

$$V\,(x_0,y)-\int_{y_0}^{y}a(x_0,\tau)V(x_0,\tau)d\tau=1.$$

Because

$$V\,(x_0,y)=\exp\int_{y_0}^{y}a(x_0,\tau)d\tau. \tag{10}$$

By virtue of (9), $C_2(y) = V(x_0, y)$. Now, it is possible to find $C_1(x)$. From (8) it follows that $V_1(x, y_0) = C_1(x)$. We suppose that in (2) $\eta = y$, $\zeta = x_0$, and then $y = y_0$, the following equation may appear

$$V(x, y_0) - \int_{x_0}^{x} [b(t, y_0) - (x - t)d(t, y_0)]V(t, y_0)dt = 1. \quad (11)$$

Therefore, three cases of solvability in quadratures:

i) $d(x, y) \equiv 0$, $\quad V(x, y_0) = \exp \int_{x_0}^{x} b(t, y_0)dt,$

ii)

$$b(x, y) + xd(x, y) \equiv 0,$$

$$V(x, y_0) = 1 - \int_{x_0}^{x} d(t, y_0)\left[\exp \int_{x}^{t} \zeta d(\zeta, y_0)d\zeta\right]dt,$$

iii) $b_x(x, y) \equiv d(x, y)$,

then from (11), the solution of the Cauchy problem:

$$V_{xx}(x, y_0) - b(x, y_0)V_x(x, y_0) = 0,$$

$$V(x_0, y_0) = 1, \qquad V_x(x_0, y_0) = b(x_0, y_0),$$

provided by the following formula

$$V(x, y_0) = 1 + b(x_0, y_0)\int_{x_0}^{x}\left[\exp \int_{x_0}^{t} b(\zeta, y_0)d\zeta\right]dt.$$

Any of the above-mentioned cases enables us to define $C_1(x)$. Now, it is only required to define $C(y)$. Since

$$V_2(x_0, y) = C(y),$$

then using the equation

$$V_x(x_0, y) - \int_{\eta}^{y} [(aV)_x(x_0, \tau) - c(x_0, \tau)V(x_0, \tau)]d\tau - b(x_0, y)V(x_0, y) = 0,$$

obtained from (2) through the differentiation with respect to x, when $x = x_0$, the following function may be calculated

$$\Theta(y) - \int_{y_0}^{y} a(x_0, \tau)\Theta(\tau)d\tau = \omega_1(y), \quad (12)$$

$$\omega_1(y) = -b(x_0, y)V(x_0, y) + \int_{y_0}^{y} [a_x(x_0, \tau) - c(x_0, \tau)]V(x_0, \tau)d\tau.$$

This enables to calculate $C(y)$ from the equation

$$V_{xy} - (\beta V)_y - \gamma V_x + \gamma\beta V = V_2.$$

Now, let us define the condition that is imposed on the coefficients of the equation (1) that provide the representation (3) in the following forms:

$$\left.\begin{array}{ll}
1)\left(\frac{\partial}{\partial y} - a\right)\left(\frac{\partial}{\partial x} - b\right)\frac{\partial V}{\partial x} = 0, & 2)\left(\frac{\partial}{\partial y} - a\right)\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x} - bV\right) = 0, \\
3)\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - b\right)\left(\frac{\partial V}{\partial y} - aV\right) = 0, & 4)\left(\frac{\partial}{\partial x} - b\right)\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y} - aV\right) = 0, \\
5)\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - a\right)\left(\frac{\partial V}{\partial y} - bV\right) = 0, & 6)\left(\frac{\partial}{\partial x} - b\right)\left(\frac{\partial}{\partial y} - a\right)\frac{\partial V}{\partial x} = 0.
\end{array}\right\} \quad (13)$$

Having completed the operations in the left side 1) from the equation (13), we have:

$$\frac{\partial^3 V}{\partial x^2 \partial y} - a\frac{\partial^2 V}{\partial x^2} - \frac{\partial b}{\partial y}\frac{\partial V}{\partial x} - b\frac{\partial^2 V}{\partial x \partial y} + ab\frac{\partial V}{\partial x} = 0. \quad (14)$$

If the equation (3) is written in the form

$$V_{xxy} - aV_{xx} - bV_{xy} + (-2a_x - b_y + c)V_x + (-b_x + d)V_y + \\ + (-a_{xx} - b_{xy} + c_x + d_y - e)V = 0. \quad (15)$$

And comparing (14) with (15), the following conditions may be obtained for the coefficients in the form

$$-b_y + ab = -2a_x - b_y + c, \quad -2a_x - ab + c \equiv 0;$$
$$-b_x + d \equiv 0; \quad -a_{xx} + c_x - e \equiv 0.$$

Similarly, we find the conditions that provide the other variants of splits (13), i.e. take the equation 3) from (13). We write all of them under the same numbers.

1) $2a_x + ab - c \equiv b_x - d \equiv a_{xx} - c_x + e \equiv 0;$

2) $2a_x + ab - c \equiv a_{xx} + ab_x - c_x + e \equiv d \equiv 0;$

3) $d \equiv b_y + ab - c \equiv e \equiv 0;$

4) $b_x - d \equiv b_y + ab - c \equiv a_x b - c_x + e \equiv 0;$

5) $d \equiv e \equiv a_x + ab - c \equiv 0;$

6) $a_{xx} - c_x + e \equiv b_x - d \equiv d_y + (ab)_x - e \equiv 0.$

Now, in order to obtain the groups of conditions of the explicit Riemann function, it is necessary to compare the identities i), ii) and iii) with each set 1) to 6).

Obviously it reduces to the supplement i), ii) or iii) into the corresponding set. Here, the sets may be simplified, sometimes coinciding with the previously obtained ones or being their special cases. Having performed the above comparison and rejected the repeating sets, we come to the following groups of the conditions:

A) With general assumptions $b \equiv d \equiv 0.$

1^0) $c - 2a_x \equiv e - a_{xx} \equiv 0;$

2^0) $c \equiv e \equiv 0;$

3^0) $e \equiv a_x - c \equiv 0.$

B) With assumptions $d \equiv 0$ and $b(x, y) = b(y).$

4^0) $2a_x + ab - c \equiv a_{xx} - c_x + e \equiv 0;$

5^0) $b_y + ab - c \equiv a_x b - c_x + e \equiv 0;$

6^0) $a_{xx} - c_x + e \equiv b_y - c + ab + a_x \equiv 0.$

C) In supposition $d \equiv 0$.

7^0) $2a_x + ab - c \equiv a_{xx} + ab_x - c_x + e \equiv 0;$

8^0) $b_y + ab - c \equiv e \equiv 0;$

9^0) $e \equiv a_x + ab - c \equiv 0.$

D) During the identity performance $b_x \equiv d$.

10^0) $2a_x + ab - c \equiv a_{xx} - c_x + e \equiv 0;$

11^0) $b_y + ab - c \equiv a_x b - c_x + e \equiv 0;$

12^0) $a_{xx} - c_x + e \equiv b_y - c + ab + a_x \equiv 0.$

E) Under the conditions $xb \equiv -xd \equiv \lambda(y) \neq 0$.

13^0) $2a_x + ab - c \equiv a_{xx} - c_x + e \equiv 0;$

14^0) $xc = \lambda' + a\lambda, \qquad x(c_x - e) = a_x \lambda.$

Consequently the above-mentioned considerations enables us to formulate the following result:

Theorem: The Goursat problem for the equation (1) is solved explicitly in all the above-mentioned cases.

REFERENCES

1. Nakhushev, A. M. (1995). *Equations of Mathematical Biology*. Moscow. Higher School.
2. Zhegalov, V. I. & Utkina, Ye. A. (1999). On a third order pseudo parabolic equation. *Izvestia VUZov: Matematika*, *10*, 73-76.
3. Utkina, Ye. A. & Maher, A. (2009). On a fourth order pseudoparabolic equation. *Italian J. of Pure and Appl. Mathematics*, *25*, 6-16.
4. Maher, A. & Utkina, Ye. A. (2009). On the Goursat problem for a sixth order equation and spatial equation. *Italian J. of Pure and APPL. Mathematics*, *25*, 37-46.
5. Utkina, Ye. A. & Maher, A. (2006). On problems reduced to the Gousat problem for a third order equation. *IJST, Trans. A*, *30*, 271-277.
6. Churikov, V. S & Mashenko, I. P. (1970). Constructing the Riemann function for the Equation

$u_{xy} + \phi(x)\psi(y)u = 0$. *Research papers of Krasnodar Polytechnic Institute, 30*, 19-25.]

7. Bitsadze, A. V. (1981). *Some classes of equations in partial derivatives*. Moscow, Nauka.
8. Zhegalov, V. I. & Kotukhov, M. P. (1998). On integral equations for the Riemann function. *Izvestia VUZov: Matematika*, *1*, 26-30.
9. Rundell, W. I. & Strecher, M. (1977). Remarks concerning of solutions of pseudoparabolic equation. *Proc. Amer. Math. Soc.*, *63*(1), 73-76.