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A GEOMETRIC-BASED NUMERICAL SOLUTION OF EIKONAL EQUATION OVER A CLOSED LEVEL CURVE*

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Abstract – This paper presents a new numerical method for solution of eikonal equation in two dimensions. In contrast to the previously developed methods which try to define the solution surface by its level sets (contour curves), the developed methodology identifies the solution surface by resorting to its characteristics. The suggested procedure is based on the geometric properties of the solution surface and does not require any mesh for computation. It works well in finding the ridge of the solution surface as well. In addition, the area of the surface and its corresponding volume can be easily determined via this method. Three examples have been provided to demonstrate the capability of the suggested method in presenting these important features of the solution. The issue of convergence has also been investigated. It has been concluded that the suggested method works well in solving the eikonal equation in problems for which the direction of characteristics of the solution surface, and its area or volume underneath are quite important.

Keywords – Eikonal equation, characteristics, non-linear partial differential equations

1. INTRODUCTION

Eikonal equation is a relatively simple first-order nonlinear partial differential equation which describes many phenomena in physical science. These include the propagation of waves in optics, acoustics, elasticity and electromagnetics. Therefore, its solution has been of great interest to many researchers.

From the classification point of view, eikonal equation belongs to the family of Hamilton-Jacobi (HJ) equations and is considered to be their stationary form. It can be derived from Maxwell's equations in electromagnetics [1]. In its general form, it can be expressed as follows:

$$\|\nabla u(x)\| = k, \tag{1}$$

where $\nabla u(x)$ is the gradient vector of the scalar function $u(x)$, and $\|\bullet\|$ represents its Euclidean norm. k is called the index of refraction. Although it is a simple equation, its solution is relatively difficult to obtain due to the nonlinearity involved. Analytical solutions are available only for very special and simple boundary conditions. Therefore, an attempt has been made to delineate a general framework for its numerical solution. The structure of the governing equation causes the resulting finite difference equation to be nonlinear. Therefore, the numerical procedure for solving the equation would also be complicated.

A very useful method developed for solving time-dependent Hamilton-Jacobi equation is the Level Set Method proposed by Osher and Sethian [2-4]. This method is based on the initial value formulation of the problem. In a sense, it takes the initial wave front curve and establishes a level set function resembling the H-J equation whose zero level set is the initial curve. The solution surface is considered as the initial state which propagates in time. At any time, the wave front is obtained by finding the zero level set (curve) of such surface. A specially designed numerical method for solving a time-independent Hamilton-

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Jacobi equation like eikonal, is the Fast Marching Method developed by Sethian [5, 6]. In contrast to the more general Level Set Method, the Fast Marching Method is essentially the boundary value formulation of the problem because the initial position of the front curve is assumed to be the boundary of the solution surface. The stationary solution surface represents the time the front passes each point. Therefore, the level sets of the arrival time surface represent the position of front in time.

The Level Set and Fast Marching Methods have been used in many applications in science and technology. These include geometric curve and surface evolution, computer graphics, geodesics, seismology, mesh generation, and fluid dynamics. While the Fast Marching Method is more effective in solving the stationary Hamilton-Jacobi equations in which the speed of front propagation does not change in sign, the Level Set Method is more general because it can handle the problems in which the front moves forward at some places and backward at some others.

Both methods require a mesh for computation which was previously a rectangular Cartesian mesh. Later, the methods were extended so that the unstructured triangular meshes can also be adopted. The basis of the Level Set and Fast Marching Methods is wave propagation. They both work on finding the level sets (contour curves) of the solution surface either in time or space. The characteristics receive scant attention in these methods. Even the Ordered Upwind Method, which was developed later, tries to find the contours and define the surface by its level sets, though it uses partial information about the characteristic directions for the solution [7, 8].

Characteristics play an important role in the solution of many physical problems [9-15]. Their application in the graphical integration of the governing equations started more than a century ago [16]. The method of characteristics is also applied for the solution of non-linear partial differential equations [17]. In some problems involving eikonal equation, characteristics are much more important than the level curves, and it is preferred to define the solution surface directly via its characteristics [11-14]. Orientation of the characteristics of the solution surface plays an important role in the solution of Cauchy boundary value problems. Values of the area of the solution surface and the volume underneath are also important and their determination is required in some problems [18, 12-14]. These features are relatively difficult, if not impossible, to be obtained via the previously mentioned methods.

A rather simple numerical method has been presented here, in this paper, for the solution of eikonal equation under constant index of refraction. The developed method is based on the geometric properties of the solution surface of eikonal equation over a closed level curve. While the previous methods work on finding the contours of the solution surface, this method is based on developing the surface by its characteristics. In contrast to the other methods, no mesh is required here for the computations. The method allows the calculation of the area of the solution surface and the corresponding volume rather simply. Examples are provided to show the capability of the method in constructing the surface over a level curve, which gives the solution to eikonal equation in many areas of physics. The issue of convergence has also been investigated via comparing the calculated values of the area and volume to the analytical solution available, or by observing the improvement in the answer with the increase in the number of divisions considered on the bounding level curve.

2. STATEMENT OF THE PROBLEM

The main purpose of this paper is to present a numerical method for solving eikonal equation in two dimensions. Mathematically the problem can be stated as: "Given a closed level curve in the x-y plane on which a scalar function $u(x, y)$ is constant; find $u(x, y)$ so that Eq. 1 holds." In a clearer form, we want to find the solution surface of $z=u(x, y)$ so that:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = k^2,\ \forall(x,y)\in\Omega\subset\Re^2, \tag{2}$$

$$u(x,y) = u_0 = \text{constant},\ \forall(x,y)\in\partial\Omega, \tag{3}$$

where k is called the constant or index of refraction, and Ω is a domain in the *x*-*y* plane bounded by the level set curve denoted by $\partial\Omega$.

As mentioned in the previous section, some numerical procedures have been developed for solving eikonal equation, but they all are based on finding the level sets (level curves) of the solution surface. Nevertheless, there are problems in physics for which the characteristic curves of the solution surface are more important than the level curves [11-14]. The area of the surface and the volume underneath are also very important in some problems involving eikonal equation. Our purpose here is to present a numerical procedure which gives these important features of the solution surface that has received scant attention in earlier studies.

3. GEOMETRIC CHARACTERISTICS OF THE SOLUTION SURFACE

In order to arrive at a numerical procedure for solving the problem, it is necessary to draw our attention to the geometric properties of the solution surface. Physically, the solution of such a problem i.e., $z=u(x, y)$, is nothing but a surface of constant slope, k, constructed over the region, Ω, bounded by the curve, denoted by $\partial\Omega$. This has been illustrated in Fig. 1.

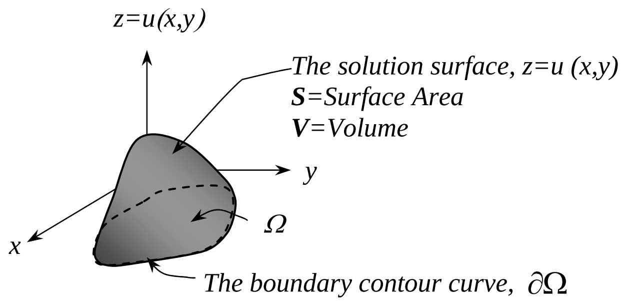

Fig. 1. The solution surface over the boundary contour curve

An analogy of the problem would be the surface of a hill of sand pile made over the same area. This analogy was first used by Nadai [15] to assess the ultimate torque that a shaft of the same cross section can carry in torsion in plastic state.

A geometric procedure suggested for solving such a problem is to start drawing contours of u inside the boundary curve towards the center of the area [18, 15]. In this way, the solution surface is defined by its contours. The precision of such a solution depends on how small are the steps between the contours. Such a surface can also be defined by its characteristic curves and this is preferred in many problems involving Cauchy boundary value data. For eikonal equation, the directions of the gradient and characteristics coincide. Moreover, the characteristic curves in this case are straight lines [17, 7, 8]. Therefore, if the gradient can be found at different points of the bounding level curve $\partial\Omega$, their direction would be known everywhere in the domain, Ω. This would simply be done by drawing lines normal to the boundary level curve since the gradient should always be perpendicular to the level curve (contour). Therefore, the inquired surface is made by moving a straight line laying on the boundary curve in the direction of the gradient, making a slope of k with the horizontal *x*-*y* plane. Obviously, such a surface may develop a ridge. If S is a point on the ridge of such a surface and S' is its projection on the *x*-*y* plane, and if f_1 and f_2 are parts of the boundary curve on its two sides (Fig. 2); there should be at least two straight lines,

SN_1 and SN_2, in the directions of steepest decent toward this perimeter curve. Figure 2 indicates that the distances SN_1 and SN_2 should be equal since the slope of the surface is constant everywhere, and the perimeter curve, $\partial\Omega$, is a contour of *u*.

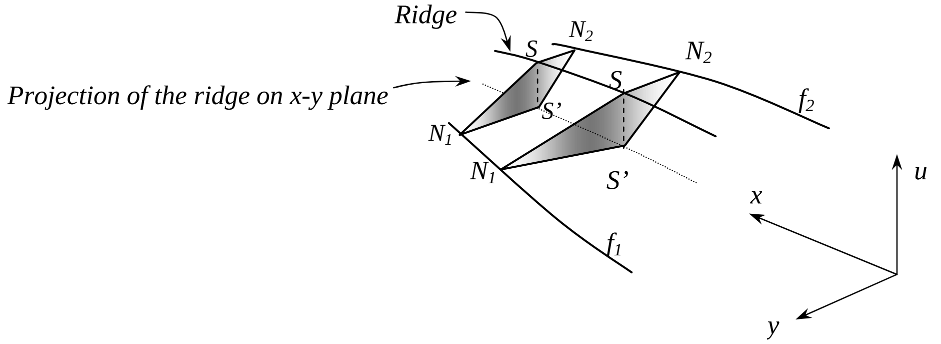

Fig. 2. The ridge of the solution surface and its projection

Figure 3 shows the *x*-*y* plane together with the segments f_1 and f_2 of the boundary curve. As shown, *S'* is the center of a circle passing through N_1 and N_2, being tangent to f_1 and f_2 at these points.

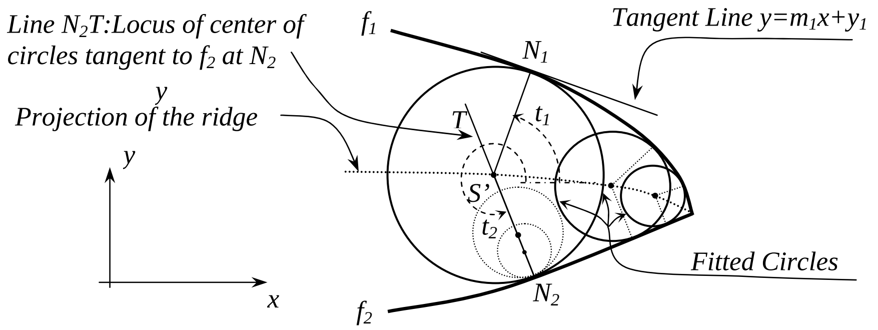

Fig. 3. Geometric considerations for finding point N_1 for an assumed N_2

The problem is, therefore, to find pairs of N_1 and N_2 on parts f_1 and f_2 of the perimeter curve. This becomes possible if the circles of different sizes are fitted between f_1 and f_2. If N_2 is a point of branch f_2 with coordinates (x_2, y_2), at which the slope of f_2 is m_2, the locus of the center of circles tangent to f_2 at N_2 would be the line N_2T of slope $\tan t_2 = -1/m_2$ (see Fig. 3). The fitted circle should be tangent to the other branch, f_1 at N_1, as well. If the equation of the tangent line to f_1 at this point is $y = m_1 x - n_1$; the radial line to that point would have the slope $\tan t_1 = -1/m_1$, where t_1 and t_2 are the angular distances of the radii to N_1 and N_2 from a fixed direction like the *x*-axis, respectively. It can simply be shown that the radius *r* of such a circle can be obtained from:

$$r = \frac{m_1 x_2 + n_1 - y_2}{m_1(\cos t_2 - \cos t_1) - (\sin t_2 - \sin t_1)} \tag{4}$$

Coordinates of N_1 can then be found from:

$$x_1 = x_2 - r\cos t_2 + r\cos t_1 \tag{5}$$

$$y_1 = y_2 - r\sin t_2 + r\sin t_1 \tag{6}$$

Therefore, we see that for the case where f_1 is a straight line, the size of the circle and coordinates of N_1 are directly obtained from these equations. Where the bounding curves are complicated, a numerical procedure involving iterations would be inevitable.

4. NUMERICAL PROCEDURE

The procedure followed here is to assume a point on one side of the perimeter curve, and find its conjugate on the other side. Different procedures might be proposed for this purpose. A geometric based iterative procedure has been suggested here. Figure 4 roughly demonstrates the steps for finding N_1 for an assumed N_2 on f_2. As mentioned before, the line N_2T is the locus of the center of all circles that pass through N_2 and are tangent to f_2. The starting point for N_1 can be taken just above N_2 on f_1 (point N). A line is drawn tangent to f_1 from this point. A circle is then drawn from N_2 tangent to f_2, so that it is tangent to this line as well. The size of this circle is found from Eq. 4. The point of tangency of this circle and the line is then found from Eqs. 5 and 6. The point located just under this point of tangency, but on f_1 is considered as the answer for N_1 for this trial. This point is used as the starting point for the next trial. The same procedure is followed for obtaining a better answer for N_1. Iterations for finding a better location for N_1 are stopped when the distance from the new location of N_1 to its previous position is less than an acceptable error. The point S on the ridge related to the pair N_1 and N_2, can be located above the center of the fitted circle by a height rk. This procedure has been implemented into a computer program written in Matlab environment.

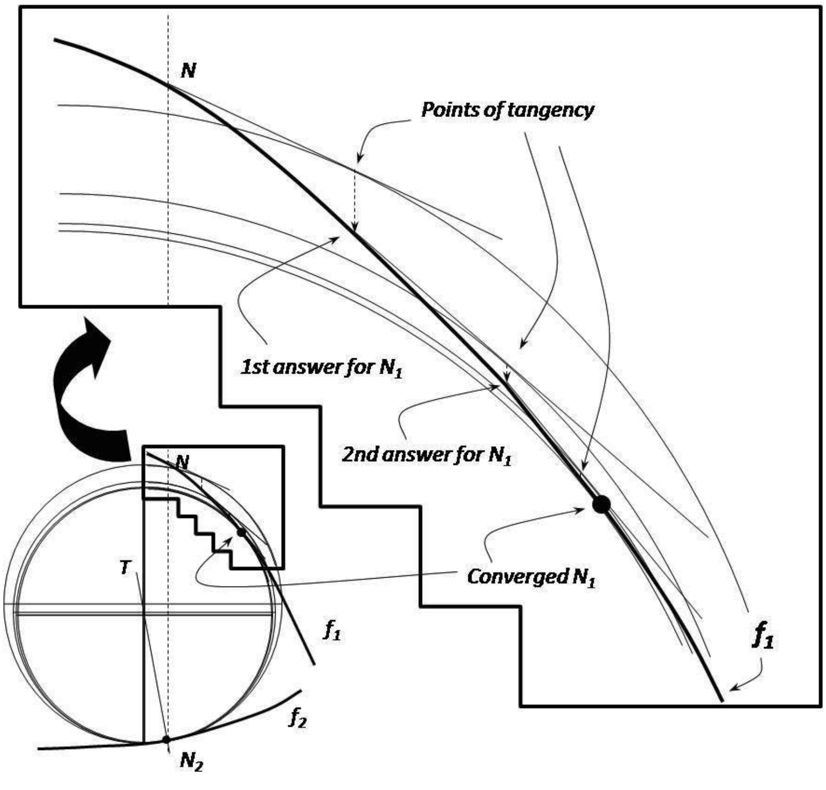

Fig. 4. Geometric based procedure for finding the conjugate of N_2 on f_1

5. ALGORITHM FOR NUMERICAL SOLUTION

In order to find the solution surface of the NPDE over a domain bounded by $\partial\Omega$, we first consider a number of N_2 points on the branch f_2 of $\partial\Omega$. For each of them we try to locate its correct conjugate N_1 on the other side using the procedure described in the previous section. In this way an answer is found for the triple (N_1, S, N_2). We then repeat the same procedure for the next N_2 and get another triple of the form (N_1, S, N_2). Proceeding in the same way will result in the solution surface, which can be constructed by connecting N_1 to S, and S to N_2 for each of the (N_1, S, N_2)-triples found (see Fig. 2). In this way, the solution for this nonlinear PDE over this contour curve is obtained. The closer the successive points N_2 taken on f_2 to each other the more precise would be the obtained solution surface.

An important application of the suggested procedure is the ease of calculating the area of the solution surface and the corresponding volume. As mentioned previously, these values are useful in some areas of physics and mechanics. Once the positions of a set of points N_1, *S* and *S'* or N_2, *S* and *S'* on successive planes they make around the perimeter contour curve are obtained, the required area and volume can readily be computed. The area of the surface is obtained by summing the areas of the triangular elements whose sides are S_iN_i's, and whose vertices are *S* or *N* of the next or previous characteristics. The volume under the surface is calculated by summing the volumes of the tetrahedrons filling the volume between successive *S'SN* planes on both sides. The computer code written, makes these calculations. The flowchart of this computer program has been shown in Fig. 5.

6. EXAMPLES OF APPLICATION

Examples are provided here, in this section, to demonstrate the capability of the suggested procedure in establishing the solution surface. The area of the surface and the volume under it have also been calculated in each case. These quantities have been used to show the progressive convergence in the answer of the problem as the number of divisions on f_1 and f_2 is increased. Results have also been compared to the analytical values where they have been available.

Example 1.

The domain over which the solution is sought in this example is the region common between two circles of equal radius, when their centers are $\sqrt{2}r$ apart. Under such a condition, the circles would be normal to each other where they intersect. Figure 6 shows the constructed surface obtained from the suggested numerical method for $k = 1$, when only 18 divisions are taken on each part of the bounding contour.

The position of the ridge for a different number of divisions is compared to that obtained from the analytical solution in Fig. 7. This figure indicates progressive convergence in the position of the ridge with an increase in the number of divisions. For this problem, the exact values of the area of the solution surface and the volume under it can be obtained analytically. These values are $0.80723r^2$ and $0.0646738r^3$ respectively. The numerically calculated values of the surface area and the volume have been compared to these exact answers in Fig. 8. The figure indicates convergence of the answer as the number of divisions is increased. As shown, there is not much improvement in the answer when the number of divisions exceeds 90.

Example 2.

The bounding contour curve in this example consists of two parts. f_1 is part of a circle of equation $y = \pm\sqrt{100-(40-x)^2}$. f_2 is part of another circle with equation $y = \pm\sqrt{40^2 - x^2}$. The intersection of these curves are points (38.75, ±9.9216). The surface representing the solution to the PDE is shown in Fig. 9. This has been obtained by the program at k =1 for 18 number of divisions on each side. The resulting volume and surface area obtained from the numerical calculations have been drawn vs. the number of divisions in Fig. 10 to demonstrate the convergence in the answer of the problem. This figure indicates rapid convergence in the answer as the numbers of divisions approach 70. Elevation of the points at the ridge for different numbers of divisions is shown in Fig. 11. The figure indicates convergence in the position of the ridge with an increase in the number of divisions.

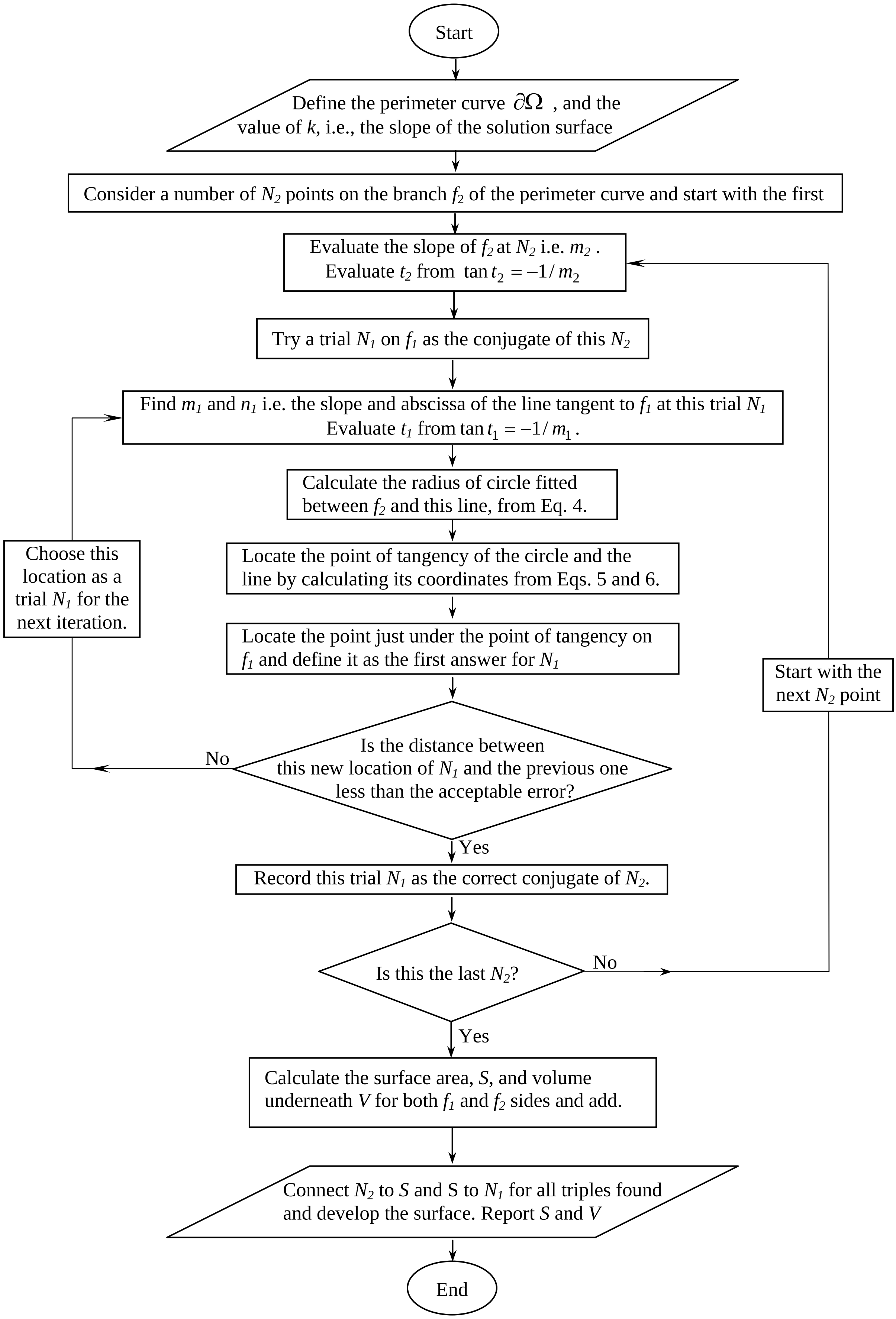

Fig. 5. Flow chart of the algorithm for finding the solution surface

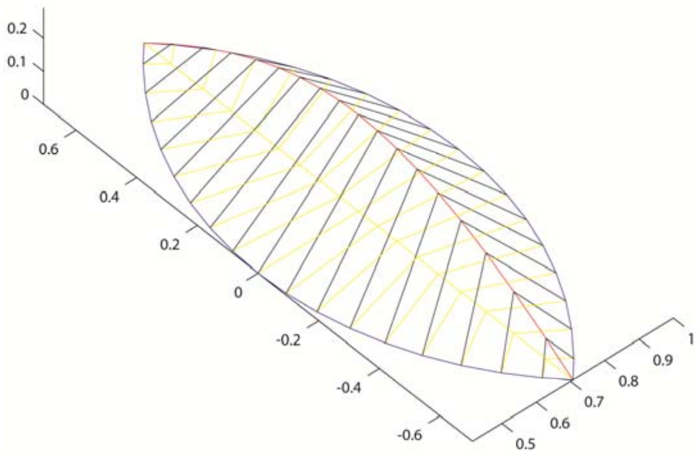

Fig. 6. The solution surface for Example 1

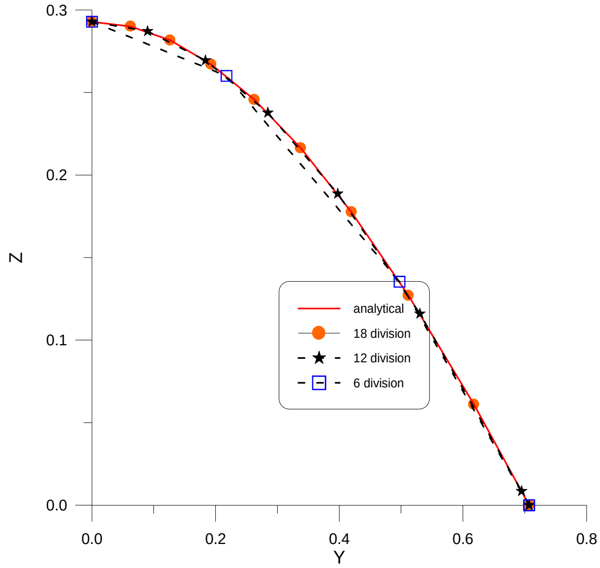

Fig. 7. Convergence in the position of the ridge for Example 1

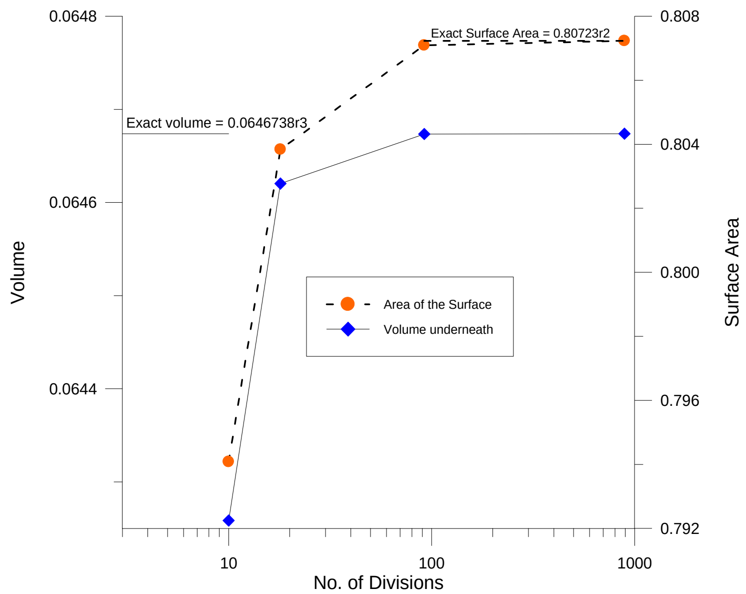

Fig. 8. Progressive convergence in the answer of Example 1

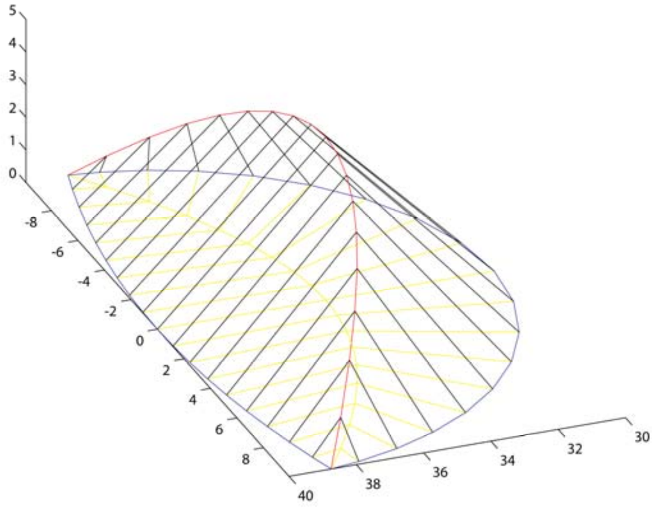

Fig. 9. The solution surface for Example 2

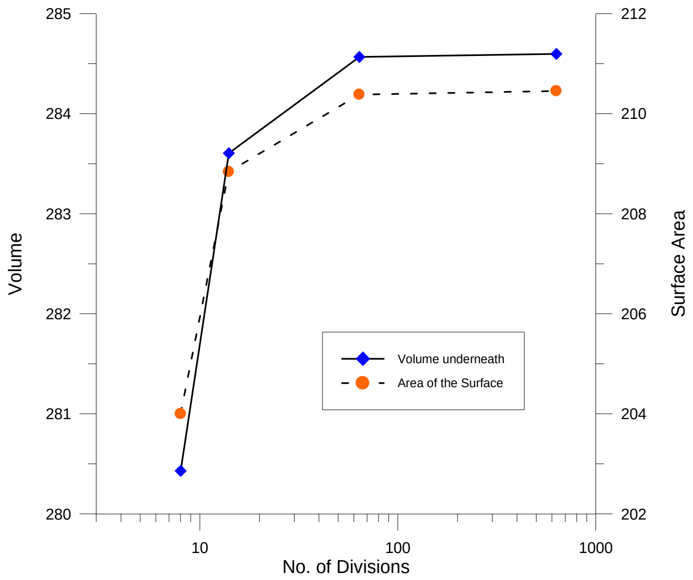

Fig. 10. Convergence in the answer of Example 2

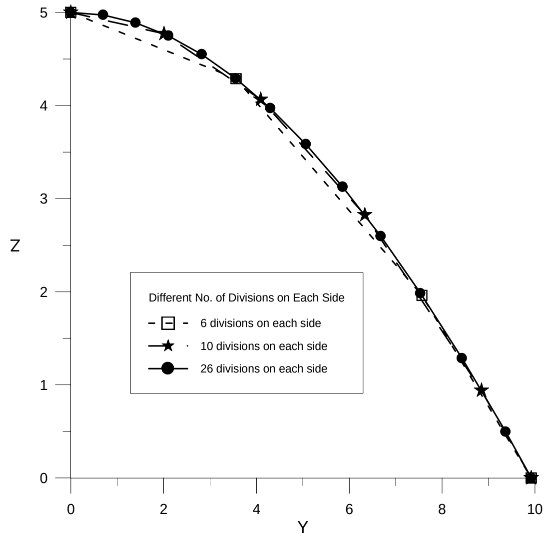

Fig. 11. Convergence in the position of the ridge for Example 2

Example 3.

Part of the bounding curve in this example has been taken concave to demonstrate the capability of the method in such a case. The domain over which the surface is constructed is the region outside the circle $y = \pm\sqrt{1^2 - x^2}$ and inside the ellipse $y = \pm\sqrt{0.75^2 - [0.75(x+0.5)]^2}$. The radius of the circle is 1 and its center is at the origin. The ellipse is a horizontal one whose center is at (-0.5, 0). The minor axis is vertical which measures 1.5. The major axis is horizontal, being 2 in length. Therefore, the ratio of minor to major axes is 0.75. The value of k is taken $1/\sqrt{3}$. This is equivalent to the slope of 30^o for the surface everywhere. Figure 12.a shows the solution surface for this case which looks like a desert sand dune, also called Barkhan. The top view of the solution surface has been shown in Fig. 12.b. The measure of the surface area and the volume underneath as a function of number of divisions taken on branches of the bounding curve has been shown in Fig. 13. The figure indicates rapid convergence in the answer as the number of divisions increases to 60.

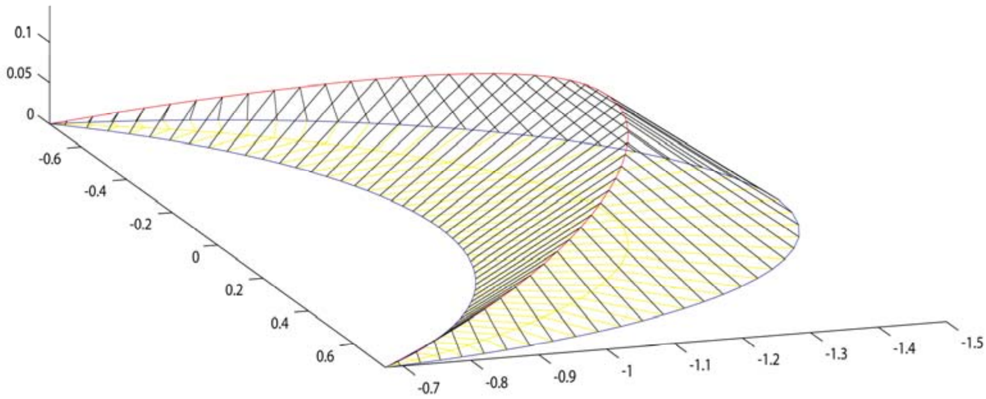

Fig. 12a The solution surface for Example 3

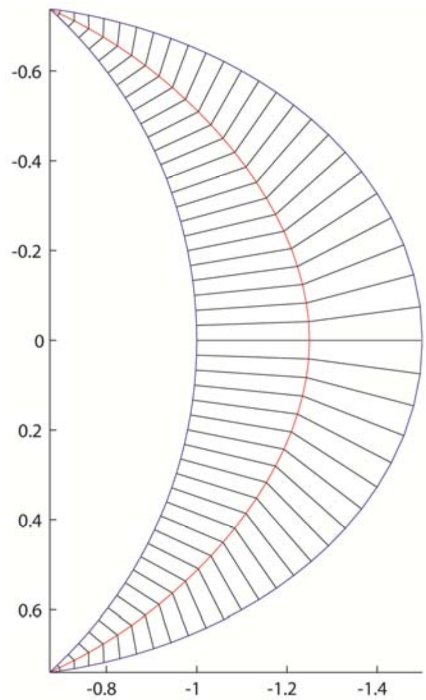

Fig. 12b Top view of the solution surface of Example 3

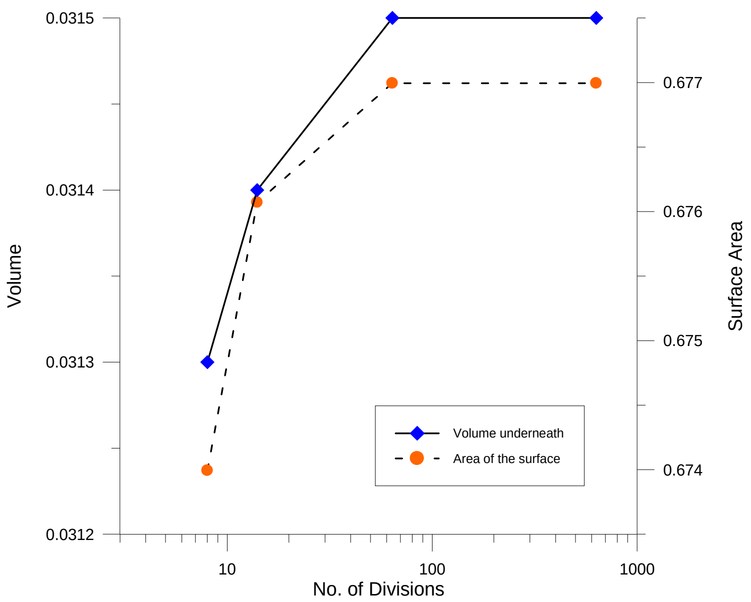

Fig. 13. Convergence in the answer of Example 3

The examples provided herein also indicate the capability of the suggested procedure in finding the characteristics of the solution surface as well (Figs. 6, 9, 12a and 12b). In this way, the essential implement in applying the method of characteristics in the solution of the boundary value problems involving eikonal equation becomes available.

7. CONCLUSION

In this paper, a new method for finding the solution surface of eikonal equation over a level curve was presented. In contrast to the previously developed methods for solving this non-linear equation, the presented geometric-based numerical procedure works on finding the characteristics of the solution surface rather than its level sets (contour curves). The method is capable of finding the surface without using any mesh in the domain. The ridge of the surface is also found simultaneously. The presented method works well in finding the area of the solution surface or the volume underneath, quantities that are difficult to be obtained by previous methods. It is concluded that the suggested procedure has a better applicability in problems involving eikonal equation where the information about the characteristics directions and the area of the solution surface or its underneath volume are essential.

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