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ON THE CANONICAL SOLUTION AND DUAL EQUATIONS OF STURM-LIOUVILLE PROBLEM WITH SINGULARITY AND TURNING POINT*

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Abstract – In this paper, we investigate the canonical property of solutions of a system of differential equations having a singularity and turning point of even order. First, by a replacement, we transform the system to the Sturm-Liouville equation with a turning point. Using the asymptotic estimates for a special fundamental system of solutions of Sturm-Liouville equation, we study the infinite product representation of solutions of the system and investigate the uniqueness of the solution for the dual equations of the Sturm-Liouville equation. Then, we transform the Sturm-Liouville equation with a turning point to the equation with a singularity, and study the asymptotic behavior of its solutions. Such representations are relevant to the inverse spectral problem.

Keywords – Turning point, singularity, sturm-liouville, infinite products, hadamard's theorem, dual equations, eigenvalues

1. INTRODUCTION

We consider the following system of differential equations

$$\frac{dy}{dt} = i\rho\frac{1}{R_1(t)}x\,, \qquad \frac{dx}{dt} = \left(i\rho R_2(t) + \frac{p(t)}{i\rho R_1(t)}\right)y\,,\ \ t\in[0,1] \tag{1}$$

with initial conditions $x(0,\rho)=0,\ \ y(0,\rho)=1$, where ρ is the spectral parameter, $R_1,\ R_2,\ p(t)$ are bounded and integrable in $I=[0,1]$ and $R_2(t)$ has one zero inside the interval I of even order.

System (1) is a canonical form for many problems in natural sciences. For example, for a wide class of problems describing the propagation of electromagnetic waves in a stratified medium, Maxwell's equations can be reduced to the canonical form (1) (see [1]). System (1) often appears in optics, spectroscopy and acoustic problems. System (1) also appears for the design of directional couplers for heterogeneous electronic lines, which constitute one of the most important classes of radio physical synthesis problems (see [2], [1]). Some aspects of synthesis problems for system (1) with $R_1 = R_2 = R > 0$ were studied in [3] and other works. Inverse problem for system (1) with the initial conditions $x(0,\rho)=1,\quad y(0,\rho)=-1$ and with $R>0$ were studied in [4, 5]. In [6], the authors studied the eigenvalues and derived a formula for the asymptotic distribution of the eigenvalues in the case when the system (1) has arbitrary order singularities and turning points inside the interval [0, T].

The importance of asymptotic analysis in obtaining information on the solution of a Sturm-Liouville equation with multiple turning points was realized by Olver [7] and Eberhard, Freiling and Schneider in [8]. Also, the inverse problem for Sturm-Liouville equation with turning points were studied by Freiling and Yurko in [5]. In [9], the asymptotic estimates for a special fundamental system of solutions of the

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corresponding differential equation and determining the asymptotic distribution of the eigenvalues with several singularities or/and turning points inside the interval [0, 1] were studied by Eberhard, Freiling and Wilchen. The results of Kazarinoff [10], Langer [11] and Olver [7] bring important innovations to the asymptotic approximation of solution of Sturm-Liouville equations with two turning points. Also in [12], the infinite product representation of solution of the equation with one turning point of odd order was obtained and the authors derived the associate dual equations by this infinite product form of solution. It is necessary to point out that applying asymptotic solutions for studying inverse problem in turning points cases, is more complicated and, practically, is not convenient to use. Especially in deriving the asymptotic formulas, one should apply Bessel function type. In addition, a more difficult and challenging task is to shape the asymptotic behavior of the solutions and corresponding eigenvalues. So the inverse problem of reconstructing the potential function from the given spectral information and corresponding dual equation cannot be studied by using the asymptotic forms. In fact, in asymptotic methods one cannot generally express the exact solution in closed form. Indeed, in methods connected with dual equations, the closed form of the solution is needed. The representing solution of the infinite product form plays an important role for investigating the corresponding dual equations. We mention that some aspects of the inverse problem with a singularity were studied in [13], also some aspects of the inverse problem with turning points were studied in [14] and other works connected with ideas of the dual equation method. In the previous article [15], the authors considered the following Sturm-Liouville equation

$$w'' + (\lambda(1-z^2) - \psi(z))w = 0, \ -\infty < a < z < 1, \tag{2}$$

with dirichlet boundary conditions $w(a) = w(z) = 0$, where the function $\psi(z)$ is continuous, $-1 \in (a,z)$ and λ is the spectral parameter, also the weight function (the coefficient of λ in (2)) has two turning points $z = \pm 1$ of odd order. For $0 < z < 1$, the solution $w(\lambda, z)$ of such an equation (2) with an initial condition $w(\lambda, a) = 0,\ \frac{\partial w}{\partial z}(\lambda, a) = 1$ was found to have the infinite product form

$$w(\lambda, z) = \frac{f^{\frac{1}{2}}(z) p^{\frac{1}{2}}(-1)}{4 i^{\frac{3}{2}} e^{3i\frac{\pi}{4}} (a^2 - 1)^{\frac{1}{4}}} \prod_{n \ge 1} \frac{(\lambda - \nu_n(z)) p^2(-1)}{j^2{}_n} \prod_{n \ge 1} \frac{(r_n(z) - \lambda) f^2(z)}{j^2{}_n},$$

where the sequence $\{r_n(z)\}_{n\ge1}$ represents the sequence of positive eigenvalues, and $\{\nu_n(z)\}_{n\ge1}$ the sequence of negative eigenvalues of the dirichlet problem associated with (2) on $[a,z]$, for each z in $(0,1)$, and

$$f(z) = \frac{\pi}{2} - \int_z^1 (1-\varsigma^2)^{\frac{1}{2}} d\varsigma, \ \ 0 < z < 1,$$

$$p(-1) = \int_a^{-1} (\varsigma^2 - 1) d\varsigma,$$

and j_n, $n = 1,2,3,...$, are the positive zeros of $J_1'(z)$.

In this paper, first, we transform (1) to the Sturm-Liouville equation with turning point of even order, then we define a fundamental system of solutions (FSS) of the equation when $|\rho| \to \infty$ (see section 2). Using these asymptotic solutions we derive a formula for the asymptotic distribution of the eigenvalues (in section 4). Further, we obtain the infinite product representation of solutions (see section 5), derive the associate dual equations by this infinite product form of solutions, and investigate the uniqueness solution of these equations (see section 6). This paper continues the investigations made in sections 2-6. In section 7, by a replacement we transform the Sturm-Liouville equation with turning point to the equation with singularity and determine the asymptotic behavior of the solution. Using the infinite representation of solutions of section 5, the canonical representation of solutions of equation with singularity are obtained (see section 8). Therefore, we define singularity's and turning's relation by upper replacements. The other missing cases will be treated in a future paper as they require different techniques.

2. NOTATIONS AND PRELIMINARY RESULTS

Let us consider the system of differential equation (1), where the function $R_1(t) = A_1 > 0$ is a constant function and

$$R_2(t) = A_2(t-t_1)^2 \ , \tag{3}$$

where the coefficient A_2 is a positive constant and $t_1 \in (0,1)$.

System (1) after the elimination of x reduces to the linear second-order Sturm-Liouville equation

$$-y'' + p(t)y = \lambda\phi^2(t)y, \tag{4}$$

with initial conditions

$$y(0,\rho) = 1, \qquad y'(0,\rho) = 0,$$

where $\lambda = \rho^2$ is a real parameter and $\varphi^2(t) = \frac{A_2}{A_1}(t-t_1)^2$ has one zero t_1 in $(0,1)$, the so called turning point. In the terminology of [8], t_1 is of Type II.

Notations 2.1.

i) Let $\delta > 0$ be fixed, sufficiently small, we define
$[1] \equiv 1 + O(\rho^{-\sigma_0})$, as $\rho \to \infty$,
where $\sigma_0 = 1 - \delta$.
ii) For $k \in Z$ we consider the sectors

$$S_k := \left\{ \rho \middle| \frac{k\pi}{4} \le \arg\rho \le \frac{(k+1)\pi}{4} \right\}.$$

Now let $C(t,\lambda)$ be the solution of (4) corresponding to the initial conditions $C(0,\rho) = 1$, $C'(0,\rho) = 0$. In order to represent the solution $C(t,\lambda)$ as an infinite product, we use a suitable fundamental system of solutions (FSS) for Equation (4) as constructed in [8].

According to the type of t_1 we know from [8, Theorem 3.2] that in the sector S_{-1} there exists an FSS of (4) $\{w_1(t,\rho), w_2(t,\rho)\}$ such that

$$w_1(t,\rho) = \begin{cases} \phi^{-\frac{1}{2}}(t) e^{i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1], \ 0 \le t < t_1, \\ \sqrt{2}\phi^{-\frac{1}{2}}(t)\left\{ e^{i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1] + \frac{\sqrt{2}}{2} i e^{-i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1] \right\}, \ t_1 < t \le 1, \end{cases} \tag{5}$$

$$w_2(t,\rho)=\begin{cases}\phi^{-\frac{1}{2}}(t)\left\{e^{-i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1]+\dfrac{\sqrt{2}}{2}ie^{i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1]\right\},\ 0\le t<t_1,\\ \dfrac{\sqrt{2}}{2}\phi^{-\frac{1}{2}}(t)e^{-i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1],\ t_1<t\le 1.\end{cases}\tag{6}$$

That leads to the following:

$$w_1'(t,\rho)=\begin{cases}i\rho\phi^{\frac{1}{2}}(t)e^{i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1],\ 0\le t<t_1,\\ \sqrt{2}\rho\phi^{\frac{1}{2}}(t)\left\{ie^{i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1]+\dfrac{\sqrt{2}}{2}e^{-i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1]\right\},\ t_1<t\le 1,\end{cases}\tag{7}$$

$$w_2'(t,\rho)=\begin{cases}\rho\phi^{\frac{1}{2}}(t)\left\{-ie^{-i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1]-\dfrac{\sqrt{2}}{2}e^{i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1]\right\},\ 0\le t<t_1,\\ -\dfrac{\sqrt{2}}{2}i\rho\phi^{\frac{1}{2}}(t)e^{-i\rho\int_{t_1}^{t}\phi(\varsigma)d\varsigma}[1],\ t_1<t\le 1.\end{cases}\tag{8}$$

We also need $\{w_1(t_1,\rho),w_2(t_1,\rho)\}$. Similarly, for $t=t_1$ from [8] we have

$$w_1(t_1,\rho)=\sqrt{\pi}\rho^{\frac{1}{4}}\left\{e^{\frac{i\pi}{8}}u_1(t_1,\rho)[1]+e^{\frac{3i\pi}{8}}u_2(t_1,\rho)[1]\right\},$$

$$w_2(t_1,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{4}}\left\{e^{\frac{i\pi}{8}}u_1(t_1,\rho)[1]-e^{\frac{3i\pi}{8}}u_2(t_1,\rho)[1]\right\},$$

where

$$u_1(t_1,\rho)=\frac{2^{\frac{1}{4}}\psi(t_1)}{\Gamma(\frac{3}{4})},\quad u_2(t_1,\rho)=0,$$

where $\psi(t_1)=\lim_{t\to t_1}\phi^{-\frac{1}{2}}(t)\left\{\int_{t_1}^{t}\phi(\varsigma)d\varsigma\right\}^{\frac{1}{4}}$. Consequently

$$w_1(t_1,\rho)=\sqrt{\pi}\rho^{\frac{1}{4}}e^{\frac{i\pi}{8}}\frac{2^{\frac{1}{4}}\psi(t_1)}{\Gamma(\frac{3}{4})},\tag{9}$$

$$w_2(t_1,\rho)=\sqrt{2\pi}\rho^{\frac{1}{4}}e^{\frac{i\pi}{8}}\frac{\psi(t_1)}{2^{\frac{3}{4}}\Gamma(\frac{3}{4})}[1].\tag{10}$$

It follows that the wronskian of FSS satisfy

$$W(\rho)\equiv W\big(w_1(t,\rho),w_2(t,\rho)\big)=-2i\rho[1],\tag{11}$$

as $\rho\to\infty$.

3. ASYMPTOTIC FORM OF THE SOLUTION

We consider the differential equation (4) with the following conditions

$$C(0,\rho)=1, \quad C'(0,\rho)=0. \tag{12}$$

Applying the FSS $\{w_1(t,\rho), w_2(t,\rho)\}$ for $t\in[0,1]$ we have

$$C(t,\rho)=c_1 w_1(t,\rho)+c_2 w_2(t,\rho), \tag{13}$$

in which using cramer's rule leads to the equation

$$C(t,\rho)=\frac{1}{W(\rho)}\left(w_2'(0,\rho)w_1(t,\rho)-w_1'(0,\rho)w_2(t,\rho)\right), \tag{14}$$

where

$$W(\rho)=W(w_1,w_2)=-2i\rho[1].$$

Taking (5)-(8) into account, we derive

$$C(t,\rho)=\begin{cases}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\left\{\cosh(i\rho\int_0^t\phi(\varsigma)d\varsigma)+O(\frac{1}{\rho^{\sigma_0}})\right\}, & 0\le t<t_1,\\ \frac{1}{2}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\left\{M_1(\rho)e^{i\rho\int_{t_1}^t\phi(\varsigma)d\varsigma}[1]+M_2(\rho)e^{-i\rho\int_{t_1}^t\phi(\varsigma)d\varsigma}[1]\right\}, & t_1<t\le 1,\end{cases} \tag{15}$$

where

$$\begin{cases}M_1(\rho)=\sqrt{2}e^{i\rho\int_0^{t_1}\phi(\varsigma)d\varsigma}-ie^{-i\rho\int_0^{t_1}\phi(\varsigma)d\varsigma},\\ M_2(\rho)=ie^{i\rho\int_0^{t_1}\phi(\varsigma)d\varsigma}+\sqrt{2}e^{-i\rho\int_0^{t_1}\phi(\varsigma)d\varsigma}.\end{cases} \tag{16}$$

By virtue of (15) and (16), the following estimates are also valid:

$$C(t,\rho)=\begin{cases}\frac{1}{2}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)e^{i\rho\int_0^t|\phi(\varsigma)|d\varsigma}E_k(t,\rho), & 0\le t<t_1,\\ \frac{\sqrt{2}}{2}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)e^{i\rho\int_0^t|\phi(\varsigma)|d\varsigma}E_k(t,\rho), & t_1<t\le 1,\end{cases} \tag{17}$$

where

$$E_k(t,\rho)=\sum_{n=1}^{v(t)}e^{\rho\alpha_k\beta_{kn}(t)}b_{kn}(t),$$

and

$\alpha_{-2}=\alpha_1=-1, \alpha_0=-\alpha_{-1}=i$, $\beta_{kn}(t)\neq 0$, $0<\delta<\beta_{k1}(t)<\beta_{k2}(t)<...\le\beta_{kv(t)}(t)\le 2R_+(1)$, where the integer-valued functions v and b_{kn} are constant in every interval $[0,t_1-\varepsilon]$ and $[t_1+\varepsilon,1]$ for ε sufficiently small and

$$R_+(t)=\int_0^t\sqrt{\max\{0,\phi^2(\varsigma)\}}d\varsigma. \tag{18}$$

Similarly, using (9), (10) and (14) for $t = t_1$ we find that

$$C(t_1,\rho) = \frac{\sqrt{\pi}\phi^{\frac{1}{2}}(0)\rho^{\frac{1}{4}}e^{\frac{i\pi}{8}}\psi(t_1)}{2^{\frac{3}{4}}\Gamma(\frac{3}{4})} e^{i\rho\int_0^{t_1}\phi(\varsigma)d\varsigma} E_k(t_1,\rho). \tag{19}$$

In addition, differentiating (15) we calculate

$$C'(t,\rho) = \begin{cases} i\rho(\phi(0)\phi(t))^{\frac{1}{2}}\left\{\sinh(i\rho\int_0^t\phi(\varsigma)d\varsigma) + O(\frac{1}{\rho^{\sigma_0}})\right\}, & 0 \le t < t_1, \\ \frac{1}{2}i\rho(\phi(0)\phi(t))^{\frac{1}{2}}\left\{M_1(\rho)e^{i\rho\int_{t_1}^t\phi(\varsigma)d\varsigma}[1] - M_2(\rho)e^{-i\rho\int_{t_1}^t\phi(\varsigma)d\varsigma}[1]\right\}, & t_1 < t \le 1. \end{cases}$$

Thus, we deduce the following theorem:

Theorem 1. Let $C(t,\rho)$ be the solution of (4) under the initial conditions $C(0,\rho) = 1, \quad C'(0,\rho) = 0$, then the following estimates hold:

$$C(t,\rho) = 2^{\frac{1}{2}\nu - 1}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)e^{i\rho\int_0^t\phi(\varsigma)d\varsigma}E_k(t,\rho), \quad t \in D_\nu, \nu = 0,1,$$

where $D_0 = [0,t_1)$ and $D_1 = (t_1,1]$, also

$$C(t_1,\rho) = \frac{\sqrt{\pi}\phi^{\frac{1}{2}}(0)\rho^{\frac{1}{4}}e^{\frac{i\pi}{8}}\psi(t_1)}{2^{\frac{3}{4}}\Gamma(\frac{3}{4})} e^{i\rho\int_0^{t_1}\phi(\varsigma)d\varsigma} E_k(t_1,\rho).$$

4. DISTTIBUTION OF THE EIGENVALUES

We consider the boundary value problem $L_1 = L_1(p(t),\phi^2(t),s)$ for Equation (4) with boundary condition

$$y(0,\lambda) = 1, \quad y'(0,\lambda) = 0, \quad y(s,\lambda) = 0.$$

The boundary value problem L_1 for $s \in (0,1)\setminus\{t_1\}$ has a countable set of positive eigenvalues $\{\lambda_n(s)\}_{n\ge 1}$. From (17), we have the following asymptotic distribution for each $\{\lambda_n(s)\}$:

$$\sqrt{\lambda_n(s)} = \frac{n\pi - \frac{\pi}{2}}{\int_0^s \phi(\varsigma)d\varsigma} + O(\frac{1}{n}). \tag{20}$$

Similarly, according to (19), the spectrum $\{\lambda_n\}_{n\ge 1}$ of boundary value problem L_1 for $s = t_1$, consists of positive eigenvalues

$$\sqrt{\lambda_n(t_1)} = \frac{n\pi - \frac{5\pi}{8}}{\int_0^{t_1} \phi(\varsigma)d\varsigma} + O(\frac{1}{n}). \tag{21}$$

5. MAIN RESULTS

Since the solution $C(t,\rho)$ of the Sturm-Liouville equation defined by a fixed set of initial conditions is an entire function of ρ for each fixed $[0,1]$, it follows from the classical Hadamard's factorization theorem (see [16, p. 24]) that such solution is expressible as an infinite product. For fixed $s \in (0,1) \setminus \{t_1\}$ by Halvorsen's result [17], $C(s,\rho)$ is an entire function of order $\frac{1}{2}$. Therefore, Hadamard's theorem can used to represent the solution in the form

$$C(s,\lambda) = h(s)\prod_{n\geq 1}\left(1-\frac{\lambda}{\lambda_n(s)}\right),$$

where $h(s)$ is a function independent of λ, but may depend on s, and the infinite number of positive eigenvalues, $\{\lambda_n(s)\}_{n=1}^{\infty}$, form the zero set of $C(s,\lambda)$ for each s. Let $\varsigma_n, n \geq 1$, be the sequence of positive zeros of $J'_{-\frac{1}{2}}(t)$. Then (see [18, 9.5.11])

$$\frac{\varsigma_n^2}{R_+^2(t)\lambda_n(t)} = 1 + O\left(\frac{1}{n^2}\right).$$

Consequently, the infinite products $\prod_{n\geq 1}\frac{\varsigma_n^2}{R^2{}_+(t)\lambda_n(t)}$ are absolutely convergent for each $s \in (0,1) \setminus \{t_1\}$. Therefore, we may

$$C(s,\lambda) = h(s)\prod_{n\geq 1}\left(1-\frac{\lambda}{\lambda_n(s)}\right) = h_1(s)\prod_{n\geq 1}\frac{(\lambda_n(s)-\lambda)R_+^2(s)}{\varsigma_n^2}, \tag{22}$$

with

$$h_1(s) := h(s)\prod_{n\geq 1}\frac{\varsigma_n^2}{R_+^2(s)\lambda_n(s)}. \tag{23}$$

Theorem 2. Let $C(t,\lambda)$ be the solution of (4) satisfying the initial conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$. Then for $t \in A_\nu$, $\nu = 0,1$,

$$C(s,\lambda) = 2^{\frac{1}{2}\nu-1}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\varsigma_n^2},$$

where $A_0 = (0,t_1)$, $A_1 = (t_1,1)$, $R_+(t) = \int_0^t \sqrt{\max\{0,\phi^2(s)\}}ds$, $\varsigma_n, n \geq 1$, is the sequence of positive zeros of $J'_{-\frac{1}{2}}$, the sequence $\lambda_n(t), n \geq 1$, represents the sequence of positive eigenvalues of the boundary value problem L_1 on $[0,t]$.

Proof: Let $\lambda_n(s)$ be the eigenvalues of the boundary value problem L_1 on $[0,s]$ for fixed $s = t, t \in A_\nu$, then according to [18, 9.5.11, 10.1.1, 10.1.11], we have

$$\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\varsigma_n^2} = 2\cos(\sqrt{\lambda}R_+(t))[1],$$

as $\lambda \to \infty$. Thus from (17) and (22), we obtain

$$h_1(t) = \frac{C(t,\lambda)}{\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2}} = 2^{\frac{1}{2}\nu - 1}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t).$$

We can proceed similarly for $s = t_1$ by Hadamard's theorem to obtain

$$C(t_1,\lambda) = A\prod_{n\geq 1}\left(1-\frac{\lambda}{\lambda_n(t_1)}\right),$$

where A is constant. Let j_n, $n \geq 1$, be the sequence of positive zeros of $J'_{\frac{1}{4}}$, then (see [18, 9.5.11])

$$\frac{j_n^2}{R_+^2(t_1)\lambda_n(t_1)} = 1 + O\left(\frac{1}{n^2}\right),$$

and so the infinite product $\prod \frac{j_n^2}{R_+^2(t_1)\lambda_n(t_1)} = 1 + O\left(\frac{1}{n^2}\right)$ is absolutely convergent. Consequently we may write as before,

$$C(t_1,\lambda) = A_1\prod_{n\geq 1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2}, \tag{24}$$

where $A_1 = A\prod \frac{j_n^2}{R_+^2(t_1)\lambda_n(t_1)}$.

Theorem 3. For $s = t_1$,

$$C(t_1,\lambda) = \frac{1}{2}\phi^{\frac{1}{2}}(0)R_+^{\frac{-1}{4}}(t_1)\psi(t_1)\prod_{n\geq 1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2},$$

where $R_+(t) = \int_0^t \sqrt{\max\{0,\phi^2(s)\}}ds$, j_n, $n = 1,2,...,$ is the sequence of positive zeros of $J'_{\frac{1}{4}}$, the sequence $\lambda_n(t_1)$, $n \geq 1$, represents the sequence of positive eigenvalues of the boundary value problem L_1 on $[0,t_1]$ and $\psi(t_1) = \lim_{t\to t_1}\phi^{-\frac{1}{2}}(t)\left\{\int_{t_1}^t \phi(s)ds\right\}^{\frac{1}{4}}$.

Proof: According to [17] the infinite product

$$\prod \frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2},$$

is an entire function of λ, whose roots are precisely $\lambda_n(t_1)$, $n \geq 1$. From [1, 9.2.11] we have

$$J_\nu'(z) = \sqrt{\frac{2\pi}{z}}\{-R(\nu,z)\sin\chi - S(\nu,z)\cos\chi\},$$

where ν is fixed and

$$\chi = z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi,$$

$$R(\nu,z)\approx\sum_{k=0}^{\infty}(-1)^k\,\frac{4\nu^2+16k^2-1}{4\nu^2-(4k+1)^2}\left\{\frac{(\nu,2k)}{(2z)^{2k}}\right\}$$
$$=1-\frac{(\mu-1)(\mu+15)}{2!(8z)^2}+\ldots,$$

$$S(\nu,z)\approx\sum_{k=0}^{\infty}(-1)^k\,\frac{4\nu^2+4(2k+1)^2-1}{4\nu^2-(4k+1)^2}\left\{\frac{(\nu,2k)}{(2z)^{2k+1}}\right\}$$
$$=\frac{\mu+3}{8z}-\frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3}+\ldots,$$

as $|z|\to\infty$, where $\mu=4\nu^2$. Now, by inserting $z=R_+(t_1)\sqrt{\lambda}$, $\nu=\dfrac{1}{4}$, and from [18, 9.5.11], we get

$$\prod_{n\geq1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2}=\sqrt{\frac{2}{\pi}}\left\{2R_+(t_1)\sqrt{\lambda}\right\}^{\frac{1}{4}}\Gamma(\frac{1}{4})\cos(R_+(t_1)\sqrt{\lambda}+\frac{\pi}{8})[1].$$

Thus it follows from (19) and (23):

$$A_1=\frac{C(t_1,\lambda)}{\prod_{n\geq1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2}}=\frac{1}{2}\phi^{\frac{1}{2}}(0)R_+^{\frac{-1}{4}}(t_1)\psi(t_1)\,.$$

6. DUAL EQUATIONS

In this section, we first derive the dual equations associated with (4) by use of infinite product representation, then we investigate the uniqueness solution of these equations.

By the implicit function theorem, $\lambda_n(t)$ is twice continuously differentiable functions. For $t\in(0,1)\setminus\{t_1\}$, the condition

$$C(t,\lambda_n(t))=0,$$

gives, as usual,

$$\frac{\partial C}{\partial t}+\frac{\partial C}{\partial\lambda}.\lambda_n'=0, \tag{25}$$

and differentiating again

$$\frac{\partial^2 C}{\partial t^2}+2\frac{\partial^2 C}{\partial t\partial\lambda}.\lambda_n'+\frac{\partial^2 C}{\partial\lambda^2}.(\lambda_n')^2+\frac{\partial C}{\partial\lambda}.\lambda_n''=0\,. \tag{26}$$

The first term in (26) is zero at $(t,\lambda_n(t))$ by virtue of (4). Thus

$$2\frac{\partial^2 C}{\partial t\partial\lambda}.\lambda_n'+\frac{\partial^2 C}{\partial\lambda^2}.(\lambda_n')^2+\frac{\partial C}{\partial\lambda}.\lambda_n''=0\,. \tag{27}$$

If we make use of the infinite product form of $C(t,\lambda)$, substitute this in (27), in the case $t\in(0,1)\setminus\{t_1\}$ the *dual* of the equation (4) will be obtained. Indeed, we need the various derivatives of $C(t,\lambda)$ at the points $(t,\lambda_n(t))$ for $t\in(0,1)\setminus\{t_1\}$.

Now, we first calculate the various derivatives of $C(t,\lambda)$ for $t \in (0,1)\setminus\{t_1\}$. From (22), it can be written

$$C(t,\lambda) = h(t)\prod_{k\geq 1}\left(1-\frac{\lambda}{\lambda_k(t)}\right), \tag{28}$$

where h is a function independent of λ. By using (23) and Theorem 2, we obtain

$$h_1 = 2^{\frac{1}{2}\nu-1}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t) = h\prod_{k\geq 1}\frac{\varsigma_k^2}{R_+^2(t)\lambda_k(t)},$$

where $\varsigma_k, k\geq 1$, is the sequence of positive zeros of $J'_{\frac{1}{2}}(t)$, the sequence $\lambda_k(t), k\geq 1,$ represents the sequence of positive eigenvalues of the boundary value problem L_1 on $[0,t]$ and $R_+(t)$ is determined in (18). Therefore,

$$h(t) = 2^{\frac{1}{2}\nu-1}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\prod_{k\geq 1}\frac{R_+^2(t)\lambda_k(t)}{\varsigma_k^2}. \tag{29}$$

We calculate $\frac{\partial C}{\partial\lambda}$, $\frac{\partial^2 C}{\partial\lambda^2}$ and $\frac{\partial^2 C}{\partial t\partial\lambda}$ at the points $(t,\lambda_n(t))$ by using (28). In determining $\frac{\partial^2 C}{\partial\lambda\partial t}$, the interchange of summation and differentiation in

$$\frac{d}{dt}\sum_{k\geq 1}\log(1-\frac{\lambda}{\lambda_k(t)})$$

is valid, because by the results of [19], the differentiated series

$$\sum_{k\neq n}\frac{-\lambda_n\lambda'_k(t)}{(\lambda_k(t)-\lambda_n)\lambda_k(t)}$$

is uniformly convergent. We define F_n by

$$F_n = F_n(t,\lambda_n(t)) = \prod_{k\neq n, k\geq 1}\left(1-\frac{\lambda_n(t)}{\lambda_k(t)}\right). \tag{30}$$

Since

$$\frac{\partial C}{\partial\lambda} = h\sum_{i=1}^{\infty}\frac{-1}{\lambda_i(t)}\prod_{k\neq i, k\geq 1}\left(1-\frac{\lambda}{\lambda_k(t)}\right),$$

we have

$$\frac{\partial C}{\partial\lambda}(t,\lambda_n(t)) = \frac{-hF_n}{\lambda_n(t)},$$

$$\frac{\partial^2 C}{\partial\lambda^2}(t,\lambda_n(t)) = \frac{2hF_n}{\lambda_n(t)}\sum_{i\neq n, i\geq 1}\frac{1}{\lambda_i}\left(1-\frac{\lambda_n(t)}{\lambda_i(t)}\right)^{-1},$$

$$\frac{\partial^2 C}{\partial\lambda\partial t}(t,\lambda_n(t)) = \frac{-h'(t)F_n}{\lambda_n(t)} + \frac{h(t)F_n\lambda_n'}{\lambda_n{}^2} - h(t)F_n \sum_{i\neq n, i\geq 1} \frac{\lambda_i'}{\lambda_i{}^2}\left(1-\frac{\lambda_n(t)}{\lambda_i(t)}\right)^{-1}$$

$$-\frac{h(t)F_n\lambda_n'}{\lambda_n} \sum_{i\neq n, i\geq 1} \frac{1}{\lambda_i}\left(1-\frac{\lambda_n(t)}{\lambda_i(t)}\right)^{-1}.$$

Placing these terms into (27), we obtain

$$\lambda_n{}'' + \frac{2h'\lambda_n'}{h} + 2\lambda_n\lambda_n' \sum_{i\neq n, i\geq 1} \frac{\lambda_i{}'}{\lambda_i{}^2}\left(1-\frac{\lambda_n(t)}{\lambda_i(t)}\right)^{-1} - 2\frac{(\lambda_n')^2}{\lambda_n} = 0. \tag{31}$$

Dividing the above equation by λ_n' and integrating from a fixed number $\alpha \neq 0$ up to t, for $t \in (0, t_1)$ we obtain

$$\lambda_n{}'(t) = \frac{\lambda_n{}^2(t)\lambda_n'(\alpha)h^2(\alpha)}{\lambda_n{}^2(\alpha)h^2(t)} e^{-2S_n(t,\lambda_n)}, \tag{32}$$

where

$$S_n(t,\lambda_n) = \sum_{i\neq n} \int_\alpha^t \frac{\lambda_i{}'\lambda_n}{\lambda_i}(\lambda_i - \lambda_n)^{-1} d\upsilon, \tag{33}$$

and $h(t)$ is determined in (29). Similarly, for $t \in (t_1, 1)$, dividing the equation (31) by λ_n' and integrating from t up to 1, we obtain

$$\lambda_n{}'(t) = \frac{\lambda_n{}^2(t)\lambda_n'(1)h^2(1)}{\lambda_n{}^2(1)h^2(t)} e^{2T_n(t,\lambda_n)}, \tag{34}$$

where

$$T_n(t,\lambda_n) = \sum_{i\neq n} \int_t^1 \frac{\lambda_i{}'\lambda_n}{\lambda_i}(\lambda_i - \lambda_n)^{-1} d\upsilon. \tag{35}$$

The system of equation (31) is dual to the original equation (4) and involves only the function $\lambda_n(t)$.

We shall establish the initial value problem consisting of this system of equations subject to the initial condition

$$\lambda_n(1) = \lambda_n, \qquad (n = 1,2,\ldots).$$

Before investigating the uniqueness theorem, we need to prove the following lemma:

Lemma 1. Let

$$\lambda_n = cn^2 - cn + O(1), \tag{36}$$

where c is a fixed number. Then

$$\sum_{k\neq n} \frac{\lambda_n}{\lambda_k - \lambda_n} = O(1). \tag{37}$$

Proof: we have

$$c^{-1}(\lambda_k - \lambda_n) = (k^2 - n^2) - (k - n) + O(1)$$
$$= (k^2 - n^2)\left\{1 - \frac{k - n + O(1)}{(k^2 - n^2)}\right\},$$

so

$$\frac{c}{(\lambda_k - \lambda_n)} = \frac{k + n + 1}{(k^2 - n^2)(k + n)} + \frac{O(1)}{(k^2 - n^2)^2} + \frac{c\{k - n + O(1)\}^2}{(k^2 - n^2)^2(\lambda_k - \lambda_n)},$$

hence

$$\sum_{k \neq n} \frac{c}{(\lambda_k - \lambda_n)} = \sum_{k \neq n} \frac{k + n + 1}{(k^2 - n^2)(k + n)} + \sum_{k \neq n} \frac{O(1)}{(k^2 - n^2)^2} + \sum_{k \neq n} \frac{c\{k - n + O(1)\}^2}{(k^2 - n^2)^2(\lambda_k - \lambda_n)}. \qquad (38)$$

Note that

$$\sum_{k \neq n} \frac{1}{(k^2 - n^2)} \leq \sum_{k \neq n} \frac{k + n + 1}{(k^2 - n^2)(k + n)} \leq \sum_{k \neq n} \frac{3}{2(k^2 - n^2)},$$

and from [20] we know that

$$\sum_{k \neq n} \frac{1}{(k^2 - n^2)} = \frac{3}{4n^2}, \qquad \sum_{k \neq n} \frac{1}{(k^2 - n^2)^2} = O(\frac{1}{n^2}).$$

Consequently, in (38), the first and second sum are $O(\frac{1}{n^2})$, while the third is $O(\frac{1}{n^3})$. Since $\lambda_n = O(n^2)$,

$$\lambda_n \sum_{k \neq n} \frac{c}{(\lambda_k - \lambda_n)} = O(1).$$

Now we investigate the uniqueness of solution of the dual equations.

Theorem 4. The initial value problem consisting of (34) subject to the initial condition

$$\lambda_n(1) = \lambda_n, \qquad (n = 1,2,...),$$

has a unique solution.

Proof: For convenience, we can write this initial value problem as

$$\frac{d\upsilon}{dt} = g(\upsilon), \qquad \upsilon(1) = \upsilon := \{\lambda_1(1), \lambda_2(1), ...\},$$

where υ, g are defined as

$$\upsilon_n = \lambda_n, \qquad n = 1,2,...,$$

$$g_n(\upsilon) = g_n \equiv \frac{H_n \lambda_n^{\,2}(t)}{h^2(t)} e^{2T_n(t,\lambda_n)}, \tag{39}$$

where $H_n = \dfrac{\lambda_n'(1)h^2(1)}{\lambda_n^{\,2}(1)}$. In order to prove the uniqueness of a solution, it suffices to prove that the function g satisfies in the Lipschitz condition. To this end, first we show that g_n is of order $O(n^4)$. Since from (29) and (20), $h(t)$ is of order $O(1)$ and $\lambda_n = O(n^2)$, $\dfrac{\lambda_n^{\,2}}{h^2(t)} = O(n^4)$. So it is enough to prove that

$$e^{2T_n(t,\lambda_n)} = O(1),$$

or equivalently we prove

$$T_n(t,\lambda_n) = O(1).$$

The interchange of summation and integration in T_n will be valid if the differentiate series $\sum_{i\neq n} \frac{\lambda_i'\lambda_n}{\lambda_i}(\lambda_i - \lambda_n)^{-1}$ is uniformly convergent (which is the case, see [19]). Next, we prove that

$$\sum_{i\neq n} \frac{\lambda_i'\lambda_n}{\lambda_i}(\lambda_i - \lambda_n)^{-1} = O(1). \tag{40}$$

Since

$$-\frac{\lambda_i'}{\lambda_i} = \frac{3(\int_0^t \phi(\varsigma)d\varsigma)^2}{tR_+^2(t)} + O(\frac{\ln(i)}{i}) = O(1),$$

it does not influence the order of the expression (40). Consequently, by Lemma 1, the estimates for (40) hold uniformly for t in a compact subset of $(0,1]$.
Note that (40) and (35) imply that $T_n(t,\lambda_n) = O(1)$. We therefore conclude that $e^{2T_n(t,\lambda_n)} = O(1)$, whence

$$g_n(\upsilon) = g_n = O(n^4).$$

To complete the proof, we define a normed space as follows: Let

$$\varphi = \left\{ \upsilon = (\upsilon_n)_1^\infty : \|\upsilon\| = \sum_1^\infty \frac{|\upsilon_n|}{n^6} < \infty \right\},$$

and define $\varphi^* \subseteq \varphi$ as a subset containing nonzero members of φ whose asymptotic distribution is of the form

$$\upsilon_n = \pm\frac{n^2\pi^2}{4I^2} \mp \frac{n\pi^2}{4I^2} + O(1),$$

where I is a constant. It is trivial to see that $\varphi^* \neq \varphi$ contains the sequence $\{\lambda_n\}_{n=1}^\infty$. Finally, g is a map from φ^* into φ, because $g_n(\upsilon) = g_n = O(n^4)$. Furthermore, φ^* is convex space, i.e., if $0 \le \alpha \le 1$ and $\sigma, \eta \in \varphi^*$, then $\alpha\sigma + (1-\alpha)\eta \in \varphi^*$, hence for each n, the function $g_n(\alpha\sigma + (1-\alpha)\eta)$ is defined on $[0,1]$. The function $g_n(\alpha\sigma + (1-\alpha)\eta)$, $n \ge 1$, is an entire function with respect to α on $[0,1]$. Consequently, for each n, we can find some α_n, $0 \le \alpha_n \le 1$, such that

$$g_n(1)-g_n(0)=g_n(\sigma)-g_n(\eta)=\frac{dg_n}{d\alpha}\Big|_{\alpha=\alpha_n}.$$

Since g_n is a function of λ_n, we have

$$g_n(\sigma)-g_n(\eta)=\frac{dg_n}{d\alpha}\Big|_{\alpha=\alpha_n}=\sum_m\frac{dg_n}{d\lambda_m}\Big|_{\alpha_n}(\sigma_m-\eta_m). \tag{41}$$

We use (39) in order to compute the coefficient $\dfrac{dg_n}{d\lambda_m}$ and obtain

$$g_n(\sigma)-g_n(\eta)=\tilde{f}_n\sum_{m\neq n}\left\{-\frac{2}{\tilde{\lambda}_m}+2\int_t^1\frac{\tilde{\lambda}_n\tilde{\lambda}''_m}{\tilde{\lambda}_m\tilde{\lambda}'_m(\tilde{\lambda}_m-\tilde{\lambda}_n)}-\frac{(2\tilde{\lambda}_n\tilde{\lambda}_m-\tilde{\lambda}_n^{\,2})\tilde{\lambda}'_m}{\tilde{\lambda}_m^{\,2}(\tilde{\lambda}_m-\tilde{\lambda}_n)^2}\,d\upsilon\right\}(\sigma_m-\eta_m)$$

$$+2\tilde{f}_n\sum_{i\neq n}\int_t^1\frac{\tilde{\lambda}'_i}{(\tilde{\lambda}_i-\tilde{\lambda}_n)^2}(\sigma_n-\eta_n), \tag{42}$$

where

$$\tilde{f}_n=g_n(\alpha_n\sigma+(1-\alpha_n)\eta),\quad \tilde{\lambda}_m=\alpha_n\sigma_m+(1-\alpha_n)\eta_m.$$

In the above calculation, the interchange of summation and derivation is valid since the differentiated series $\sum_{i\neq n}\frac{\lambda'_i\lambda_n}{\lambda_i}(\lambda_i-\lambda_n)^{-1}$ is uniformly convergent (see [19]). Dividing the above equation by $(2n)^6$ and summing with respect to n, we obtain

$$\sum_n\frac{g_n(\sigma)-g_n(\eta)}{(2n)^6}\leq M\sum_n\frac{|\sigma_n-\eta_n|}{(2n)^6}, \tag{43}$$

where M is a constant number. In order to derive the inequality, we have made use of the fact that the dependence of the various quantities on α_n can be ignored for large n's and that the sum $\sum_{i\neq n}\int_t^1\frac{\tilde{\lambda}'_i}{(\tilde{\lambda}_i-\tilde{\lambda}_n)^2}$ is of $O(n^{-4})$ and $\frac{\lambda_n}{\lambda'_n}=O(1)$. Thus, (43) give

$$\|g_n(\sigma)-g_n(\eta)\|\leq M\|\sigma-\eta\|,$$

where M is a constant. Thus g satisfies a Lipsschitz condition, and consequently, the equation (34) has an unique solution.

7. THE BOUNDARY VALUE PROBLEM WITH A SINGULARITY

In this section we transform (4) by a replacement to the differential equation with a singular point and we study the asymptotic behavior of the solutions.
Denote $T=\int_0^1\phi(\varsigma)d\varsigma$.
We transform (4) by means of the replacement

$$z=\int_0^t\phi(\varsigma)d\varsigma,\ \mathrm{u(z)}=\phi^{\frac{1}{2}}(t)y(t), \tag{44}$$

to the differential equation

$$-u''(z)+q(z)u(z)=\lambda u(z), \qquad z\in[0,T], \tag{45}$$

with initial conditions

$$u(0,\lambda)=r_1, \qquad u'(0,\lambda)=r_2, \tag{46}$$

where $r_1=\phi^{\frac{1}{2}}(0),\ r_2=\frac{1}{2}\phi^{\frac{-3}{2}}(0)\phi'(0)$, and $q(z)$ has quadratic singularity in the interval $(0,T)$ and has the form:

$$q(z)=\frac{-\frac{3}{16}}{(z-z_1)^2}+q_0(z),$$

where $z_1=\int_0^{t_1}\phi(\varsigma)d\varsigma$. We also assume that

$$q_0(z)(z-z_1)^{\frac{1}{2}}\in L(0,T).$$

Since the solutions of equation (45) have singularity at $z=z_1$, and therefore, in general, the values of the solutions and their derivatives at $z=z_1$ are not defined.

Remark 1. In [13], fundamental system of solutions $\{S_k(z,\lambda)\},\ k=1,2$, of equation (45) were constructed with the following properties:
i) For each fixed $z\in[0,T]$, the functions $S_k^{(\nu)}(z,\lambda),\ \nu=0,1$, are entire in λ of order $\frac{1}{2}$.
ii) Denote $\mu_k=\frac{(-1)^k}{4}+\frac{1}{2},\ k=1,2$. Then

$$S_k(z,\lambda)\le C\left|\rho(z-z_1)^{\mu_k}\right|,$$

for $|\rho(z-z_1)|\le 1$, where C is a positive constant in the estimate, not depending on z and ρ.
iii) The following relation holds

$$<S_1(z,\lambda),S_2(z,\lambda)>\equiv 1,$$

where $<y(z),\tilde{y}(z)>:=y(z)\tilde{y}'(z)-y'(z)\tilde{y}(z)$ is the wronskian of y and $\tilde{y}$.
Let $\omega_0=(0,z_1),\ \omega_1=(z_1,T)$, from [2] for $z\in\omega_0\cup\omega_1$,

$$S_k(z,\lambda)=(z-z_1)^{\mu_k}\sum_{m=0}^{\infty}S_{km}\left(\rho(z-z_1)\right)^{2m}, \qquad k=1,2,$$

where $S_{10}S_{20}=2,\ S_{km}=(-1)^m S_{k0}\left(\prod_{s=1}^{m}\left((2s+\mu_k)(2s+\mu_k-1)+\frac{3}{16}\right)\right)^{-1}$.
From [5], we have the following Lemma:

Lemma 2. For $(\rho,z)\in\Omega:=\{(\rho,z):|\rho(z-z_1)|\ge 1\},\ z\in\omega_s, s=0,1$:

$$\begin{aligned}S_k^{(m)}(z,\lambda)=\beta_k\rho^{-\mu_k}\{&(-i\rho)^m\exp(-i\rho(z-z_1))[1]_\gamma+(i\rho)^m\exp((-1)^s i\pi\mu_k)\exp(i\rho(z-z_1))[1]_\gamma\\&+(-1)^s\sqrt{2}i(i\rho)^m\exp(i\rho(z-z_1))[1]_\gamma\}\end{aligned} \tag{47}$$

where $[1]_\gamma=1+O\left((\rho(z-z_1))^{-1}\right),\ \beta_1\beta_2=(-2\sqrt{2}i)^{-1}$.
Denote

$$\varphi_k(z,\lambda) = (-1)^{k-1}\left(S_2^{(2-k)}(0,\lambda)S_1(z,\lambda) - S_1^{(2-k)}(0,\lambda)S_2(z,\lambda)\right), k = 1,2\,.$$

The functions $\varphi_k(z,\lambda)$ are solutions of (45) and

$$\varphi_k^{(m-1)}(0,\lambda) = \delta_{k,m}, \;\; k,m = 1,2\,, \tag{48}$$

($\delta_{k,m}$ is the Kronecker delta). Moreover,

$$<\varphi_1(z,\lambda),\varphi_2(z,\lambda)>\equiv 1\,.$$

Using (47) we get for $z\in\omega_s$, $(\rho,z)\in\Omega$:

$$\varphi_k^{(m-1)}(z,\lambda) = \frac{1}{2}(i\rho)^{m-k}\{\exp(i\rho z)[1]_\gamma + (-1)^{m-k}\exp(-i\rho z)[1]_\gamma$$
$$+(-1)^k\sqrt{2}i\exp(i\rho(z-2z_1))[1]_\gamma\}, \qquad |\rho|\to\infty\;,\; k,m=1,2\,. \tag{49}$$

Using the preceding results, from (46) and (48), we have

$$u(z,\rho) = r_1\varphi_1(z,\lambda) + r_2\varphi_2(z,\lambda)\,. \tag{50}$$

Now, from (49) and (50) we obtain the asymptotic solution of equation (45) in the following theorem :

Theorem 5. For $z\in\omega_s$, $(\rho,z)\in\Omega$, $|\rho|\to\infty$, $\operatorname{Im}\rho\ge 0$, $m=0,1$:

$$u^{(m)}(z,\rho) = \frac{1}{2}(i\rho)^{m-1}(i\rho r_1 + r_2)\exp(i\rho z)[1]_\gamma + \frac{1}{2}(-i\rho)^{m-1}(-i\rho r_1 + r_2)\exp(-i\rho z)[1]_\gamma$$
$$+\frac{\sqrt{2}}{2}(i\rho)^{m-1}(\rho r_1 + ir_2)\exp(i\rho(z-2z_1))[1]_\gamma\,. \tag{51}$$

8. CANONICAL PRODUCT REPRESENTATION OF SOLUTION IN THE SINGULARITY CASE

According to (24), the boundary value problem $L_1 = L_1(p(t),\phi^2(t),s)$ defined by equation (4) with boundary conditions $y(0,\lambda)=1, y'(0,\lambda)=0, y(s,\lambda)=0$, transform to the boundary value problem $L_2 = L_2(q(z),b)$ with boundary conditions

$$u(0,\lambda) = r_1, \qquad u'(0,\lambda) = r_2, \qquad u(b,\lambda) = 0, \tag{52}$$

where $b = \int_0^s \phi(\varsigma)d\varsigma$, $s\in(0,1)\setminus\{t_1\}$, $\mathrm{r}_1 = \phi^{\frac{1}{2}}(0)$ and $\mathrm{r}_2 = \frac{1}{2}\phi^{\frac{-3}{2}}(0)\phi'(0)$.

Thus, according to (20) and (44) for $b\in(0,T)\setminus\{z_1\}$, the boundary value problem L_2 has a countable set of positive eigenvalues $\{\lambda_{1n}\}_{n\ge1}$:

$$\sqrt{\lambda_{1n}(b)} = \frac{n\pi - \frac{\pi}{2}}{b} + O(\frac{1}{n}). \tag{53}$$

According to remark 1, the solution $u(z,\rho)$ of Sturm-Liouville equation (45) defined by initial conditions (52) is an entire function of ρ for each fixed $z\in[0,T]$, thus it follows from the Hadamard's theorem (see [16, p. 24]) that such solution is expressible as an infinite product.

To complete the investigation of the last sections, we want to prove the following theorem. Theorem 2 is a useful tool for the proof of this result:

Theorem 6. Let $u(z,\lambda)$ be the solution of (45) satisfying the initial conditions $u(0,\lambda)=r_1$, $u'(0,\lambda)=r_2$. Then for $z\in B_\nu, \nu=0,1$,

$$u(z,\lambda)=2^{\frac{1}{2}\nu-1}r_1\prod_{n\geq 1}\frac{(\lambda_{1n}(z)-\lambda)z^2}{\varsigma_n^2}, \tag{54}$$

where $B_0=(0,z_1)$, $B_1=(z_1,T)$, $\lambda_{1n}(z), n\geq 1$, represents the sequence of positive eigenvalues of the boundary value problem L_1 on $[0,z]$, and $\varsigma_n, n=1,2,...,$ is the sequence of positive zeros of $J'_{\frac{1}{2}}$.

Proof: From (18) and (44) we obtain $R_+(t)=z$. Thus, according to (44), (53), $r_1=\phi^{\frac{1}{2}}(0)$ and Theorem 2, we arrive at (54).
This completes the representation of the solution of (45) with the initial conditions (46) as an infinite product.

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