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APPLICATION OF DIFFERENTIAL TRANSFORMS FOR SOLVING THE VOLTERRA INTEGRO-PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract – In this paper, first the properties of one and two-dimensional differential transforms are presented. Next, by using the idea of differential transform, we will present a method to find an approximate solution for a Volterra integro-partial differential equations. This method can be easily applied to many linear and nonlinear problems and is capable of reducing computational works. In some particular cases, the exact solution may be achieved. Finally, the convergence and efficiency of this method will be discussed with some examples which indicate the ability and accuracy of the method.

Keywords – Volterra, integro-partial differential equations, differential transforms

1. INTRODUCTION

The purpose of this paper is to employ the two-dimensional differential transforms method for Volterra integro-partial differential equations, which are often encountered in many branches of physics, chemistry and engineering. The solution of integral and integro-differential equations has a major role in the fields of science and engineering. When a physical system is modelled under the differential sense, it finally gives a differential equation, an integral equation or an integro-differential equation. There are various techniques for solving an integral or integro-differential equation, e. g. Galerkin, wavelet Galerkin, Haar wavelet method [1-5], Adomian decomposition method [6, 7], homotopy perturbation method (HPM) [8], polynomial solution [9] and multi-level iteration method [10]. Such methods are based on developing and analyzing numerical methods for solving one-dimensional integral equations. But in two-dimensional cases so far, a small amount of work has been done (see [11, 12]).

Differential transforms method (DTM) is a semi analytical-numerical technique and is an iterative procedure that depends on Taylor series expansion. The concept of the differential transforms was first proposed by Zhou [13]. By useing this method, it is possible to solve differential equations [14-24], difference equations [24], differential-difference equations [25], KDV equations [26] fractional differential equations [27, 28], two-dimensional integral equations [11, 12] and integro-differential equations [29, 30]. In this paper, we apply one and two-dimensional differential transform method to solve the Volterra integro-partial differential equations of the form:

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$$\alpha \frac{\partial^2 u(x,y)}{\partial x^2} + \beta \frac{\partial^2 u(x,y)}{\partial x \, \partial y} + \gamma \frac{\partial^2 u(x,y)}{\partial y^2} + \delta \frac{\partial u(x,y)}{\partial x} + \varepsilon \frac{\partial u(x,y)}{\partial y} + \eta u(x,y)$$
$$-\lambda \int_0^{\,x} \int_0^y K(x,y,s,t,u(s,t)) ds dt = f(x,y), \tag{1}$$

where $u(x,y)$ is an unknown function, α, β, γ, δ, ε and η are some two variable given functions, functions $f(x,y)$ and $K(x,y,s,t,u)$ are two known continuous functions defined respectively on $D = [0,X] \times [0,Y]$ and $E = \{(x,y,s,t,u): 0 \le s \le x \le X, 0 \le t \le y \le Y, -\infty \le u \le \infty\}$ and λ is a given constant. The completion conditions related to Eq. (1) are as follows:

$$\sum_{p=1}^{2} \sum_{j=0}^{1} \sum_{k=0}^{1-j} d_{ijk}^{(p)} \frac{\partial^{j+k} u(x,y)}{\partial x^k \, \partial y^j} |_{x=x_0} = a_i(x) \, , x \in [a,b], \ i = 1,2, \tag{2}$$

and

$$\sum_{p=1}^{2} \sum_{j=0}^{1} \sum_{k=0}^{1-j} e_{ijk}^{(p)} \frac{\partial^{j+k} u(x,y)}{\partial x^k \, \partial y^j} |_{y=y_0} = b_i(y) \, , y \in [c,d] \, , i = 1,2, \tag{3}$$

where $d_{ijk}^{(p)}$ and $e_{ijk}^{(p)}$ are known constant coefficients, and $b_i(y)$, $a_i(x)$ for $i = 1,2$ are known functions.

2. DIFFERENTIAL TRANSFORM

a) One- Dimensional Differential Transform

Definition 2.1. Consider the analytical function of one variable $u(x)$, which is defined on $D = [0,X] \subseteq \mathbb{R}$ and $x_0 \in D$. One- dimensional differential transform of $u(x)$ is denoted by $U(k)$ and is defined on $\mathbb{N} \cup \{0\}$ as the following*:*

$$U(k) = \frac{1}{k!} [\frac{d^k u(x)}{dx^k}] |_{x=x_0}, \tag{4}$$

where $u(x)$ is the original function and $U(k)$ is called the transformed function.
Inverse differential transform of $U(k)$ in the Eq. (4) is defined as follows:

$$u_0(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k. \tag{5}$$

Since $u(x)$ is an analytical function, it is clear that $u(x) = u_0(x)$. By combination of Eqs. (4) and (5), with $x_0 = 0$, the function $u(x)$ can be written as:

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} [\frac{d^k u(x)}{dx^k}] |_{x=0} x^k. \tag{6}$$

The fundamental mathematical properties of one-dimensional differential transform can readily be obtained and are summarized in the following theorem.

Theorem 2.2. If $U(k), F(k)$ and $G(k)$ are one-dimensional differential transforms of the functions $u(x), f(x)$ and $g(x)$ respectively, then*:*
1. *If* $u(x) = f(x) \pm g(x)$ *then* $U(k) = F(k) \pm G(k)$.
2. *If* $u(x) = \alpha f(x)$ *then* $U(k) = \alpha F(k)$.
3. *If* $u(x) = f(x) g(x)$ *then* $U(k) = \sum_{l=1}^{k} F(l) G(k-l)$.
4. *If* $u(x) = \frac{df(x)}{dx}$ *then* $U(k) = (k+1) F(k+1)$.
5. *If* $u(x) = \frac{d^m f(x)}{dx^m})$ *then* $U(k) = (k+1)(k+2) \dots (k+m) F(k+m)$.
6. *If* $u(x) = \int_0^x f(t) dt$ *then* $U(k) = \frac{F(k-1)}{k}$, $k \ge 1$, $U(0) = 0$.

7. *If* $u(x) = x^m$ *then* $U(k) = \delta(k-m) = \begin{cases} 1, & k = m; \\ 0, & O.W. \end{cases}$
8. *If* $u(x) = sin(\omega x + \alpha)$ *then* $U(k) = \frac{\omega^k}{k!} sin(\frac{k\pi}{2} + \alpha)$.
9. *If* $u(x) = cos(\omega x + \alpha)$ *then* $U(k) = \frac{\omega^k}{k!} cos(\frac{k\pi}{2} + \alpha)$.

b) Two- Dimensional Differential Transform

Definition 2.3. Consider the analytical function of two variable $u(x,y)$, which is defined on $D = [0,X] \times [0,Y] \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in D$. The two- dimensional differential transform of $u(x,y)$ is denoted by $U(k,h)$ and is defined on $\mathbb{N}^2 \cup \{(0,0)\}$ as the following*:*

$$U(k,h) = \frac{1}{k!h!} [\frac{\partial^{k+h} u(x,y)}{\partial x^k \partial y^h}]|_{(x_0,y_0)}, \tag{7}$$

where $u(x,y)$ is the original function, and $U(k,h)$ is called the transformed function. Inverse differential transform of $U(k,h)$ in Eq. (7) is defined as follows:

$$u_0(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k,h)(x-x_0)^k (y-y_0)^h. \tag{8}$$

Since $u(x,y)$ is an analytical function, it is clear that $u(x,y) = u_0(x,y)$. By combining Eqs. (7) and (8), with $(x_0, y_0) = (0,0)$, the function $u(x,y)$ can be written as:

$$u(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} [\frac{\partial^{k+h} u(x,y)}{\partial x^k \partial y^h}]_{(0,0)} x^k y^h. \tag{9}$$

The fundamental mathematical properties of two-dimensional differential transform can readily be obtained and are expressed in the following theorem.

Theorem 2.4. If $U(k,h), F(k,h)$ and $G(k,h)$ are two-dimensional differential transforms of the functions $u(x,y), f(x,y)$ and $g(x,y)$ respectively, then:
1. *If* $u(x,y) = f(x,y) \pm g(x,y)$ *then* $U(k,h) = F(k,h) \pm G(k,h)$.
2. *If* $u(x,y) = \alpha f(x,y)$ *then* $U(k,h) = \alpha F(k,h)$.
3. *If* $u(x,y) = f(x,y) g(x,y)$ *then* $U(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} F(r, h-s) G(k-r, s)$.
4. *If* $u(x,y) = \frac{\partial f(x,y)}{\partial x}$ *then* $U(k,h) = (k+1)F(k+1,h)$.
5. *If* $u(x,y) = \frac{\partial f(x,y)}{\partial y}$ *then* $U(k,h) = (h+1)F(k,h+1)$.
6. *If* $u(x,y) = x^m y^n$ *then* $U(k,h) = \delta(k-m, h-n) = \delta(k-m)\delta(h-n)$.
7. *If* $u(x,y) = \frac{\partial^{r+s} f(x,y)}{\partial^r x \partial^s y}$ *then* $U(k,h) = (k+1)(k+2) \dots (k+r)(h+1)(h+2) \dots (h+s) F(k+r, h+s)$.

In the following, we prove some fundamental theorems, which will be used in section three. It should be pointed out that these theorems may be found in [11, 12, 17], but we claim that our approaches are simpler and more preferable.

Theorem 2.5. If $u(x,y) = \int_0^x \int_0^y f(s,t) ds dt$, $U(k,h)$ and $F(k,h)$ are differential transforms of the functions $u(x,y)$ and $f(x,y)$, respectively, then:

$$U(k,h) = \begin{cases} 0, & if \;\; k = 0 \;\; or \;\; h = 0; \\ \frac{F(k-1,h-1)}{kh} & if \;\; k, \;\; h = 1,2,3, \dots . \end{cases} \tag{10}$$

Proof: By using the Leibniz's formula and mathematical induction on k and h we have:

$$\frac{\partial^k u(x,y)}{\partial x^k} = \int_0^y \frac{\partial^{k-1} f(s,x)}{\partial x^{k-1}} ds, \tag{11}$$

and

$$\frac{\partial^h u(x,y)}{\partial x^h} = \int_0^x \frac{\partial^{h-1} f(y,t)}{\partial x^{h-1}} dt, \tag{12}$$

respectively. Hence, by applying Eqs. (11) and (12) in definition (2.3), with $(x_0, y_0) = (0,0)$, we have:

$$F(k,h) = 0, \text{ if } k = 0 \text{ or } h = 0.$$

Since for $k \geq 1$, $h \geq 1$, we get:

$$\begin{aligned}\frac{\partial^{k+h} u(x,y)}{\partial x^k \, \partial y^h} &= \frac{\partial^k}{\partial x^k}\left[\frac{\partial^h}{\partial x^h} u(x,y)\right] \\ &= \frac{\partial^k}{\partial x^k}\left[\int_0^x \frac{\partial^{h-1} f(y,t)}{\partial x^{h-1}} dt\right] \\ &= \frac{\partial^{k-1}}{\partial x^{k-1}}\left[\frac{\partial^{h-1}}{\partial x^{h-1}} f(x,y)\right] \\ &= \frac{\partial^{k+h-2} f(x,y)}{\partial x^{k-1}\,\partial y^{h-1}},\end{aligned}$$

therefore by definition (2.3), with $(x_0, y_0) = (0,0)$, we will have:

$$k!\,h!\,U(k,h) = (k-1)!\,(h-1)!\,F(k-1,h-1).$$

Hence:

$$U(k,h) = \frac{F(k-1,h-1)}{kh}.$$

Theorem 2.6. If $u(x,y) = \int_0^x \int_0^y f(s,t) g(s,t) ds dt$, $U(k,h)$, $F(k,h)$ and $G(k,h)$ are differential transforms of the functions $u(x,y)$, $f(x,y)$ and $g(x,y)$, respectively, then:

$$U(k,h) = \begin{cases} 0 & if\ \ k = 0\ \ or\ \ h = 0; \\ \frac{1}{kh} \sum_{r=0}^{k-1} \sum_{s=0}^{h-1} F(r, h-s-1) G(k-r-1, s), & if\ \ k,\ h = 1,2,3,\ldots. \end{cases} \tag{13}$$

Proof: Define $v(x,y) = f(x,y)g(x,y)$. By using theorem (2.5), where the corresponding function f is replaced by v, and applying theorem 2.4(3), the assertion is immediately obtained.

Theorem 2.7. Suppose that $u(x,y) = \prod_{i=1}^{n} f_i(x,y)$. If $U(k,h)$ and $F_i(k,h)$ for $i = 1,2,\ldots n$ are differential transforms of the functions $u(x,y)$ and $f_i(x,y)$ for $i = 1,2,\ldots n$, respectively, then:

$$\begin{aligned}&U(k,h) = \sum_{r_{n-1}=0}^{k} \sum_{s_{n-1}=0}^{h} \sum_{r_{n-2}=0}^{r_{n-1}} \sum_{s_{n-2}=0}^{s_{n-1}} \cdots \sum_{r_2=0}^{r_3} \sum_{s_2=0}^{s_3} \sum_{r_1=0}^{r_2} \sum_{s_1=0}^{s_2} \\ &F_1(r_1,s_1) F_2(r_2-r_1, s_2-s_1) \ldots \\ &F_{n-1}(r_{n-1}-r_{n-2}, s_{n-1}-s_{n-2}) F_n(k-r_{n-1}, h-s_{n-1})],\ \ k \geq 0, h \geq 0.\end{aligned} \tag{14}$$

Proof: The assertion is obviously obtained by induction on n and using theorem 2.4(3).
The next corollary is the direct result of the theorems 2.5 and 2.7.

Corollary 2.8. If the assumptions of theorem(2.7) are satisfied and

$$u(x,y) = \int_0^x \int_0^y \prod_{i=1}^{n} f_i(s,t) ds dt,$$

then $U(k,h) = 0$, if $k = 0$ or $h = 0$ and

$$U(k,h)=\frac{1}{kh}\sum_{r_{n-1}=0}^{k-1}\sum_{s_{n-1}=0}^{h-1}\sum_{r_{n-2}=0}^{r_{n-1}}\sum_{s_{n-2}=0}^{s_{n-1}}\dots\sum_{r_2=0}^{r_3}\sum_{s_2=0}^{s_3}\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}$$
$$[F_1(r_1,s_1)F_2(r_2-r_1,s_2-s_1)\dots F_{n-1}(r_{n-1}-r_{n-2},s_{n-1}-s_{n-2})F_n(k-r_{n-1}-1,h-s_{n-1}-1)] \quad (15)$$
$$k\geq 1, h\geq 1.$$

Now we prove the following complementary theorems as the basis of our method.

Theorem 2.9. Let $u(x,y)=f(\alpha x+\beta y)$, where f is a one variable function and α, β are two constants. If $U(k,h)$ and $F(k)$ are differential transforms of the functions $u(x,y)$ and $f(x)$, respectively, then:

$$U(k,h)=F(k+h)\binom{k+h}{h}\alpha^k\beta^h. \quad (16)$$

Proof: We know that $f(t)=\sum_{k=0}^{\infty}F(k)t^k$. Therefore:

$$\begin{aligned}f(\alpha x+\beta y)&=\sum_{k=0}^{\infty}F(k)(\alpha x+\beta y)^k\\&=\sum_{k=0}^{\infty}F(k)\sum_{h=0}^{k}\binom{k}{h}(\alpha x)^h(\beta y)^{k-h}\\&=\sum_{k=0}^{\infty}\sum_{h=0}^{k}F(k)\binom{k}{h}(\alpha x)^h(\beta y)^{k-h}.\end{aligned} \quad (17)$$

The result is obtained by comparing the coefficients of x^iy^j, for $i,j=0,1,2,\dots$ in (9) and (17).

Corollary 2.10. *If* $u(x,y)=sin(\alpha x+\beta y)$ *and* $v(x,y)=e^{\alpha x+\beta y}$ *then:*

$$U(k,h)=\frac{\alpha^k\beta^h}{k!h!}\sin\left(\frac{(k+h)\pi}{2}\right),\quad V(k,h)=\frac{\alpha^k\beta^h}{k!h!}. \quad (18)$$

Theorem 2.11. Let $u(x,y)=f(x,y)g(x)$ where g is a one variable function of x. If $U(k,h)$, $F(k,h)$ and $G(k)$ are differential transforms of the functions $u(x,y)$, $f(x,y)$ and $g(x)$, respectively, then:

$$U(k,h)=\sum_{r=0}^{k}F(k-r,h)G(r). \quad (19)$$

Proof: By defining $g_0(x,y):=g(x)$, we will have $u(x,y)=f(x,y)g_0(x,y)$. We know that $G_0(k,h)=G(k)\delta(h)$. By applying theorem 2.4(3), we have:

$$\begin{aligned}U(k,h)&=\sum_{r=0}^{k}\sum_{s=0}^{h}F(k-r,s)G_0(r,h-s)\\&=\sum_{r=0}^{k}\sum_{s=0}^{h}F(k-r,s)G(r)\delta(h-s).\end{aligned} \quad (20)$$

According to the definition of δ, equation (20) implies that:

$$U(k,h)=\sum_{r=0}^{k}F(k-r,h)G(r).$$

Theorem 2.12. Let $u(x,y)=f(x,y)g(y)$, where g is a one variable function of y. If $U(k,h)$, $F(k,h)$ and $G(h)$ are differential transforms of the functions $u(x,y)$, $f(x,y)$ and $g(y)$, respectively, then:

$$U(k,h)=\sum_{s=0}^{h}F(k,h-s)G(s). \quad (21)$$

Proof: The proof is similar to the previous theorem.

Theorem 2.13. Let $u(x,y) = f(x)g(y)$, where f and g are one variable functions of x and y, respectively. If $U(k,h)$, $F(k)$ and $G(h)$ are differential transforms of the functions $u(x,y)$, $f(x)$ and $g(y)$, respectively, then:

$$U(k,h) = F(k)G(h). \tag{22}$$

Proof: By defining $f_0(x,y) := f(x)$ and $g_0(x,y) := g(y)$, we will have $u(x,y) = f_0(x,y)g_0(x,y)$, $F_0(k,h) = F(k)\delta(h)$ and $G_0(k,h) = G(h)\delta(k)$. By using theorem 2.4(3) and definition of δ, we have:

$$\begin{aligned} U(k,h) &= \sum_{r=0}^{k}\sum_{s=0}^{h} F_0(k-r,s)G_0(r,h-s) \\ &= \sum_{r=0}^{k}\sum_{s=0}^{h} F(r)\delta(h-s)G(s)\delta(k-r) \\ &= F(k)G(h). \end{aligned}$$

The following theorem will imply the differential transform of the completion conditions (2), (3). In this theorem, we consider the applicable case $x_0, y_0 = 0$ or 1.

Theorem 2.14. Assume that $F(k)$, $G(h)$ and $U(k,h)$ are differential transforms of $f(x)$, $g(x)$ and $u(x,y)$, so we have:

1. *If* $f(x) = \frac{\partial u(x,y)}{\partial y}|_{y=0}$ *then F(k)=U(k,1), k=0,1,2,... .*
2. *If* $f(x) = \frac{\partial u(x,y)}{\partial y}|_{y=1}$ *then* $F(k) = \sum_{s=0}^{n} sU(k,s)$, $k = 0,1,2, \dots$.
3. *If* $f(x) = \frac{\partial u(x,y)}{\partial x}|_{y=0}$ *then* $F(k) = (k+1)U(k+1,0)$, $k = 0,1,2, \dots$.
4. *If* $f(x) = \frac{\partial u(x,y)}{\partial x}|_{y=1}$ *then* $F(k) = (k+1)\sum_{r=0}^{n} U(k+1,r)$, $k = 0,1,2, \dots$.
5. *If* $g(y) = \frac{\partial u(x,y)}{\partial y}|_{x=0}$ *then* $G(h) = (h+1)U(0,h+1)$, $h = 0,1,2, \dots$.
6. *If* $g(y) = \frac{\partial u(x,y)}{\partial y}|_{x=1}$ *then* $G(h) = (h+1)\sum_{r=0}^{m} U(r,h+1)$, $h = 0,1,2, \dots$.
7. *If* $g(y) = \frac{\partial u(x,y)}{\partial x}|_{x=0}$ *then* $G(h) = U(1,h)$, $h = 0,1,2, \dots$.
8. *If* $g(y) = \frac{\partial u(x,y)}{\partial x}|_{x=1}$ *then* $G(h) = \sum_{s=0}^{m} sU(s,h)$, $h = 0,1,2, \dots$.

Proof: The assertions are obtained easily, by use of theorem 2.4(4) and 2.4(5) and comparing the results to the equation (5).

3. APPLICATIONS AND NUMERICAL RESULTS

In this section, the differential transform is applied to solve Volterra integro-partial differential equations of the form equation (1). To this end, we consider the solution of equation (1) in the form of Taylor series as equation (8). Since the truncated Taylor series or the corresponding polynomial expansion is an approximate solution of equation (1), by substituting the solutions $U(k,h)$, for $k = 0,1,2, \dots m$ and $h = 0,1,2, \dots n$ in Eq. (8) we have:

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{m}\sum_{h=0}^{n} U(k,h)(x-x_0)^k(y-y_0)^h + e_{m,n}(x,y) \\ &= \tilde{u}_{m,n}(x,y) + e_{m,n}(x,y), \end{aligned}$$

where $e_{m,n}(x,y)$ is the error function and $\tilde{u}_{m,n}(x,y)$ is the approximate function. Now, for $(x,y) = (x_i, x_j)$, we define the absolute error by:

$$e_{m,n}(x_i, y_j) = |u(x_i, y_j) - \tilde{u}_{m,n}(x_i, y_j)|. \tag{23}$$

Corresponding to Eq. (1), we define $\varphi(x, y)$ and $\psi(x, y)$ as the follows:

$$\varphi(x,y) := \alpha \frac{\partial^2 u(x,y)}{\partial x^2} + \beta \frac{\partial^2 u(x,y)}{\partial x \, \partial y} + \gamma \frac{\partial^2 u(x,y)}{\partial y^2} + \delta \frac{\partial u(x,y)}{\partial x} + \varepsilon \frac{\partial u(x,y)}{\partial y} + \eta u(x,y),$$

$$\psi(x,y) := -\lambda \int_0^x \int_0^y K(x,y,s,t,u(s,t)) ds ds.$$

Therefore, equation (1) can be written as:

$$\varphi(x,y) + \psi(x,y) = f(x,y). \tag{24}$$

In order to obtain the approximate solution, $\tilde{u}$, it is sufficient to determine $U(k,h)$ for $k = 0,1,2, \dots m$, $h = 0,1,2, \dots n$. By applying differential transform on the equation (24) and the completion conditions, (2) and (3), they will be transformed into a system of recursive algebraic equations. This process is called differential transform method (DTM).

Remark 3.1. In most of the practical cases, the exact solution is not available, hence the computation of the absolute error is impossible. One way to show the convergence of the iterative solution $\tilde{u}_{m,n}$ to the exact solution, $u(x,y)$, is as follows:
1. Compute $\tilde{u}_{m,n}(x,y)$ and $\tilde{u}_{m+1,n+1}(x,y)$.
2. Compute $\tilde{u}_{p,q}(x,y)$ and $\tilde{u}_{p+1,q+1}(x,y)$, for $m,n \le p,q$.
3. Define:

$$E_{m,n}(x_i, y_j) = |\tilde{u}_{m,n}(x_i, y_j) - \tilde{u}_{m+1,n+1}(x_i, y_j)|$$

and:

$$E_{p,q}(x_i, y_j) = |\tilde{u}_{p,q}(x_i, y_j) - \tilde{u}_{p+1,q+1}(x_i, y_j)|$$

for some x_i and y_j.
4. If $E_{m,n}(x_i, y_j) \ge E_{p,q}(x_i, y_j)$ it is concluded that $\tilde{u}_{m,n}(x,y)$ converges to the exact solution when $m,n \to \infty$.
The above technique will be applied in example (3.4).
In the following, we consider the case of:

$$K(x,y,s,t,u(s,t)) = \sum_{i=1}^{p} w_i(x,y) v_i(s,t) u_i^i(s,t), \tag{25}$$

which is solvable by using differential transform method.

Example 3.2. Consider the following Volterra Integro-Partial Differential Equation:

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} = -1 + e^x + e^y + e^{x+y} + \int_0^x \int_0^y u(s,t) ds dt, \tag{26}$$

subject to the initial conditions:

$$\begin{cases} u(x,0) = e^x, \\ u(0,y) = e^y. \end{cases} \tag{27}$$

Transformed versions of Eqs. (26) and (27) are:

$$(k+1)U(k+1,h) + (h+1)U(k,h+1) = -\delta(k)\delta(h) + \frac{\delta(h)}{k!} + \frac{\delta(k)}{h!} + \frac{1}{k!h!} \frac{U(k-1,h-1)}{kh}, \tag{28}$$

and

$$\begin{cases} U(k,0) = \frac{1}{k!}, & k = 0,1,2,\dots, \\ U(0,h) = \frac{1}{h!}, & h = 0,1,2,\dots. \end{cases} \tag{29}$$

respectively. By substituting (29) in (28), we obtain the closed form of the solution series as the follow:

$$u(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k,h) x^k y^h = (1 + x + \frac{x^2}{2!} + \cdots)(1 + y + \frac{y^2}{2!} + \cdots) = e^{x+y},$$

which is the exact solution.

Example 3.3. Consider the following Volterra integro-partial differential equation:

$$\varphi(x,y) + \psi(x,y) = f(x,y), \tag{30}$$

where

$$\varphi(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} \quad , \qquad \psi(x,y) = -\int_0^x \int_0^y (e^s + t) u(s,t) ds dt,$$

$$f(x,y) = e^x - \frac{1}{2} y e^{2x} - xye^x + ye^x - \frac{1}{4} x^2 y^2 + \frac{1}{3} xy^3 - \frac{1}{2} y,$$

is subject to the initial conditions:

$$\begin{cases} u(0,y) = 1 - t, \\ u(x,0)) = x + e^x, \\ \frac{\partial u(x,y)}{\partial y}|_{y=0} = -1. \end{cases} \tag{31}$$

If x indicates the location and y indicates the time, then the above equation will be the wave equation which is affected by the power dependent on the location and the time. The exact solution is $u(x,y) = e^x + x - y$.

Applying differential transform on Eqs. (30) and (31) and using the Theorems of section 2, and theorem 2.14 for the initial conditions, we obtain:

$$\Phi(k,h) + \Psi(k,h) = F(k,h), \tag{32}$$

where:

$$\Phi(k,h) = (k+1)(k+2)U(k+2,h) + (h+1)(h+2)U(k,h+2),$$

$$\Psi(k,h) = \begin{cases} 0, & if\ \ k = 0\ \ or\ \ h = 0; \\ \frac{1}{kh} \sum_{r=0}^{k-1} \frac{U(r,h-1)}{k-r-1}, & if\ \ h = 0\ \ and\ \ k \geq\ 1; \\ \frac{1}{kh} [U(k-1,h-2) + \sum_{r=0}^{k-1} \frac{U(r,h-1)}{k-r-1}], & o.w. \end{cases}$$

$$F(k,h) = \frac{\delta(h)}{k!} - \frac{2^k \delta(h-1)}{2k!} - \begin{cases} \frac{1}{(k-1)!}, & if\ \ k \geq\ 1\ \ and\ \ h = 1; \\ 0, & o.w. \end{cases} + \frac{\delta(h-1)}{k!}$$
$$- \frac{\delta(k-2,h-2)}{4} + \frac{\delta(k-1,h-3)}{3} - \frac{\delta(k,h-1)}{2},$$

and:

$$\begin{cases} U(0,h) = \delta(h) - \delta(h-1), & h \geq 0; \\ U(k,0) = \delta(k-1) + \frac{1}{k!}, & k \geq 0; \\ U(k,1) = -\delta(k), & k \geq 0. \end{cases}$$

By solving the above recursive equations for cases n=m=4, n=m=8 and n=m=16 we obtain:

$$\tilde{u}_{4,4}(x,y) = 1 + 2x - y + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!},$$

$$\tilde{u}_{8,8}(x,y) = 1 + 2x - y + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}$$

and:

$$\tilde{u}_{8,8}(x,y) = 1 + 2x - y + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}\frac{x^9}{9!} + \ldots + \frac{x^{16}}{16!},$$

which are the truncated Taylor series of the exact solution. Table 1 shows the absolute errors at some particular points.

Table 1. Numerical results for example 3.3

(x_i, y_j)	$e_{4,4}(x_i, y_j)$	$e_{8,8}(x_i, y_j)$	$e_{16,16}(x_i, y_j)$
(0,0)	0	0	0
(0.1,0.3)	$8.474e-008$	$2.997e-015$	$2.22e-016$
(0.2,0.2)	$2.758e-006$	$1.439e-012$	0
(0.3,0.1)	$2.130e-005$	$5.591e-011$	$2.220e-016$
(0.4,0.6)	$9.136e-005$	$7.523e-010$	$2.220e-016$
(0.5,0.4)	$2.837e-004$	$5.664e-009$	$4.440e-016$
(0.6,0.9)	$7.188e-004$	$2.953e-008$	0
(0.7,0.6)	$1.581e-003$	$1.195e-007$	$4.440e-016$
(0.8,0.5)	$3.140e-003$	$4.017e-007$	$4.440e-016$
(0.9,0.2)	$5.765e-003$	$1.172e-006$	0
(1,1)	$9.948e-003$	$03.0586e-006$	$2.664e-015$

Example 3.4. Consider the following Volterra integro-partial differential equation:

$$\varphi(x,y) + \psi(x,y) = f(x,y), \tag{33}$$

where:

$$\varphi(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial u(x,y)}{\partial y}, \qquad \psi(x,y) = -\int_0^x \int_0^y x\cos(s-t)u(s,t)dsdt,$$

$$f(x,y) = 2cos(x+y) - xcos(x+y) - 1 + t - \frac{x^3}{8} + xsinx - xsiny + xycosy - \frac{xysin(2x)}{8} + \frac{x^2ycos(2x)}{4} + \frac{x^3cos(2y)}{8} - xsin(x-y) - xycos(x-y),$$

subject to the initial conditions:

$$\begin{cases} \left[u(x,y) + \frac{\partial u(x,y)}{\partial y}\right]|_{x=0} = 1 + t, \\ u(x,y)|_{y=0} = xsinx. \end{cases} \tag{34}$$

This equation indicates that the heat transform process depends on the location and time.
Applying differential transform on Eqs. (33) and (34) and using the Theorems of section 2 and the theorem 2.14 we obtain:

$$\Phi(k,h) = (k+2)(k+1)U(k+2,h) - (h+1)U(k,h+1) + U(k,h).$$

$$\Psi(k,h) = -\begin{cases} 0, & if\ \ k=0,1\ \ or\ \ h=0; \\ \dfrac{1}{(k-1)h}\displaystyle\sum_{r=0}^{k-2}\sum_{s=0}^{h-1}\dfrac{(-1)^{h-s-1}\cos(\frac{(h+r-1)\pi}{2})U(k-r-2,s)}{r!\,(h-s-1)!}, & o.w. \end{cases},$$

and:

$$F(k,h) = 2\frac{\cos(\frac{(k+h)\pi}{2})}{h!\,k!} - \begin{cases} 0, & if\ \ k=0; \\ \dfrac{\cos(\frac{(k+h-1)\pi}{2})}{h!\,(k-1)!}, & o.w. \end{cases} + \delta(k,h) + \delta(k,h-1)$$

$$-\frac{1}{8}\delta(k-3,h) + \begin{cases} \dfrac{1}{(k-1)!}\sin(\dfrac{(k-1)\pi}{2}), & if\ \ h=0\ \ and\ \ k\geq 1; \\ 0, & o.w. \end{cases}$$

$$-\delta(k-1)\frac{1}{h!}\sin(\frac{h\pi}{2}) + \begin{cases} \dfrac{1}{(h-1)!}\cos(\dfrac{(h-1)\pi}{2}), & if\ \ k=1\ \ and\ \ h\geq 1; \\ 0, & o.w. \end{cases}$$

$$-\frac{1}{8}\begin{cases} \dfrac{2^{k-1}}{(k-1)!}\sin(\dfrac{(k-1)\pi}{2}), & if\ \ h=1\ \ and\ \ k\geq 1; \\ 0, & o.w. \end{cases}$$

$$+\frac{1}{4}\begin{cases} \dfrac{2^{k-2}}{(k-2)!}\cos(\dfrac{(k-2)\pi}{2}), & if\ \ h=1\ \ and\ \ k\geq 2; \\ 0, & o.w. \end{cases} + \frac{2^h\delta(k-3)\cos(\frac{h\pi}{2})}{8h!}$$

$$-\begin{cases} 0, & if\ \ k=0; \\ \dfrac{(-1)^h\sin(\frac{(k+h-1)\pi}{2})}{(k-1)!\,(h-1)!}, & o.w. \end{cases} - \begin{cases} 0, & if\ \ k=0\ \ or\ \ h=0; \\ \dfrac{(-1)^{h-1}\cos(\frac{(k+h-2)\pi}{2})}{(k-1)!\,h!}, & o.w. \end{cases}$$

and

$$\begin{cases} U(0,h) + (h+1)U(0,h+1) = \delta(h) + \delta(h-1), \quad h\geq 0; \\ U(k,0) = \begin{cases} 0, & if\ \ k=0; \\ \dfrac{1}{(k-1)!}\sin\left(\dfrac{(k-1)\pi}{2}\right), & if\ \ k\geq 1. \end{cases} \end{cases}$$

By solving the above recursive equations for the cases n=m=4, n=m=8 and n=m=16 we obtain

$$\tilde{u}_{4,4}(x,y) = x^2 - \frac{x^4}{6} + y + xy - \frac{x^3y}{2} - \frac{x^2y^2}{2} + \frac{x^4y^2}{12} - \frac{xy^3}{6} + \frac{x^3y^3}{12} + \frac{x^2y^4}{24} - \frac{x^4y^4}{144},$$

$$\tilde{u}_{8,8}(x,y) = x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \frac{x^8}{5040} + y + xy - \frac{x^3y}{2} + \frac{x^5y}{24} - \frac{x^7y}{720} - \frac{x^2y^2}{2} + \frac{x^4y^2}{12} - \frac{x^6y^2}{240} + \frac{x^8y^2}{10080} - \frac{xy^3}{6} + \frac{x^3y^3}{12} - \frac{x^5y^3}{144} + \frac{x^3y^3}{12} + \frac{x^7y^3}{4320} - \frac{x^4y^4}{144} + \ldots$$

$$\tilde{u}_{16,16}(x,y) = x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \frac{x^8}{5040} + y + xy - \frac{x^3y}{2} + \frac{x^5y}{24} - \frac{x^7y}{720} - \frac{x^2y^2}{2} + \frac{x^4y^2}{12} - \frac{x^6y^2}{240} + \frac{x^8y^2}{10080} - \frac{xy^3}{6} + \frac{x^3y^3}{12} - \frac{x^5y^3}{144} + \ldots$$

Table 2 shows $E_{m,n}$, which is introduced in remark (3.1), at some points.

Table 2. Numerical results for example 3.4

(x_i, y_j)	$E_{4,4}(x_i, y_j)$	$E_{8,8}(x_i, y_j)$	$E_{16,16}(x_i, y_j)$
(0,0)	0	0	0
(0.1,0.3)	$2.1380e-006$	$5.4043e-012$	0
(0.2,0.2)	$3.1716e-006$	$2.7994e-012$	0
(0.3,0.1)	$1.0132e-005$	$4.8736e-011$	0
(0.4,0.6)	$4.7938e-004$	$1.3903e-008$	0
(0.5,0.4)	$5.4439e-004$	$1.9181e-008$	0
(0.6,0.9)	0.0050	$7.2448e-007$	$4.4409e-016$
(0.7,0.6)	0.0043	$5.7998e-007$	0
(0.8,0.5)	0.0067	$1.5989e-006$	$4.4409e-016$
(0.9,0.2)	0.0049	$1.9089e-006$	$1.5543e-015$
(1,1)	0.0392	$2.2359e-005$	$4.2188e-014$

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