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a - MINIMAL SETS AND THEIR PROPERTIES*

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Abstract – $a-$minimal sets' approach introduced some closed right ideals of the Ellis semigroup of a transformation semigroup which behave like minimal right ideals of an Ellis semigroup in some senses. From 1997 till now they have caused some new ideas in distality, proximal relation, transformed dimension, Here we will compare the above mentioned ideas and will improve them.

Keywords – Almost periodicity, $a-$minimal set, transformation group, transformation semigroup

1. PRELIMINARIES IN TRANSFORMATIONSEMIGROUPS

Let X be a compact Hausdorff topological space, S be a topological discrete semigroup with identity e and $\pi : X \times S \to X$ ($\pi(x,s) = xs$ $(\forall x \in X, \forall s \in S)$) be a continuous map such that for all $x \in X$ and for all $s,t \in S$, we have $xe = x$ and $x(st) = (xs)t$, then the triple (X,S,π) or simply (X,S) is called a transformation semigroup. In a transformation semigroup (X,S) we have the following definitions:
1. For each $s \in S$, define the continuous map $\pi^s : X \to X$ by $x\pi^s = xs$ ($\forall x \in X$), then $\mathrm{E}(X,S)$ or simply $\mathrm{E}(X)$ is the closure of $\{\pi^s \mid s \in S\}$ in X^X with pointwise convergence. Moreover, it is called the enveloping semigroup (or Ellis semigroup) of (X,S). $\mathrm{E}(X)$ has a semigroup structure (see [1] and [2]). A nonempty subset K of $\mathrm{E}(X)$ is called a right ideal if $K\mathrm{E}(X) \subseteq K$, and it is called a minimal right ideal if none of the right ideals of $\mathrm{E}(X)$ is a proper subset of K. The set of all minimal right ideals of $\mathrm{E}(X)$ will be denoted by $\mathrm{Min}(\mathrm{E}(X))$.
2. A nonempty subset Z of X is called invariant if $ZS \subseteq Z$. Moreover, it is called minimal if it is closed and none of the closed invariant subsets of X is a proper subset of Z. The element $a \in X$ is called almost periodic if $a\mathrm{E}(X)$ is a minimal subset of X.
3. Let $a \in X$, A be a nonempty subset of X, $p \in \mathrm{E}(X)$, C be a nonempty subset of $\mathrm{E}(X)$, and K be a right ideal of $\mathrm{E}(X)$, then $\mathrm{L}_p : \mathrm{E}(X) \to \mathrm{E}(X)$ such that $\mathrm{L}_p(q) = pq$ $(\forall q \in \mathrm{E}(X))$ is a continuous map. The following sets are introduced:

$$\mathrm{B}(K) = \{p \in K \mid \mathrm{L}_p : K \to K \text{ is bijective}\}, \quad \mathrm{F}(a,C) = \{p \in C : ap = a\},$$

$$\mathrm{S}(K) = \{p \in K \mid \mathrm{L}_p : K \to K \text{ is surjective}\}, \quad \mathrm{F}(A,C) = \bigcap_{b \in A} \mathrm{F}(b,C),$$

$$\mathrm{I}(K) = \{p \in K \mid \mathrm{L}_p : K \to K \text{ is injective}\}, \quad \mathrm{J}(C) = \{p \in K \mid p^2 = p\}.$$

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2. A BRIEF SURVEY

2.1. a - minimal sets: introduction and their idempotents

$a-$minimal sets have been introduced in [3]. From some points of view they behave very similar to a minimal right ideal of an Ellis semigroup; these similarities lead us to other collections of a closed right ideal of Ellis semigroup, i.e., $\underline{A-\text{minimal}}$ sets and $\underline{\underline{A-\text{minimal}}}$ sets which have been introduced in [4].

In the transformation semigroup (X,S), let $a\in X$, A be a nonempty subset of X and K be a closed right ideal of $\mathrm{E}(X)$, then:

- K is called an $a-$minimal set if:

  o $aK = a\mathrm{E}(X)$,

  o K does not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X)$ with $aL = a\mathrm{E}(X)$,

the set of all $a-$minimal sets is denoted by $\mathrm{M}(a)$ and it is nonempty [3, Proposition 3],

- K is called an $\underline{A-\text{minimal}}$ set if:

  o $\forall b\in A \quad bK = b\mathrm{E}(X)$,

  o K does not have any proper subset like L, such that L be a closed right ideal of $\mathrm{E}(X)$ with $bL = b\mathrm{E}(X)$ for all $b\in A$,

the set of all $\underline{A-\text{minimal}}$ sets is denoted by $\overline{\mathrm{M}}(A)$ and it is nonempty [4, Theorem 2],

- K is called an $\underline{\underline{A-\text{minimal}}}$ set if:

  o $AK = A\mathrm{E}(X)$,

  o K does not have any proper subset like L, such that L be a closed right ideal of $\mathrm{E}(X)$ with $AL = A\mathrm{E}(X)$, the set of all $\underline{\underline{A-\text{minimal}}}$ sets is denoted by $\overline{\overline{\mathrm{M}}}(A)$, there are examples in which $\overline{\overline{\mathrm{M}}}(A)$ is empty and the others in which $\overline{\overline{\mathrm{M}}}(A)$ is nonempty [5].

- The following sets are introduced:

  o $\overline{\mathcal{M}}(X,S) = \{A\subseteq X \mid A\neq\varnothing \wedge (\forall K\in\overline{\mathrm{M}}(A)\ \mathrm{J}(\mathrm{F}(A,K))\neq\varnothing)\}$,

  o $\overline{\overline{\mathcal{M}}}(X,S) = \{A\subseteq X \mid A\neq\varnothing \wedge \overline{\overline{\mathrm{M}}}(A)\neq\varnothing \wedge (\forall K\in\overline{\overline{\mathrm{M}}}(A)\ \mathrm{J}(\overline{\mathrm{F}}(A,K))\neq\varnothing)\}$.

In the transformation semigroup (X,S), let $a\in A\subseteq X$, then we have:

1. let K be a closed right ideal of $\mathrm{E}(X)$, $I\in\overline{\mathrm{M}}(A)$ and $J\in\overline{\overline{\mathrm{M}}}(A)$ ($\overline{\overline{\mathrm{M}}}(A)$ may be empty in which case the last item will be disregarded), then we have: In the following table, the mark ✓ indicates that for the corresponding case $\pi(\mathrm{Q})$ is true, where

- α is: (If $\mathrm{Q}\neq\varnothing$ then Q is a subsemigroup of C),
- β is: ($\forall u,v\in\mathrm{J}(C)\quad (u$ is a left identity of $C \wedge (u$ is the identity of $Cu) \wedge (Cu\cong Cv)$),
- γ is: (($\forall u\in\mathrm{J}(\mathrm{Q})\quad (\mathrm{Q}u$ is a group with identity $u)) \wedge (\{\mathrm{Q}v \mid v\in\mathrm{J}(\mathrm{Q})\}$ is a partition of Q into some of its disjoint isomorphic subgroups$) \wedge \mathrm{card}(\{\mathrm{Q}v \mid v\in\mathrm{J}(\mathrm{Q})\}) = \mathrm{card}(\mathrm{J}(\mathrm{Q}))$),

([4, Theorem 4], similar to [1, Proposition 3.5]).

$\pi(Q)$	Q / C	$F(A,C)$	$\overline{F}(A,C)$	$B(C)$	$S(C)$	$I(C)$
α	K or I or J	✓	✓	✓	✓	✓
β	K			✓	✓	✓
β	I or J	✓	✓	✓	✓	✓
γ	K			✓		
γ	I	✓		✓		
γ	J	✓	✓	✓		

2. For each $K, L \in \overline{M}(A)$, we have:

i. $\forall p \in F(A,K) \quad pL = K$,
ii. $\forall u \in J(F(A,K)) \quad \exists! v \in J(F(A,L)) \quad uv = u \wedge vu = v$,
iii. $\forall u \in J(F(A,K)) \quad \exists! v \in J(F(A,L)) \quad uv = u$,
iv. $\forall u \in J(F(A,K)) \quad \mathrm{card}(J((L_u|_L)^{-1}(u))) = 1$,
v. $\mathrm{card}(J(F(A,K))) = \mathrm{card}(J(F(A,L)))$,
vi. $\mathrm{card}(\overline{M}(A))\mathrm{card}(J(F(A,K))) = \mathrm{card}(\bigcup_{N\in\overline{M}(A)} J(F(A,N)))$.

And for each $K, L \in \overline{\overline{M}}(A)$:

$$(\forall p \in \overline{F}(A,K) \quad pL = K) \wedge \mathrm{card}(\overline{\overline{M}}(A))\mathrm{card}(J(F(A,K))) = \mathrm{card}(\bigcup_{N\in\overline{\overline{M}}(A)} J(F(A,N))),$$

and the same as (ii), (iii), (iv), and (v) in the above mentioned items ([4, Theorem 7], similar to [1, Proposition 3.6]).

3. One of the best similarities is: "for each $b \in A$, b is almost periodic", iff "$\overline{M}(A) = \mathrm{Min}(E(X))$", iff "$\overline{M}(A) \cap \mathrm{Min}(E(X)) \neq \varnothing$", iff "$\overline{\overline{M}}(A) = \mathrm{Min}(E(X))$", iff "$\overline{\overline{M}}(A) \cap \mathrm{Min}(E(X)) \neq \varnothing$" [4, Note 12].

2.2. Distality and proximal relations in a - minimal sets

In the transformation semigroup (X,S), let $d \in A \subseteq X$. (X,S) is called distal if $E(X) \in \mathrm{Min}(E(X))$, (X,S) is called $d-$distal if $E(X) \in M(d)$, and for $Q \in \{\overline{M}, \overline{\overline{M}}\}$, (X,S) is called $A^{\underline{(Q)}}$ distal if $E(X) \in Q(A)$. Sets:

$$P(X,S) = \{(x,y) \in X \times X \mid \exists I \in \mathrm{Min}(E(X)) \quad \forall p \in I \quad xp = yp\}$$

$$P_d(X,S) = \{(x,y) \in X \times X \mid \exists I \in M(d) \quad \forall p \in I \quad xp = yp\}$$

$$\overline{P}_A(X,S) = \{(x,y) \in X \times X \mid \exists I \in \overline{M}(A) \quad \forall p \in I \quad xp = yp\}$$

$$\overline{\overline{P}}_A(X,S) = \{(x,y) \in X \times X \mid \exists I \in \overline{\overline{M}}(A) \quad \forall p \in I \quad xp = yp\}$$

are called respectively proximal relation, $d-$proximal relation, $A^{\underline{(\overline{M})}}$ proximal relation, and $A^{\underline{(\overline{\overline{M}})}}$ proximal relation (on X). Suppose $n \in \mathbf{N}$, we have:

1. It is well-known that "(X,S) is distal", iff "$\mathrm{Min}(E(X)) = \{E(X)\}$" iff "$P(X,S) = \Delta_X$" (see [1, Proposition 5.3, Lemma 5.12] also iff "$\forall x \in X \quad (X,S)$ is $x-$distal" iff "$\exists x \in X \quad ((x$ is almost periodic$) \wedge ((X,S)$ is $x-$distal$))$", (in these cases $E(X)$ is a group). On the other hand, ([4, Theorem 18] and [6, Theorem 4]):

a. if $A^n \in \overline{\mathcal{M}}(X^n, S^n)$, then "$(X,S)$ is $A^{\underline{(\overline{\mathrm{M}})}}$ distal", iff "$\overline{\mathrm{M}}(A) = \{\mathrm{E}(X)\}$", iff "$\mathrm{F}(A, \mathrm{E}(X))$ is a subgroup of $\mathrm{E}(X)$", iff "$\mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))$ is a subgroup of $\mathrm{E}(X)$", iff "$\mathrm{J}(\mathrm{F}(A, \mathrm{E}(X))) = \{e\}$", iff "$(X^n, S^n)$ is $A^{n\,\underline{(\mathrm{M})}}$ distal", iff "$\overline{\mathrm{P}}_A(X,S) = \Delta_X$";
(in these cases $\mathrm{F}(A, \mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \mathrm{F}(A, \mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \overline{\mathrm{F}}(A, \mathrm{E}(X))$ and $\mathrm{S}(\mathrm{E}(X)) \cap \mathrm{F}(A, \mathrm{E}(X))$ are subgroups of $\mathrm{E}(X)$);
b. if $A \in \overline{\overline{\mathcal{M}}}(X,S)$, then "$(X,S)$ is $A^{\underline{(\overline{\overline{\mathrm{M}}})}}$ distal", iff "$\overline{\overline{\mathrm{M}}}(A) = \{\mathrm{E}(X)\}$", iff "$\overline{\mathrm{F}}(A, \mathrm{E}(X))$ is a subgroup of $\mathrm{E}(X)$", iff "$\mathrm{F}(A, \mathrm{E}(X))$ is a subgroup of $\mathrm{E}(X)$", iff "$\mathrm{J}(\mathrm{F}(A, \mathrm{E}(X)))(= \mathrm{J}(\overline{\mathrm{F}}(A, \mathrm{E}(X))))$ is a subgroup of $\mathrm{E}(X)$", iff "$\mathrm{J}(\mathrm{F}(A, \mathrm{E}(X))) = \{e\}$", iff "$\overline{\overline{\mathrm{P}}}_A(X,S) = \Delta_X$";
(in these cases $\mathrm{F}(A, \mathrm{E}(X))$, $\overline{\mathrm{F}}(A, \mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \mathrm{F}(A, \mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \overline{\mathrm{F}}(A, \mathrm{E}(X))$, $\mathrm{S}(\mathrm{E}(X)) \cap \mathrm{F}(A, \mathrm{E}(X))$ and $\mathrm{S}(\mathrm{E}(X)) \cap \overline{\mathrm{F}}(A, \mathrm{E}(X))$ are subgroups of $\mathrm{E}(X)$;
(Note the fact that $\{d\} \in \overline{\overline{\mathcal{M}}}(X,S) \subseteq \overline{\mathcal{M}}(X,S)$ (if $A \in \overline{\overline{\mathcal{M}}}(X,S)$, then $\overline{\overline{\mathrm{M}}}(A) \subseteq \overline{\mathrm{M}}(A)$)).
2. It is well-known that "$\mathrm{Min}(\mathrm{E}(X))$ is singleton", iff "$\mathrm{P}(X)$ is a transitive relation on X", iff "$\mathrm{P}(X)$ is an equivalence relation on X" (see [1, Proposition 5.16]). On the other hand, we have [6, Theorem 5]:
c. if $A \in \overline{\mathcal{M}}(X,S)$, then "$\overline{\mathrm{M}}(A)$ is singleton", iff "$\overline{\mathrm{P}}_A(X)$ is a transitive relation on X", iff "$\overline{\mathrm{P}}_A(X)$ is an equivalence relation on X";
d. if $A \in \overline{\overline{\mathcal{M}}}(X,S)$, then "$\overline{\overline{\mathrm{M}}}(A)$ is singleton", iff "$\overline{\overline{\mathrm{P}}}_A(X)$ is a transitive relation on X", iff "$\overline{\overline{\mathrm{P}}}_A(X)$ is an equivalence relation on X".
3. Let S be abelian. It is well-known that if "$\mathrm{P}(X)$ is a closed relation on X" then "$\mathrm{P}(X)$ is an equivalence relation on X" (see [1, Lemma 5.18]). On the other hand, we have [6, Theorem 5]:
e. if $A \in \overline{\mathcal{M}}(X,S)$ and $\overline{\mathrm{P}}_A(X)$ is a closed relation on X, then $\overline{\mathrm{P}}_A(X)$ is an equivalence relation on X,
f. if $A \in \overline{\overline{\mathcal{M}}}(X,S)$ and $\overline{\overline{\mathrm{P}}}_A(X)$ is a closed relation on X, then $\overline{\overline{\mathrm{P}}}_A(X)$ is an equivalence relation on X.

2.3. a - minimal sets and product spaces

Let (X,S), (X_1,S_1), ..., (X_n,S_n) be transformation semigroups and $\{(X_\alpha, S_\alpha) \mid \alpha \in \Gamma\}$ be a nonempty collection of transformation semigroups.

1. For $(a_1,\dots,a_n) \in \prod_{1\le i\le n} X_i$ we have $\mathrm{M}_{(\prod_{i=1}^n X_i, \prod_{i=1}^n S_i)}(a_1,\dots,a_n) = \prod_{i=1}^n \mathrm{M}_{(X_i,S_i)}(a_i)$ $(= \{\prod_{i=1}^n K_i \mid \forall i \in \{1,\dots,n\}\ K_i \in \mathrm{M}_{(X_i,S_i)}(a_i)\})$ ([7, Theorem 5]), as a matter of fact, if $\prod_{\alpha\in\Gamma} A_\alpha \in \overline{\mathcal{M}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$, then $\overline{\mathrm{M}}_{(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)}(\prod_{\alpha\in\Gamma} A_\alpha) = \prod_{\alpha\in\Gamma} \overline{\mathrm{M}}_{(X_\alpha,S_\alpha)}(A_\alpha)$. Note to the fact that $\{\{x\} \mid x \in X\} \subseteq \overline{\mathcal{M}}(X,S)$, and conclude $\mathrm{Min}(\mathrm{E}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)) = \prod_{\alpha\in\Gamma} \mathrm{Min}(\mathrm{E}(X_\alpha, S_\alpha))$.
2. $\mathrm{M}_{(X^\Gamma,S)}(\hat{a}) = \{\Delta_{K^\Gamma} \mid K \in \mathrm{M}_{(X,S)}(a)\} = \{\{(p)_{\alpha\in\Gamma} \mid p \in K\} \mid K \in \mathrm{M}_{(X,S)}(a)\}$ and $\mathrm{M}_{(X,S)}(a) = \pi_\beta(\mathrm{M}_{(X^\Gamma,S)}(\hat{a}))$, $(\forall \beta \in \Gamma)$ (where π_β is the projection map on β's component and $\hat{a} = (a)_{\alpha\in\Gamma} \in X^\Gamma$) ([7, Theorem 5]).

2.4. Other notes

The reader may find new generalizations about almost periodicity related to $a-$minimal sets in [4], related topics between bitransformation semigroups and $a-$minimal sets in [8], transformed dimensions (in $a-$minimal sets) in [9], interesting examples and counterexamples in [5], and more related topics in [10-15]. The reader may find interesting theorems in the above mentioned articles which have been disregarded in this note for briefness, e.g., in [10] it has been proved that in the transformation semigroup (X,S) if $S_1,\dots,S_n$ are subsemigroups of S such that $e \in \bigcap_{1\le i\le n} S_i$, then (X,S) is distal (resp. $a-$distal) iff for each $i \in \{1,\dots,n\}$, (X,S_i) is distal (resp. $a-$distal).

3. AN IMPROVEMENT: SOME EQUIVALENCE RELATIONS AND a - MINEMAL SETS

In this section some simple equivalence relations based on $a-$minimal sets' definition are found, which help the reader to feel much closer to $a-$minimal sets' approach.

Definition 1.
In the transformation semigroup (X,S), for nonempty subsets A and B of X, if $\mathrm{D}\in\{\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}}\}$, define:

- A and B are equivalent if $\{\mathrm{M}(a):a\in A\}=\{\mathrm{M}(b):b\in B\}$.
- A and B are $\mathrm{D}-$equivalent if $\mathrm{D}(A)=\mathrm{D}(B)$.

Moreover:

- A is called subq-independent (or subq$\xrightarrow{(-)}$ independent) if there is not any nonempty proper subset C of A such that A and C are equivalent,
- A is called subq$\xrightarrow{(\mathrm{D})}$ independent if there is not any nonempty proper subset C of A such that A and C are $\mathrm{D}-$equivalent,
- A is called supq-independent (or supq$\xrightarrow{(-)}$ independent) if there is not any proper supset C of A such that A and C are equivalent,
- A is called supq$\xrightarrow{(\mathrm{D})}$ independent if there is not any proper supset C of A such that A and C are $\mathrm{D}-$equivalent.

Remark 2.

- In the transformation semigroup (X,S), "equivalent", "$\overline{\mathrm{M}}-$equivalent", and "$\overline{\overline{\mathrm{M}}}-$equivalent" relations are equivalence relations on the collection of nonempty subsets of X.
- The "equivalent", "$\overline{\mathrm{M}}-$equivalent", and "$\overline{\overline{\mathrm{M}}}-$equivalent" relations are invariable under isomorphisms of transformation semigroups (a continuous function $\varphi:(X,S)\to(Y,S)$ is called homomorphism if for each $x\in X$ and each $s\in S$ we have $\varphi(xs)=\varphi(x)s$, and if in addition it is onto and 1-1, then it is called an isomorphism).

Theorem 3.
In the transformation semigroup (X,S), let $a\in A\subseteq X$ and $b\in B\subseteq X$.

1. If for each $c\in A$, c and a are equivalent (i.e., $\{c\}$ and $\{a\}$ are equivalent), then for each nonempty subset C of A, A and C are equivalent and $\overline{\mathrm{M}}-$equivalent.
2. a and b are equivalent if and only if a is $b-$almost periodic and b is $a-$almost periodic.
3. A and B are $\overline{\mathrm{M}}-$equivalent if and only if A is $B\xrightarrow{(\overline{\mathrm{M}},\overline{\mathrm{M}})}$ almost periodic and B is $A\xrightarrow{(\overline{\mathrm{M}},\overline{\mathrm{M}})}$ almost periodic.
4. A and B are $\overline{\overline{\mathrm{M}}}-$equivalent if and only if A is $B\xrightarrow{(\overline{\overline{\mathrm{M}}},\overline{\overline{\mathrm{M}}})}$ almost periodic and B is $A\xrightarrow{(\overline{\overline{\mathrm{M}}},\overline{\overline{\mathrm{M}}})}$ almost periodic or $\overline{\overline{\mathrm{M}}}(A)=\overline{\overline{\mathrm{M}}}(B)=\varnothing$.

(**Remark:** Let $\mathrm{Q},\mathrm{R}\in\{\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}}\}$ and A, B be nonempty subsets of X, such that whenever $\mathrm{R}=\overline{\overline{\mathrm{M}}}$, then $\overline{\overline{\mathrm{M}}}(A)\neq\varnothing$, and let $a,b\in X$. We say (see [4, Definition 13]):

- b is $a-$almost periodic if:

$$\forall K\in\mathrm{M}(a)\quad \exists L\in\mathrm{M}(b)\quad L\subseteq K,$$

- B is $A\xrightarrow{(\mathrm{Q},\mathrm{R})}$ almost periodic if:

$$\forall K\in\mathrm{R}(A)\quad \exists L\in\mathrm{Q}(B)\quad L\subseteq K.)$$

Proof: In the transformation semigroup (X,S), let $a\in A\subseteq X$ and $b\in B\subseteq X$.

1. For each each nonempty subset C of A we have $\overline{\mathrm{M}}(C) = \mathrm{M}(a)$.
2. Suppose a be $b-$almost periodic and b be $a-$almost periodic. If $K \in \mathrm{M}(a)$, there exists $L \in \mathrm{M}(b)$ and $K' \in \mathrm{M}(a)$ such that $L \subseteq K$ and $K' \subseteq L$, so if $K' \subseteq K$ by $K, K' \in \mathrm{M}(a)$ we have $K = K'$ and $K = L \in \mathrm{M}(b)$; therefore $\mathrm{M}(a) \subseteq \mathrm{M}(b)$, by a similar method $\mathrm{M}(b) \subseteq \mathrm{M}(a)$, thus a and b are equivalent.

- For (3) and (4), use a similar method described for (2).

Note 4.

In the transformation semigroup (X, S), let $a \in A \subseteq X$.

- o $\{b \in X \mid \mathrm{M}(b) = \mathrm{M}(a)\}$ is supq-independent and the maximum element of $(\{B \subseteq X \mid B \neq \varnothing \wedge (a$ and B are equivalent$)\}, \subseteq)$.
- o $\bigcup\{B \subseteq X \mid B \neq \varnothing \wedge (A$ and B are equivalent$)\}(= \{b \in X \mid \exists c \in A \quad \mathrm{M}(b) = \mathrm{M}(c)\})$ is supq-independent and the maximum element of $(\{B \subseteq X \mid B \neq \varnothing \wedge (A$ and B are equivalent$)\}, \subseteq)$.
- o $\bigcup\{B \subseteq X \mid B \neq \varnothing \wedge (A$ and B are $\overline{\mathrm{M}}-$equivalent$)\}$ is supq$\frac{(\overline{\mathrm{M}})}{}$independent and the maximum element of $(\{B \subseteq X \mid B \neq \varnothing \wedge (A$ and B are $\overline{\mathrm{M}}-$equivalent$)\}, \subseteq)$.

- The following sets are directed:

$$(\{B \subseteq X \mid B \neq \varnothing \wedge (A \text{ and } B \text{ are equivalent})\}, \subseteq),$$

$$(\{B \subseteq X \mid B \neq \varnothing \wedge (A \text{ and } B \text{ are } \overline{\mathrm{M}}-\text{equivalent})\}, \subseteq).$$

(Note: Using [9, Theorem 12], for each nonempty subset B and C of X, we have $\overline{\mathrm{M}}(B \cup C) = \min(\{K_1 \cup K_2 \mid K_1 \in \overline{\mathrm{M}}(B), K_2 \in \overline{\mathrm{M}}(C)\}, \subseteq)$, which leads us to the desired result.)

- For $A \subseteq C \subseteq B \subseteq X$, we have:
- o if A and B are equivalent, then A and C are equivalent,
- o if A and B are $\overline{\mathrm{M}}-$equivalent, then A and C are $\overline{\mathrm{M}}-$equivalent.

Theorem 5.

In the transformation semigroup (X, S), let A and B be nonempty subsets of X.

A. If A is the set of all almost periodic points, then:

- For each subq-independent subset B of X, $\mathrm{card}(A \cap B) \leq 1$.
- For each subq$\frac{(\overline{\mathrm{M}})}{}$independent subset B of X, $\mathrm{card}(A \cap B) \leq 1$ (moreover, in this case $A \cap B \in \{B, \varnothing\}$).
- For each subq$\frac{(\overline{\overline{\mathrm{M}}})}{}$independent subset B of X, $\mathrm{card}(A \cap B) \leq 1$ (moreover, in this case $A \cap B \in \{B, \varnothing\}$).
- A is supq-independent, supq$\frac{(\overline{\mathrm{M}})}{}$independent and supq$\frac{(\overline{\overline{\mathrm{M}}})}{}$independent.

B. If for each $a \in A$, a is almost periodic, then the following statements are pairwise equivalent:

1. A and B are equivalent.
2. A and B are $\overline{\overline{\mathrm{M}}}-$equivalent.
3. A and B are $\overline{\mathrm{M}}-$equivalent.
4. for each $c \in B$, c is almost periodic.

C. The following statements are pairwise equivalent:

1. For each $a \in A$, a is almost periodic.
2. For each nonempty subset B of X, B and $A \cup B$ are $\overline{\mathrm{M}}-$quivalent.
3. For each supq$\frac{(\overline{\overline{\mathrm{M}}})}{}$independent subset B of X, $A \subseteq B$.
4. For each supq$\frac{(\overline{\mathrm{M}})}{}$independent subset B of X, $A \subseteq B$.

Proof:

First note:

- A is supq-independent if and only if for each $d \in X - A$, $\mathrm{M}(d) \notin \{\mathrm{M}(c) \mid c \in A\}$.
- Due to [4, Note 12] the statements "for each $a \in A$, a is almost periodic", "for each $a \in A$, $\mathrm{M}(a) = \mathrm{Min}(\mathrm{E}(X))$", "$\overline{\mathrm{M}}(A) = \mathrm{Min}(\mathrm{E}(X))$", and "$\overline{\overline{\mathrm{M}}}(A) = \mathrm{Min}(\mathrm{E}(X))$" are pairwise equivalent.

Now we have:

A. Let A be the set of all almost periodic points of (X, S).

Suppose B be a nonempty subset of X and $a, b \in A \cap B$ be such that $a \neq b$, then B and $B - \{b\}$ are equivalent, since $a \in B - \{b\}$ and $\mathrm{M}(b) = \mathrm{M}(a) = \mathrm{Min}(\mathrm{E}(X))$; thus B is not subq-independent. Moreover, if $b \in A \cap B$ and $B - \{b\} \neq \varnothing$, then $\overline{\mathrm{M}}(B) = \overline{\mathrm{M}}(B - \{b\})$ and $\overline{\overline{\mathrm{M}}}(B) = \overline{\overline{\mathrm{M}}}(B - \{b\})$, therefore B is not subq$\overline{(\mathrm{M})}$ independent and B is not subq$\overline{\overline{(\mathrm{M})}}$ independent.

C.

- $(1) \Rightarrow (2)$: By [9, Theorem 12] we have $\overline{\mathrm{M}}(A \cup B) = \min(\{K_1 \cup K_2 \mid K_1 \in \overline{\mathrm{M}}(A), K_2 \in \overline{\mathrm{M}}(B)\}, \subseteq)$. If for each $a \in A$, a is almost periodic, then $\overline{\mathrm{M}}(A) = \mathrm{Min}(\mathrm{E}(X))$ and

$$\overline{\mathrm{M}}(A \cup B) = \min(\{K_1 \cup K_2 \mid K_1 \in \mathrm{Min}(\mathrm{E}(X)), K_2 \in \overline{\mathrm{M}}(B)\}, \subseteq) = \overline{\mathrm{M}}(B)$$

(since each closed right ideal of $\mathrm{E}(X)$ contains a minimal right ideal of $\mathrm{E}(X)$).

- $(2) \Rightarrow (3)$: It is clear.
- $(3) \Rightarrow (4)$: The set of all almost periodic points is supq$\overline{(\mathrm{M})}$ independent, so all of the points of A are almost periodic and for each nonempty subset B of X, $\overline{\overline{\mathrm{M}}}(B) = \overline{\overline{\mathrm{M}}}(A \cap B)$.
- $(4) \Rightarrow (1)$: The set of all almost periodic points is supq$\overline{(\mathrm{M})}$ independent, so all of the points of A are almost periodic.

Corollary 6.

In the transformation semigroup (X, S), let A be a nonempty subset of X and $\mathrm{D} \in \{\overline{\mathrm{M}}, \overline{\overline{\mathrm{M}}}\}$.

- If for each nonempty subset B of X we have: "A and B are equivalent if and only if $A = B$", then:

$$\forall a \in A \quad \forall x \in X \quad (\mathrm{M}(a) = \mathrm{M}(x) \Rightarrow a = x).$$

- If for each nonempty subset B of X we have: "A and B are $\mathrm{D}-$equivalent if and only if $A = B$", then A is a singleton set which contains the unique almost periodic point of X.

Theorem 7.

Let $\{(X_\alpha, S_\alpha) \mid \alpha \in \Gamma\}$ be a nonempty collection of transformation semigroups and for each $\alpha \in \Gamma$, A_α and B_α be nonempty subsets of X_α, then in the transformation semigroup $(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$ (where for each $(x_\alpha)_{\alpha\in\Gamma} \in \prod_{\alpha\in\Gamma} X_\alpha$ and $(s_\alpha)_{\alpha\in\Gamma} \in \prod_{\alpha\in\Gamma} S_\alpha$, $(x_\alpha)_{\alpha\in\Gamma}(s_\alpha)_{\alpha\in\Gamma} := (x_\alpha s_\alpha)_{\alpha\in\Gamma}$) we have:

- If W is one of the terms: "equivalent", "$\overline{\mathrm{M}}-$equivalent (under the assumption $\prod_{\alpha\in\Gamma} A_\alpha, \prod_{\alpha\in\Gamma} B_\alpha \in \overline{\mathcal{M}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$)", and "$\overline{\overline{\mathrm{M}}}-$equivalent (under the assumption $\prod_{\alpha\in\Gamma} A_\alpha, \prod_{\alpha\in\Gamma} B_\alpha \in \overline{\overline{\mathcal{M}}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$)", then we have:

$$\text{"}\prod_{\alpha\in\Gamma} A_\alpha \text{ and } \prod_{\alpha\in\Gamma} B_\alpha \text{ are W" if and only if}$$

$$\text{"for each } \alpha \in \Gamma,\ A_\alpha \text{ and } B_\alpha \text{ are W".}$$

- If W_0 is one of the signs: "-", "$\overline{\mathrm{M}}$ (under the assumption $\prod_{\alpha\in\Gamma} A_\alpha \in \overline{\mathcal{M}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$)", "$\overline{\overline{\mathrm{M}}}$ (under the assumption $\prod_{\alpha\in\Gamma} A_\alpha \in \overline{\overline{\mathcal{M}}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$)", then we have:

"$\prod_{\alpha\in\Gamma} A_\alpha$ is supq$\underline{(\mathcal{W}_0)}$ independent" if and only if

"for each $\alpha \in \Gamma$, A_α is supq$\underline{(\mathcal{W}_0)}$ independent".

- If $\mathrm{W}_0 \in \{-, \overline{\mathrm{M}}, \overline{\overline{\mathrm{M}}}\}$, then we have:

If "$\prod_{\alpha\in\Gamma} A_\alpha$ is supq$\underline{(\mathcal{W}_0)}$ independent", then

"for each $\alpha \in \Gamma$, A_α is supq$\underline{(\mathcal{W}_0)}$ independent".

- If W is one of the terms: "equivalent", "$\overline{\mathrm{M}}-$ equivalent (under the assumption $C \in \overline{\mathcal{M}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$)", and "$\overline{\overline{\mathrm{M}}}-$ equivalent (under the assumption $C \in \overline{\overline{\mathcal{M}}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$)", then: "$C$ and $\prod_{\alpha\in\Gamma} \pi_\alpha(C)$ (where π_α is the projection map on the α's coordinate) are W".
- If W_0 is one of the signs: "-", $\overline{\mathrm{M}}$ (under the assumption $C \in \overline{\mathcal{M}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$)", "$\overline{\overline{\mathrm{M}}}$ (under the assumption $C \in \overline{\overline{\mathcal{M}}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$", then we have:

If "C is supq$\underline{(\mathcal{W}_0)}$ independent", then "$C = \prod_{\alpha\in\Gamma} \pi_\alpha(C)$".

Proof:

Use the following notes:

- if $\prod_{\alpha\in\Gamma} A_\alpha \in \overline{\mathcal{M}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$, then $\overline{\mathrm{M}}_{(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)}(\prod_{\alpha\in\Gamma} A_\alpha) = \{\prod_{\alpha\in\Gamma} K_\alpha \mid \forall\alpha\in\Gamma \ \ K_\alpha \in \overline{\mathrm{M}}_{(X_\alpha, S_\alpha)}(A_\alpha)\}$,
- if $\prod_{\alpha\in\Gamma} A_\alpha \in \overline{\overline{\mathcal{M}}}(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$, then $\overline{\overline{\mathrm{M}}}_{(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)}(\prod_{\alpha\in\Gamma} A_\alpha) = \{\prod_{\alpha\in\Gamma} K_\alpha \mid \forall\alpha\in\Gamma \ \ K_\alpha \in \overline{\overline{\mathrm{M}}}_{(X_\alpha, S_\alpha)}(A_\alpha)\}$.

Theorem 8.

In the transformation semigroup (X,S), let $\pi : X \to \frac{X}{\Re}$ be the projection map (where $\Re = \{(a,b) \mid a$ and b are equivalent$\}$), then we have:

1. $\frac{X}{\Re}$ is a singleton set if and only if all of the points of X are almost periodic.
2. If, in addition, S is an abelian group, $a \in X$, and $\Re$ is closed and invariant, then in the transformation semigroup $(\frac{X}{\Re}, S)$ (where $\pi(x)s := \pi(xs)$ $(x \in X, s \in S)$ (for the projection map $\pi : X \to \frac{X}{\Re}$):

- a is almost periodic if and only if $\pi(a)$ is almost periodic,
- If a is an almost periodic point of X, then $\pi(a)$ is the unique almost periodic point of $\frac{X}{\Re}$.

Example 9.

Let us call a closed right ideal K of $\mathrm{E}(X)$ particular if for each $a,b \in X$, $K \in \mathrm{M}(a) \cap \mathrm{M}(b)$ if and only if a and b are equivalent, and for $\mathrm{D} \in \{\overline{\mathrm{M}}, \overline{\overline{\mathrm{M}}}\}$ call K D-particular if for each nonempty subsets A, B of X, $K \in \mathrm{D}(A) \cap \mathrm{D}(B)$ if and only if A and B are D-equivalent. In the transformation semigroup (X,S), we have:

- $\mathrm{Min}(\mathrm{E}(X)) \cup \{\mathrm{E}(X)\}$ is a subset of the set of particular, $\overline{\mathrm{M}}-$particular and $\overline{\overline{\mathrm{M}}}-$particular closed right ideals of $\mathrm{E}(X)$.
- If K is $\overline{\mathrm{M}}-$particular or $\overline{\overline{\mathrm{M}}}-$particular, then K is particular (since for each $a \in X$ we have $\mathrm{M}(a) = \overline{\mathrm{M}}(\{a\}) = \overline{\overline{\mathrm{M}}}(\{a\})$.

Let $\{(X_\alpha, S_\alpha) \mid \alpha \in \Gamma\}$ be a nonempty collection of transformation semigroups, and for each $\alpha \in \Gamma$, K_α be particular a closed right ideal of $\mathrm{E}(X_\alpha, S_\alpha)$, then $\prod_{\alpha\in\Gamma} K_\alpha$ is particular in the transformation semigroup $(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$.

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